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SEQUENTIAL CONVERGENCES IN BOOLEAN ALGEBRAS

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In this paper sequential convergences in Boolean algedras are investigated which are compatible with the Boolean operations. Analogous questions for lattice ordered groups were studied by M. Harminc [1], [2], [3] and the author [4].

Several types of sequential convergences in abelian lattice ordered groups and in Boolean algebras were dealt with by F. Papangelou [8]; for convergences in Boolean algebras cf. also H. Löwig [5]. Some questions on sequential convergences in σ-fields of sets were investigated by J. Novák a M. Novotný [7].

1. CONVERGENCES AND 0-CONVERGENCES

Let B be a Boolean algebra. In this section the notion of sequential convergence in B will be introduced. It will be proved that a sequential convergence is uniquely determined by the system of all sequences which converge to the zero element of B.

We denote by S the system of all sequences of elements of B. Let α be a subset of $S \times B$. If $((x_n), x) \in \alpha$, then we shall write $x_n \rightarrow_\alpha x$. Let N be the set of all positive integers. If there exists $a \in B$ such that $x_n = a$ for each $n \in N$, then we write $(x_n) = $ $= \text{const}\ a$.

1.1. Definition. A subset α of $S \times B$ is said to be a *convergence in* B, if the following conditions are satisfied:

(i) If $x_n \rightarrow_\alpha x$ and (y_n) is a subsequence of (x_n), then $y_n \rightarrow_\alpha x$.

(ii) If $(x_n) \in S$ and if for each subsequence (y_n) of (x_n) there exists a subseqeunce (z_n) of (y_n) such that $z_n \rightarrow_\alpha x$, then $x_n \rightarrow_\alpha x$.

(iii) For each $x \in B$, const $x \rightarrow_\alpha x$.

(iv) If $x_n \rightarrow_\alpha x$ and $x_n \rightarrow_\alpha y$, then $x = y$.

(v) If $x_n \rightarrow_\alpha x$ and $y_n \rightarrow_\alpha y$, then $x_n \wedge y_n \rightarrow_\alpha x \wedge y$, $x_n \vee y_n \rightarrow_\alpha x \vee y$ and $x'_n \rightarrow_\alpha x'$.

(vi) If $x_n \leqq y_n \leqq z_n$ is valid for each $n \in N$, and if $x_n \rightarrow_\alpha x$, $z_n \rightarrow_\alpha x$, then $y_n \rightarrow_\alpha x$.

The system of all convergences on B will be denoted by Conv B. Let α be a fixed element of Conv B.

1.2. Lemma. *The following conditions are equivalent:*

(a) $x_n \rightarrow_\alpha x$.

(b) $x_n \wedge x \rightarrow_\alpha x$ *and* $x_n \vee x \rightarrow_\alpha x$.

Proof. In view of (iii) and (v) we have (a) $\Rightarrow$ (b). According to (vi), the relation (b) $\Rightarrow$ (a) is valid.

1.3. Lemma. *The condition* (a) *from* 1.2 *is equivalent to the following condition*: (c) $x_n \wedge x' \to_\alpha 0$ *and* $x'_n \wedge x \to_\alpha 0$.

Proof. Let (a) be valid. Then in view of (iii) and (v) the condition (c) holds. Conversely, let (c) be satisfied. Then we have

$$(x_n \wedge x') \vee x \to_\alpha x \quad \text{and} \quad (x'_n \wedge x) \vee x' \to_\alpha x',$$

hence $x_n \vee x \to_\alpha x$ and $x'_n \vee x' \to_\alpha x'$. In view of (v), $x_n \wedge x = (x'_n \vee x')' \to_\alpha x$. Thus by applying 1.2 we obtain that (a) holds.

Let us denote by α_0 the set of all $(x_n) \in S$ such that $x_n \to_\alpha 0$. From 1.3 we infer:

1.4. Corollary. *The set* α_0 *uniquely determines the convergence* α.

A natural problem arises, to characterize those subsets T of S for which there exists $\alpha \in \operatorname{Conv} B$ such that $T = \alpha_0$.

1.5. Lemma. *Let* T *be a nonempty subset of* S. *There exists* $\alpha \in \operatorname{Conv} B$ *with* $T = \alpha_0$ *if and only if the following conditions are satisfied*:

(i_1) *If* $(x_n) \in T$, *then each subsequence of* (x_n) *belongs to* T.

(ii_1) *If* $(x_n) \in S$ *and if for each subsequence* (y_n) *of* (x_n) *there exists a subsequence* (z_n) *of* (y_n) *such that* $(z_n) \in T$, *then* $(x_n) \in T$.

(iii_1) *For* $a \in B$ *we have* $\operatorname{const} a \in T$ *if and only if* $a = 0$.

(iv_1) *If* (x_n) *and* (y_n) *belong to* T, *then* $(x_n \vee y_n)$ *also belongs to* T.

(v_1) *If* (x_n) *belongs to* T *and if* $(y_n) \in S$, $y_n \leqq x_n$ *for all* $n \in N$, *then* $(y_n) \in T$.

Proof. If there is $\alpha \in \operatorname{Conv} G$ such that $T = \alpha_0$, then from 1.1 we immediately obtain that the conditions (i_1)–(v_1) are satisfied.

Conversely, suppose that $T \subseteq S$ is such that (i_1)–(v_1) hold. For $(x_n) \in S$ and $x \in B$ we put $x_n \to_\alpha x$ if $(x_n \wedge x') \in T$ and $(x'_n \wedge x) \in T$.

First we observe that the relation

$$(x_n) \in T \Leftrightarrow x_n \to_\alpha 0$$

is valid for each $(x_n) \in S$.

Indeed, let $(x_n) \in T$. We have $(x_n) = (x_n \wedge 0') \in T$ and $\operatorname{const} 0 = (x'_n \wedge 0) \in T$, whence $x_n \to_\alpha 0$. Conversely, let $x_n \to_\alpha 0$. Then $(x_n \wedge 0') \in T$, whence $(x_n) \in T$.

Now we have to verify that the conditions (i)–(vi) from 1.1 are satisfied.

The conditions (i), (ii) and (iii) are consequences of (i_1), (ii_1) and (iii_1), respectively.

(v): Let $x_n \to_\alpha x$ and $y_n \to_\alpha y$. In view of the first relation we have $(x_n \wedge x') \in T$ and $(x'_n \wedge x) \in T$, whence $x'_n \to_\alpha x'$. Denote $z_n = x_n \vee y_n$, $z = x \vee y$. Then

$$z_n \wedge z' = (x_n \vee y_n) \wedge (x \vee y)' = (x_n \vee y_n) \wedge (x' \wedge y') =$$
$$= [x_n \wedge (x' \wedge y')] \vee [y_n \wedge (x' \wedge y')] .$$

According to (v_1), both $(x_n \wedge (x' \wedge y'))$ and $(y_n \wedge (x' \wedge y'))$ belong to T; hence in view of (iv_1), $(z_n \wedge z')$ belongs to T. Similarly we obtain that $(z'_n \wedge z)$ belongs to T. Thus $z_n \to_\alpha z$.

Next, let $v_n = x_n \wedge y_n$, $v = x \wedge y$. Then

$$v_n \wedge v' = (x_n \wedge y_n) \wedge (x \wedge y)' = (x_n \wedge y_n) \wedge (x' \vee y') =$$
$$= [(x_n \wedge y_n) \wedge x'] \vee [(x_n \wedge y_n) \wedge y'] .$$

By applying (v_1) and (iv_1) we obtain that $(v_n \wedge v') \in T$. Similarly, $(v'_n \wedge v) \in T$. Thus $v_n \to_\alpha v$.

(vi): Let $x_n \to_\alpha x$, $z_n \to_\alpha x$ and suppose that $x_n \leqq y_n \leqq z_n$ is valid for each $n \in N$. Hence the sequences $(x'_n \wedge x)$ and $(z_n \wedge x')$ belong to T. Then according to (v_1) we have $y'_n \wedge x \in T$ and $y_n \wedge x' \in T$; therefore $y_n \to_\alpha x$.

(iv): First we shall verify that if $(a_n) = \text{const } 0$ and if $a_n \to_\alpha a$ then $a = 0$. In fact, in view of the assumption we have $a'_n \wedge a \in T$, hence const $a \in T$. Thus according to (iii_1), $a = 0$. Now assume that $x_n \to_\alpha x$ and $x_n \to_\alpha y$. Hence $x'_n \to_\alpha y'$ and therefore $x_n \wedge x'_n \to_\alpha x \wedge y'$. Since $(x_n \wedge x'_n) = \text{const } 0$, we infer that $x \wedge y' = 0$ and hence $x \leqq y$. Similarly we obtain that $y \leqq x$. Hence $y = x$. The proof is complete.

Denote $\text{Conv}_0 B = \{\alpha_0 : \alpha \in \text{Conv } B\}$. The elements of $\text{Conv}_0 B$ are said to be 0-convergences in B. For $\alpha, \beta \in \text{Conv } B$ we put $\alpha \leqq \beta$ if, whenever $(x_n) \in S$, $x \in B$ and $x_n \to_\alpha x$, then $x_n \to_\beta x$. Further, we put $\alpha_0 \leqq \beta_0$ if α_0 is a subset of β_0. Then we have

$$\alpha \leqq \beta \Leftrightarrow \alpha_0 \leqq \beta_0 .$$

Let $(x_n) \in S$, $x \in B$. We put $x_n \to_d x$ if there is $m \in N$ such that $x_n = x$ for each $n \in N$ with $n > m$. The following assertion is easy to verify.

1.6. Lemma. $d \in \text{Conv } B$ *and for each* $\alpha \in \text{Conv } B$ *we have* $d \leqq \alpha$.

1.7. Corollary. d_0 *is the least element of* $\text{Conv}_0 B$.

2. REGULAR SETS OF SEQUENCES

A nonempty subset A of $S \times B$ will be called *regular* if there exists $\alpha \in \text{Conv } B$ such that $A \subseteq \alpha$. A set A is regular if and only if $A \cup \{(\text{const } 0, 0)\}$ is regular.

Analogously, a nonempty subset T of S will be said to be *regular* if there exists $\alpha_0 \in \text{Conv}_0 B$ such that $T \subseteq \alpha_0$. The set C is regular if and only if $C \cup \{\text{const } 0\}$ is regular.

Let $\emptyset \neq A \subseteq S \times B$. Denote

$$A_1 = \{(x_n \wedge x') : ((x_n), x) \in A\} ,$$
$$A_2 = \{(x'_n \wedge x) : ((x_n), x) \in A\} , \quad A_3 = A_1 \cup A_2 .$$

Let $\emptyset \neq C \subseteq S$. We put

$$C_1 = \{((x_n), x) : (x_n \wedge x') \in C \text{ and } (x'_n \wedge x) \in C\} .$$

In view of the results of Section 1 we have

2.1. Lemma. (i) *Let* (const 0, 0) $\in A \subseteq S \times B$. *Then A is regular if and only if A_3 is regular.*

(ii) *Let* const $0 \in C \subseteq S$. *Then C is regular if and only if C_1 is regular.*

Thus it suffices to investigate the regularity of subsets C of S such that const $0 \in C$.

Let (x_n) and (y_n) be elements of S. We put $(x_n) \wedge (y_n) = (x_n \wedge y_n)$, $(x_n) \vee (z_n) = (x_n \vee y_n)$, $(x_n)' = (x_n')$. Then S turns out to be a Boolean algebra.

Let A be a nonempty subset of S. We denote by

δA – the set of all subsequences of sequences belonging to A;

A^* – the set of all $(x_n) \in S$ such that for each subsequence (y_n) of (x_n) there exists a subsequence (z_n) of (y_n) which belongs to A;

$[A]$ – the ideal of the Boolean algebra S generated by the set A.

The following assertions 2.2–2.4 are easy to verify; the proofs will be omitted.

2.2. Lemma. *Let $b \in B$. Then* const $b \in A$ *if and only if* const $b \in A^*$.

2.3. Lemma. $\delta[\delta A] = [\delta A]$.

2.4. Lemma. $\delta(A^*) \subseteq (\delta A)^*$ *and* $[A^*] \subseteq [A]^*$.

2.5. Corollary. *Put $C = [\delta A]^*$. Then $C = \delta C = [C] = C^*$.*

From 1.5 and 2.5 we infer:

2.6. Corollary. $[\delta A]^*$ *belongs to* $\text{Conv}_0 B$ *if and only if for each nonzero element b of B we have* const $b \notin [\delta A]^*$.

2.7. Proposition. *A nonempty subset A of S is regular if and only if for each nonzero element b of B we have* const $b \notin [\delta A]$.

Proof. This is a consequence of 2.6 and 2.2.

2.8. Proposition. *Let A be a regular subset of S. Let $\alpha \in \text{Conv}_0 B$, $A \subseteq \alpha$. Then $[\delta A]^* \subseteq \alpha$.*

Proof. This is an immediate consequence of 1.5.

In view of 2.5 and 2.8, for a regular subset A of S the 0-convergence $[\delta A]^*$ will be said to be *generated by the set A*. If $A = \{(x_n)\}$ and A is regular, then A is said to be *generated by* (x_n); in such a case $[\delta A]^*$ is called *principal*.

If $\emptyset \neq A \subseteq S$, then $[A]$ is the set of all $(x_n) \in S$ which have the following property: there exist $(y_n^1), (y_n^2), \ldots, (y_n^m)$ in A such that $(x_n) \leqq (y_n^1) \vee (y_n^2) \vee \ldots \vee (y_n^m)$. From 2.7 we obtain:

2.9. Proposition. *Let $\emptyset \neq A \subseteq S$. Then the following conditions are equivalent:*

(i) *A is regular.*

(ii) *If $(y_n^1), (y_n^2), \ldots, (y_n^m)$ are elements of δA and if b is an element of B such that $b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ is valid for each $n \in N$, then $b = 0$.*

2.10. Lemma. *Let $I \neq \emptyset$ and for each $i \in I$ let $\alpha_i^0 \in \mathrm{Conv}_0 B$. Put $A = \bigcup_{i \in I} \alpha_i^0$ Then the following conditions are equivalent:*

(i) *A is regular.*

(ii) *If $i_1, i_2, \ldots, i_m$ are distinct elements of I and if $(y_n^k) \in \alpha_k^0$ for each $k \in \{i_1, i_2, \ldots \ldots, i_m\}$, $b \in B$, $b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ for each $n \in N$, then $b = 0$.*

Proof. This follows from 2.9 and from the fact that $\delta\alpha_i^0 = \alpha_i^0 = [\alpha_i^0]$ for each $i \in I$.

2.11. Lemma. *Let I, α_i^0 and A be as in 2.10. Assume that A is regular. Put $\alpha = [A]^*$. Then*

(i) *$\alpha \in \mathrm{Conv}_0 B$;*

(ii) *$\alpha_i^0 \leqq \alpha$ for each $i \in I$;*

(iii) *if $\beta^0 \in \mathrm{Conv}_0 B$ and $\alpha_i^0 \leqq \beta^0$ for each $i \in I$, then $\alpha \subseteq \beta^0$.*

Proof. Because $\alpha_i^0 \in \mathrm{Conv}_0 B$ for each $i \in I$, we have $\delta\alpha_i^0 = \alpha_i^0$, whence $\delta A = A$. Hence $[\delta A]^* = \alpha$. According to 2.6 and 2.8, $\alpha \in \mathrm{Conv}_0 B$. The assertions (ii) and (iii) are obvious.

A sequence (x_n) in S is said to be *decreasing* if $x_n \geqq x_{n+1}$ for each $n \in N$.

2.12. Lemma. *Let (x_n) be a decreasing sequence in B and let $A = \{(x_n)\}$. Then A is regular if and only if $\bigwedge x_n = 0$.*

Proof. If A s regular, then in view of 2.7 we must have $\bigwedge x_n = 0$. Conversely, assume that $\bigwedge x_n = 0$. Let $(y_n^1), (y_n^2), \ldots, (y_n^m)$ be subsequences of (x_n). Let $b \in B$ and suppose that $b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ is valid for each $n \in N$. We have $y_n^k \leqq x_n$ for $k = 1, 2, \ldots, m$, whence $b \leqq x_n$ for each $n \in N$. Therefore $b = 0$. Thus according to 2.9, A is regular.

3. THE PARTIALLY ORDERED SET $\mathrm{Conv}_0 B$

As we already remarked in Section 1, the set $\mathrm{Conv}_0 B$ is considered to be partially ordered by inclusion. Each nonempty subset of $\mathrm{Conv}_0 B$ is partially ordered by the induced partial order. Let $\mathrm{Conv}_p B$ be the set of all principal elements of $\mathrm{Conv}_0 B$.

Let $I \neq \emptyset$ and for each $i \in I$ let $\alpha_i^0 \in \mathrm{Conv}_0 B$. If the set $\{\alpha_i^0\}_{i \in I}$ has the infimum or the supremum in $\mathrm{Conv}_0 B$, then these elements will be denoted by $\bigwedge_{i \in I} \alpha_i^0$ or $\bigwedge_{i \in I} \alpha_i^0$, respectively.

3.1. Lemma. *Let $\{\alpha_i^0\}_{i \in I}$ be a nonempty subset of* $\mathrm{Conv}_0 B$. *Then $\bigwedge_{i \in I} \alpha_i^0 = \bigcap_{i \in I} \alpha_i^0$.*

Proof. This is a consequence of the fact that $\bigcap_{i \in I} \alpha_i^0$ satisfies the conditions from 1.5.

3.2. Corollary. *Let $\alpha^0 \in \mathrm{Conv}_0 B$. Then the interval $[d, \alpha^0]$ of* $\mathrm{Conv}_0 B$ *is a complete lattice.* $\mathrm{Conv}_0 B$ *is a $\bigwedge$-semilattice.*

In Section 4 it will be shown that $\mathrm{Conv}_0 B$ need not be a lattice.

3.3. Lemma. *Let* $\{\alpha_i^0\}_{i\in I}$ *be a nonempty subset of* $\mathrm{Conv}_0 B$. *Put* $A = \bigcup_{i\in I} \alpha_i^0$. *Then the following conditions are equivalent*:

(i) *A is regular.*

(ii) $[A]^* = \bigvee_{i\in I} \alpha_i^0$.

Proof. The implication (i) ⇒ (ii) is a consequence of 2.11. The implication (ii) ⇒ (i) is obvious.

3.4. Lemma. *The following conditions are equivalent*:

(i) $\mathrm{Conv}_0 B$ *has no greatest element.*

(ii) *There are* $\beta_1^0, \beta_2^0 \in \mathrm{Conv}_p B$ *such that the set* $\{\beta_1^0, \beta_2^0\}$ *is not upper bounded in* $\mathrm{Conv}_0 B$.

Proof. The implication (ii) ⇒ (i) is trivial. Assume that (i) holds. Let $\mathrm{Conv}_0 B = \{\alpha_j^0\}_{j\in J}$. Put $A = \bigcup_{j\in J} \alpha_j^0$. In view of 3.3, A fails to be regular. Hence according to 2.10 there exists a positive integer m, elements $j_1, j_2, \ldots, j_m \in J$, sequences $(y_n^1) \in \alpha_{j_1}^0, \ldots, (y_n^m) \in \alpha_{j_m}^0$ and an element $b \neq 0$ in B such that $b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ is valid for each $n \in N$.

Let m be the least positive integer having the just mentioned property. We must have $m \geqq 2$. Assume that $m > 2$. In view of this assumption, the set $(\{y_n^2), (y_n^3), \ldots, (y_n^m)\}$ is regular. From this we infer that the one-lement set $\{(y_n^2 \vee y_n^3 \vee \ldots \vee y_n^m)\} = \{(z_n)\}$ is regular as well. Since $b \leqq y_n^1 \vee z_n$ holds for each $n \in N$, by virtue of the relation $m > 2$ we have $b = 0$, which is a contradiction. Hence we have $m = 2$. Both the sets $A_1 = \{(y_n^1)\}$, $A_2 = \{(z_n)\}$ are regular, hence $\beta_1^0 = [\delta A_1]^*$ and $\beta_2^0 = [\delta A_2]^*$ belong to $\mathrm{Conv}_0 B$. But the set $\{(y_n^1), (z_n)\}$ is not regular. Thus the set $\{\beta_1^0, \beta_2^0\}$ fails to be upper bounded in $\mathrm{Conv}_0 B$.

3.5. Lemma. *Let* α_1^0 *and* α_2^0 *be principal elements of* $\mathrm{Conv}_0 B$ *generated by the sequences* (x_n^1) *and* (x_n^2), *respectively. Assume that the set* $\{\alpha_1^0, \alpha_2^0\}$ *is upper bounded in* $\mathrm{Conv}_0 B$. *Then* $\alpha_1^0 \vee \alpha_2^0$ *is principal and it is generated by* $(x_n^1 \vee x_n^2)$.

Proof. In view of 3.2, $\alpha_1^0 \vee \alpha_2^0$ does exist in $\mathrm{Conv}_0 B$. Hence the one-element set $A = \{(x_n^1 \vee x_n^2)\}$ is regular. Thus there exists $\beta^0 \in \mathrm{Conv}_0 G$ such that β^0 is generated by A. Clearly $\beta^0 \leqq \alpha_1^0 \vee \alpha_2^0$ isnce $x_n^1 \vee x_n^2 \to_\gamma 0$, where $\gamma = \alpha_1^0 \vee \alpha_2^0$. On the other αand, from $[\delta\{(x_n^1)\}] \subseteq [\delta\{(x_n^1 \vee x_n^2)\}]$ we obtain that $\alpha_1^0 \leqq \beta^0$; similarly we have $h_2^0 \leqq \beta^0$. Thus $\beta^0 = \alpha_1^0 \vee \alpha_2^0$.

From 3.2, 3.4 and 3.5 we infer:

3.6. Theorem. *Let B be a Boolean algebra. The following conditions are equivalent*:

(i) $\mathrm{Conv}_0 B$ *has the greatest element.*

(ii) $\mathrm{Conv}_p B$ *is a* $\bigvee$*-semilattice.*

(iii) $\mathrm{Conv}_0 B$ *is a lattice.*

(iv) $\mathrm{Conv}_0 B$ *is a complete lattice.*

For a related result concerning lattice ordered groups cf. [3].

Let us remark that if α_1^0 and α_2^0 are as in 3.5, then the element $\alpha_1^0 \wedge \alpha_2^0 = \alpha_1^0 \cap \alpha_2^0$ of $\mathrm{Conv}_0\, B$ need not be generated by the sequence $(x_n \wedge y_n)$. Also, if $\beta \in \mathrm{Conv}_0\, B$ such that $\beta < \beta_1^0$, then β need not be principal.

4. COMPLETE DISTRIBUTIVITY

In this section the following result will be proved:

4.1. Theorem. *Let B be a Boolean algebra. Assume that B is completely distributive. Then* $\mathrm{Conv}_0\, B$ *has the greatest element.*

Next it will be shown that there exists a Boolean algebra B such that $\mathrm{Conv}_0\, B$ has no greatest element.

Proof of 4.1. Since B is completely distributive, there exists a set I such that there is an isomorphism φ of B into a complete field C of subsets of I such that, whenever $\bigwedge_{i\in I} x_i = x$ is valid in B, then $\bigcap_{i\in I} \varphi(x_i) = \varphi(x)$ is valid (and dually). Without loss of generality we can assume that $\varphi(0) = \emptyset$ and $\varphi(1) = I$. Let A be the set of all $(x_n) \in S$ which have the following property: for each $i \in I$ there exists a positive integer $n(i)$ such that $i \notin \varphi(x_n)$ whenever $n \geqq n(i)$. Then we clearly have $[\delta A]^* = A$. Let $(y_n^1), (y_n^2), \ldots, (y_n^m) \in A$, $b \in B$ and suppose that $b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$ is valid for each $n \in N$. Assume that $b > 0$. Then there exists $i \in I$ such that $i \in \varphi(b)$. On the other hand, there exists $n_0 \in N$ such that for each $n \geqq n_0$ and each $k \in \{1, 2, \ldots, m\}$ we have $i \notin \varphi(y_n^k)$. Thus $i \notin \varphi(y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m)$ for $n \geqq n_0$, which is a contradiction. Therefore in view of 2.9, A is regular. Hence $A \in \mathrm{Conv}_0\, B$.

If $\alpha \in \mathrm{Conv}_0\, B$, $(x_n) \in \alpha$, then $\{(x_n)\}$ is regular and therefore for each $i \in I$ there is $n_0 \in N$ such that $i \notin \varphi(x_n)$ whenever $n \geqq n_0$. Hence $(x_n) \in A$ and thus A is the greatest element of $\mathrm{Conv}_0\, B$.

An analogous result for convergences in archimedean lattice ordered groups was established in [4].

The following example shows that $\mathrm{Conv}_0\, B$ need not have the greatest element.

4.2. Example. Let Q be the set of all rational numbers and let e be a fixed irrational number. Put $Q_1 = \{x \in Q: e < x < e + 1\}$. Let B be the set of all mappings f of Q_1 into the set $\{0, 1\}$ having the property that there are irrational numbers $a_0 < a_1 < \ldots \ldots < a_n$ (depending on f), $a_0 = e$, $a_n = e + 1$ such that, whenever $j \in \{0, 1, 2, \ldots, n - 1\}$, then f is a constant on the set $\{x \in Q: a_j < x < a_{j+1}\}$. The set B is pointwise partially ordered; then B is a Boolean algebra. Let $(S(n))$ and $(T(n))$ be as in [1], Section 5. From 2.7 and from the results of [1], Section 5 (cf. also [3], Section 7.6) it follows that the sets $(S(n))$ and $(T(n))$ are regular (with respect to B), but the set $\{(S(n)), (T(n))\}$ fails to be upper bounded in $\mathrm{Conv}_0\, B$. Hence $\mathrm{Conv}_0\, B$ has no greatest element.

For any partially ordered set P with the least element 0_P we define a subset P_1 of P to be disjoint if $p > 0_P$ for each $p \in P_1$ and $p \wedge q = 0_P$ whenever p and q are distinct elements of P_1. Denote

$$D(P) = \sup \{\mathrm{card}\, A_i \colon A_i \in \mathscr{A}\} ,$$

where $\mathscr{A}$ is the system of all disjoint subsets of P.

Now let $\mathscr{A}_1$ be the set of all linearly ordered subsets of a partially ordered set P. Put

$$L(P) = \sup \{\mathrm{card}\, A_i \colon A_i \in \mathscr{A}_i\} .$$

Let B be a Boolean algebra. The cardinals $D(B)$ and $L(B)$ were dealt with in several papers, cf., e.g., Pierce [9] and Monk [6].

In the present section it will be proved that for each infinite Boolean algebra B the relations

$$D(B) \leqq D(\mathrm{Conv}_0\, B) , \quad D(B) \leqq L(\mathrm{Conv}_0\, B)$$

are valid. Also it will be shown that $\mathrm{Conv}_0\, B$ has no atom.

Throughout this section we assume that B is an infinite Boolean algebra. A sequence (x_n) in B is said to be *disjoint* if $x_n > 0$ for each $n \in N$ and $x_n \wedge x_m = 0$ whenever m n are distinct positive integers.

5.1. Lemma. *Let* $A = \{(x_n^i)\}_{i\in I}$ *be a system of sequences in* B *such that* $x_{n(1)}^{i(1)} \wedge \wedge\, x_{n(2)}^{i(2)} = 0$ *whenever* $(n(1), i(1))$ *and* $(n(2), i(2))$ *are distinct elements of the set* $N \times I$. *Then the set* A *is regular.*

Proof. By way of contradiction, assume that A fails to be regular. Hence in view of 2.9 there are elements $i(1), i(2), \ldots, i(m)$, subsequences (y_n^t) of $(x_n^{i(t)})$ $(t = 1, 2, \ldots, m)$ and an element $0 < b \in B$ such that

$$b \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m$$

is valid for each $n \in N$.

In particular, we have

$$b \leqq y_1^1 \vee y_1^2 \vee \ldots \vee y_1^m .$$

There exists $n \in N$ such that for each $t \in \{1, 2, \ldots, m\}$ and for each $i \in I$ we have $y_1^t \wedge x_n^i = 0$. Let n have the just mentioned property. Then $y_1^t \wedge y_n^{t(1)} = 0$ for each t, $t(1) \in \{1, 2, \ldots, m\}$. Hence

$$b = b \wedge (y_n^1 \wedge y_n^2 \wedge \ldots \wedge y_n^m) \leqq$$
$$\leqq (y_1^1 \wedge y_1^2 \wedge \ldots \wedge y_1^m) \wedge (y_n^1 \wedge y_n^2 \wedge \ldots \wedge y_n^m) = 0 ,$$

which is a contradiction.

From 5.1, 2.6 and 2.8 we obtain:

5.2. Corollary. *For each disjoint sequence* (x_n) *in* B *there exists* $\alpha(x_n) \in \mathrm{Conv}_0\, B$ *such that* $\alpha(x_n)$ *is generated by* (x_n).

In the sequel, the notation $\alpha(x_n)$ from 5.2 will be applied whenever the set $\{(x_n)\}$ will be regular.

5.3. Lemma. *Let (x_n) and (y_n) be disjoint sequences in B such that $x_n \wedge y_m = 0$ for each m, $n \in N$. Then $\alpha(x_n) \wedge \alpha(y_n) = d$.*

Proof. By way of contradiction, assume that there exists $(s_n) \in \alpha(x_n) \wedge \alpha(y_n)$ such that $(s_n) \notin d$. Then without loss of generality we can assume that $s_n > 0$ for each $n \in N$.

From $(s_n) \in \alpha(x_n)$ we infer that there is a subsequence (s_n^1) of (s_n) with $(s_n^1) \in [\delta(x_n)]$. Hence there are subsequences $(x_n^1), (x_n^2), \ldots, (x_n^k)$ of (x_n) such that

(1) $$s_n^1 \leqq x_n^1 \vee x_n^2 \vee \ldots \vee x_n^k \quad \text{for each} \quad n \in N\,.$$

We have $(s_n^1) \in \alpha(y_n)$. Hence by an analogous reasoning we deduce that there are subsequences $(y_n^1), (y_n^2), \ldots, (y_n^m)$ of (y_n) and a subsequence (s_n^2) of (s_n^1) such that

(2) $$s_n^2 \leqq y_n^1 \vee y_n^2 \vee \ldots \vee y_n^m \quad \text{for each} \quad n \in N\,.$$

In view of (1) and (2) the relation $s_n^2 = 0$ is valid for each $n \in N$, which is a contradiction.

Since B is infinite, there exists an infinite disjoint subset of B.

5.4. Theorem. *Let B be a Boolean algebra. Let X be an infinite disjoint subset of B,* card $X = \varkappa$. *Then there exists a system $S_1 = \{\alpha_i^0\}_{i \in I}$ in* $\text{Conv}_0\, B$ *such that*

(i) *the system S_1 is disjoint and* card $S_1 = \varkappa$;

(ii) *for each $i \in I$, the 0-convergence α_i^0 is generated by a disjoint sequence.*

Proof. Without loss of generality we can assume that we have $X = \{x_n^i\}_{i \in I, n \in N}$, card $I = \varkappa$, and that $x_{n(1)}^{i(1)} \neq x_{n(2)}^{i(2)}$ whenever $(i(1), n(1)) \neq (i(2), n(2))$. For each $i \in I$ we put $\alpha_i^0 = \alpha(x_n^i)$. In view of 5.2, $\alpha_i^0 \in \text{Conv}_0\, B$ for each $i \in I$. According to 5.3, the system S_1 is disjoint in $\text{Conv}_0\, B$.

We clearly have card $S_1 = \varkappa$. Thus we obtain:

5.5. Corollary. *Let B be an infinite Boolean algebra. Then $D(B) \leqq D(\text{Conv}_0\, B)$.*

5.6. Lemma. *Let (x_n) be a disjoint sequence in B. Assume that $y_n = \bigvee_{m \geqq n} x_m$ is valid for each $n \in N$. Then (y_n) is decreasing and $\bigwedge y_n = 0$.*

Proof. Let $z \in B$, $z \leqq y_n$ for each $n \in N$. First suppose that there exists $n \in N$ such that $0 < z_1 = z \wedge x_n$. There exists $z_2 \in B$ such that $z_1 \wedge z_2 = 0$ and $z_1 \vee z_2 = z$. Then $z_1 \wedge x_m = 0$ for each $m \in N \setminus \{n\}$ and hence $z_1 \wedge y_m = 0$ for each $m > n$. Hence for $m > n$ we have $z \wedge y_m = z_2 \wedge y_m < z$, which is a contradiction. Hence $z \wedge x_n = 0$ for each $n \in N$. Thus $z = z \wedge y_m = z \wedge (\bigvee_{m \geqq n} x_m) = \bigvee_{m \geqq n} (z \wedge x_m) = 0$ and therefore $\bigwedge y_n = 0$. It is obvious that (y_n) is decreasing.

5.7. Theorem. *Let B be a complete Boolean algebra. Let X be an infinite disjoint subset of B,* card $X = \varkappa$. *Then there exists a system $S_2 = \{\beta_i^0\}_{i \in I}$ in* $\text{Conv}_0\, B$ *such that*

(i) *the system S_2 is disjoint and* card $S_2 = \varkappa$;

(ii) *for each* $i \in I$, *the* 0-*convergence* β_i^0 *is generated by a decreasing sequence.*

Proof. Let X be as in the proof of 5.4. For each $i \in I$ and each $n \in N$ put $y_n^i = \bigvee_{m \geqq n} x_m^i$. Then for each $i \in I$, $\{y_n^i\}$ is a decreasing sequence and $\bigwedge_n y_n^i = 0$ (cf. 5.6). Hence according to 2.12 there exists $\beta_i^0 = \alpha(y_n^i)$ in $\text{Conv}_0\, B$. From the fact that X is a disjoint system and from 5.3 we infer that the system S_2 is disjoint. Clearly card $S_2 = \varkappa$.

5.8. Remark. The question whether the assumption of completeness of B can be cancelled in 5.7 remains open.

5.9. Theorem. *Let B be a Boolean algebra. Let X be an infinite disjoint subset of B,* card $X = \varkappa$. *Then there exists a system* $S_3 = \{\beta_i^0\}_{i \in I}$ *in* $\text{Conv}_0\, B$ *such that* S_3 *is a chain and* card $S_3 = \varkappa$.

Proof. Let X be expressed as in the proof of 5.4. Without loss of generality we may suppose that the set I is linearly ordered. For each $i \in I$ put

$$A_i = \{(x_n^j) : j \in I,\ j \leqq i\}\ .$$

Then for each $i \in I$, the set A_i is regular. Moreover, if $i(1)$ and $i(2)$ are elements of I such that $i(1) < i(2)$, then $\alpha(A_{i(1)}) \subset \alpha(A_{i(2)})$. (we denote $\alpha(A_{i(1)}) = [\delta A_{i(1)}]^*$, and similarly for $A_{i(2)}$.) Hence the system $S_3 = \{\alpha(A_i)\}_{i \in I}$ is a chain and card $S_3 = \varkappa$.

5.10. Corollary. *Let B be an infinite Boolean algebra. Then* $D(B) \leqq L(\text{Conv}_0\, B)$.

5.11. Theorem. *Let B be an infinite Boolean algebra. Then the partially ordered set* $\text{Conv}_0\, B$ *has no atom.*

Proof. Let $A \in \text{Conv}_0\, B$. Then for each $(x_n) \in A$, the set $\{(x_n)\}$ is regular, hence $\alpha(x_n) \in \text{Conv}_0\, B$ and $\alpha(x_n) \leqq A$. If $\alpha(x_n) = d$ is valid for each $(x_n) \in A$, then $A = d$.

Thus it suffices to verify that no principal element of $\text{Conv}_0\, B$ is an atom of $\text{Conv}_0\, B$.

To each sequence (x_n) such that $\{(x_n)\}$ is regular and $\alpha(x_n) \neq d$ we shall assign in a constructive way a sequence (z_n) such that $\{(z_n)\}$ is regular and $d < \alpha(z_n) < \alpha(x_n)$.

The construction proceeds as follows. Let (x_n) have the above mentioned properties. We denote by $n(1)$ the first positive integer n with $x_n \neq 0$. Since $\{(x_n)\}$ is regular, there exists $n \in N$ such that $n > n(1)$, $x_n \neq 0$ and $x_n \ngeqq x_{n(1)}$; let $n(2)$ be the least positive integer having this property. Then $x_{n(1)} \wedge x_{n(2)} < x_{n(1)}$. Let y_1 be the relative complement of $x_{n(1)} \wedge x_{n(2)}$ in the interval $[0, x_{n(1)}]$. We have $0 < y_1 \leqq x_{n(1)}$ and $y_1 \wedge x_{n(2)} = 0$.

There exists $n \in N$ such that $n > n(2)$, $x_n \neq 0$ and $x_n \neq x_{n(2)}$; let $n(3)$ be the least n having this property. We construct y_2 by means of $x_{n(2)}$ and $x_{n(3)}$ in the same way as we did y_1 by means of $x_{n(1)}$ and $x_{n(2)}$. Then $0 < y_2 \leqq x_{n(2)}$ and $y_2 \wedge x_{n(3)} = 0$. We have also $y_1 \wedge y_2 = 0$.

We proceed by the obvious induction, obtaining a disjoint sequence (y_n) in B

such that $y_1 \leqq x_{n(1)}, y_2 \leqq x_{n(2)}, \ldots$. Hence $\alpha(y_n) \leqq \alpha(x_n)$. For each $n \in N$ let $z_n = y_{2n}$ and $t_n = y_{2n+1}$. Then $\{(z_n)\}$ is regular and $d < (z_n) \leqq \alpha(x_n)$. Moreover, $(t_n) \in (x_n)$, but (t_n) does not belong to $\alpha(z_n)$. Thus $\alpha(z_n) < \alpha(x_n)$.

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