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# ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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## 1. INTRODUCTION

We consider nonlinear differential equations of the form

$$(1)\qquad y^{(n)} + f(t, y) = 0\,, \quad n \geqq 2\,,$$

where $f: [0, \infty) \times \mathbb{R} \to (0, \infty)$ is continuous and nondecreasing in the second variable. In some of the results presented below the following condition will be needed:

$$(2)\qquad \lim_{y \to -\infty} f(t, y) = 0 \quad \text{for each} \quad t \in [0, \infty)\,.$$

A prototype of such equations is

$$(3)\qquad y^{(n)} + p(t) \exp\left(|y|^{\gamma-1} y\right) = 0\,,$$

where $\gamma > 0$ is a constant and $p: [0, \infty) \to (0, \infty)$ is a continuous function.

We are concerned with the asymptotic behavior of solutions of equation (1). We begin by observing that all solutions of (1) can be continued indefinitely to the right; more precisely, for any $a \geqq 0$ and $(\eta_0, \eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^n$, the solution $y(t)$ of (1) satisfying the initial conditions

$$(4)\qquad y^{(i)}(a) = \eta_i\,, \quad i = 0, 1, \ldots, n - 1\,,$$

exists throughout the interval $[a, \infty)$. In fact, let $[a, T)$ be the maximal interval of existence of $y(t)$ and suppose that $T < \infty$. Since $y^{(n)}(t) = -f(t, y(t)) < 0$, $y^{(n-1)}(t)$ is decreasing and $\lim_{t \to T-} y^{(n-1)}(t) = -\infty$. It follows that $\lim_{t \to T-} y(t)$ exists in $\mathbb{R} \cup \{-\infty\}$, and so there are constants $c$ and $t_0 \in (a, T)$ such that $y(t) \leqq c$ for $t \in [t_0, T)$. Integrating (1) and using this inequality, we have

$$y^{(n-1)}(t_0) - y^{(n-1)}(t) = \int_{t_0}^{t} f(s, y(s))\, \mathrm{d}s \leqq \int_{t_0}^{t} f(s, c)\, \mathrm{d}s\,,$$

which in the limit as $t \to T-$ gives

$$\infty = \int_{t_0}^{T} f(s, c)\, \mathrm{d}s < \infty\,.$$

This contradiction shows that $T$ must be $\infty$, that is, $y(t)$ exists on the entire interval $[a, \infty)$.

Let $a \geqq 0$ be fixed and let $S$ denote the set of all solutions of (1) existing on $[a, \infty)$. To classify $S$ according to orders of increase or decrease of its members as $t \to \infty$ we introduce the following notation:

(i) $S_+^{n-1} = \{y \in S: y^{(n-1)}(\infty) = \lim_{t\to\infty} y^{(n-1)}(t) > 0\}$,
$S_-^{n-1} = \{y \in S: y^{(n-1)}(\infty) < 0\}$,
$S_0^{n-1} = \{y \in S: y^{(n-1)}(\infty) = 0\}$;

(ii) for $k = 1, 2, \ldots, n-2$,
$S_+^k = \{y \in S: y^{(n-1)}(\infty) = \ldots = y^{(k+1)}(\infty) = 0,\ y^{(k)}(\infty) > 0\}$,
$S_-^k = \{y \in S: y^{(n-1)}(\infty) = \ldots = y^{(k+1)}(\infty) = 0,\ y^{(k)}(\infty) < 0\}$,
$S_0^k = \{y \in S: y^{(n-1)}(\infty) = \ldots = y^{(k+1)}(\infty) = y^{(k)}(\infty) = 0\}$;

(iii) for $k = 1, 2, \ldots, n-1$,
$S_{+b}^k = \{y \in S_+^k: y^{(k)}(\infty) < \infty\}$, $\quad S_{+u}^k = \{y \in S_+^k: y^{(k)}(\infty) = \infty\}$,
$S_{-b}^k = \{y \in S_-^k: y^{(k)}(\infty) > -\infty\}$, $S_{-u}^k = \{y \in S_-^k: y^{(k)}(\infty) = -\infty\}$.

If $y \in S$, then $y^{(n-1)}(t)$ is decreasing, so that $y^{(n-1)}(\infty)$ exists in $\mathbb{R} \cup \{-\infty\}$. It may happen that $y \in S_+^{n-1}$, $y \in S_0^{n-1}$ or $y \in S_-^{n-1}$, that is, we have a decomposition

$$S = S_+^{n-1} \cup S_-^{n-1} \cup S_0^{n-1}\,, \tag{5}$$

where

$$S_+^{n-1} = S_{+b}^{n-1}\,, \quad S_-^{n-1} = S_{-b}^{n-1} \cup S_{-u}^{n-1}\,. \tag{6}$$

Let $y \in S_0^{n-1}$. Then, $y^{(n-1)}(t) > 0$ on $[a, \infty)$, and so $y^{(n-2)}(t)$ is increasing and $y^{(n-2)}(\infty)$ exists in $\mathbb{R} \cup \{\infty\}$. Therefore,

$$S_0^{n-1} = S_+^{n-2} \cup S_-^{n-2} \cup S_0^{n-2}\,, \tag{7}$$

where

$$S_+^{n-2} = S_{+b}^{n-2} \cup S_{+u}^{n-2}\,, \quad S_-^{n-2} = S_{-b}^{n-2}\,. \tag{8}$$

From (5) and (7) it follows that

$$S = (S_+^{n-1} \cup S_+^{n-2}) \cup (S_-^{n-1} \cup S_-^{n-2}) \cup S_0^{n-2}\,,$$

where $S_\pm^{n-1}$ and $S_\pm^{n-2}$ satisfy (6) and (8). Next we express $S_0^{n-2}$ as the union of $S_+^{n-3}$, $S_-^{n-3}$ and $S_0^{n-3}$, and then analyse $S_0^{n-3}$. Continuing in this manner, we arrive at a decomposition

$$S = (S_+^{n-1} \cup S_+^{n-2} \cup \ldots \cup S_+^1) \cup (S_-^{n-1} \cup S_-^{n-2} \cup \ldots \cup S_-^1) \cup S_0^1\,, \tag{9}$$

where

$$S_+^k = S_{+b}^k\,, \quad S_-^k = S_{-b}^k \cup S_{-u}^k \quad \text{for} \quad n \not\equiv k \pmod 2\,, \tag{10}$$

$$S_+^k = S_{+b}^k \cup S_{+u}^k\,, \quad S_-^k = S_{-b}^k \quad \text{for} \quad n \equiv k \pmod 2\,. \tag{11}$$

Finally, noting that in case $n$ is even [resp. odd] $y \in S_0^1$ satisfies $y'(t) > 0$ [resp.

$y'(t) < 0]$ for all large $t$, we can decompose $S_0^1$ as follows:

$$S_0^1 = S_b^0 \cup S_+^0 \quad \text{for} \quad n \text{ even}, \tag{12}$$

$$S_0^1 = S_b^0 \cup S_-^0 \quad \text{for} \quad n \text{ odd}, \tag{13}$$

where

$$\begin{aligned} S_+^0 &= S_{+u}^0 = \{y \in S_0^1 : y(\infty) = \infty\}, \\ S_-^0 &= S_{-u}^0 = \{y \in S_0^1 : y(\infty) = -\infty\}, \\ S_b^0 &= \{y \in S_0^1 : -\infty < y(\infty) < \infty\}. \end{aligned} \tag{14}$$

This leads us to a refinement of (9):

$$\begin{aligned} S = {} & (S_+^{n-1} \cup S_+^{n-2} \cup \ldots \cup S_+^1 \cup S_+^0) \cup S_b^0 \cup \\ & \cup (S_-^{n-1} \cup S_-^{n-2} \cup \ldots \cup S_-^1) \quad \text{for} \quad n \text{ even}, \end{aligned} \tag{15}$$

$$\begin{aligned} S = {} & (S_+^{n-1} \cup S_+^{n-2} \cup \ldots \cup S_+^1) \cup S_b^0 \cup \\ & \cup (S_-^{n-1} \cup S_-^{n-2} \cup \ldots \cup S_-^1 \cup S_-^0) \quad \text{for} \quad n \text{ odd}. \end{aligned} \tag{16}$$

The objective of this paper is to establish criteria for the existence (or nonexistence) of members of the subclasses of $S$ appearing in (15) and (16), and then to obtain detailed information about the structure of the solution set of equation (1). Our main tool is the Schauder-Tychonoff fixed point theorem applied to a nonlinear integral equation whose solution gives rise to a solution of (1) belonging to one of the subclasses under consideration. We present examples illustrating the main results, and show that our theory is applicable to the qualitative study of certain semilinear elliptic equations in exterior domains.

The qualitative behavior of equations of the form (1) generalizing the Emden-Fowler equation

$$y^{(n)} + p(t)\,|y|^\gamma \operatorname{sgn} y = 0, \quad n \geqq 2, \tag{17}$$

with $\gamma > 0$ and $p: [0, \infty) \to (0, \infty)$ continuous, has been investigated in great detail by many authors; see, for example, the papers [1–3, 6–11]. However, equation (1) in our setting seems to have been ignored in the literature, and the present work was motivated by this observation. For the first attempt in this direction we refer to the paper [5] dealing with the equation $(p(t)\,y')' + f(t, y, y') = 0$.

## 2. MAIN RESULTS

**A.** We start with the analysis of $S_+^{n-1} \cup S_-^{n-1}$.

**Theorem 1.** *The class $S_-^{n-1}$ is always nonempty.*

Proof. It is clear that any solution $y(t)$ of (1) satisfying (4) with $\eta_{n-1} \leqq 0$ is a member of $S_-^{n-1}$.

**Theorem 2.** $S_{-b}^{n-1} \neq \emptyset$ *if and only if*

$$\int_a^\infty f(t, -\lambda t^{n-1})\, dt < \infty \quad \textit{for some} \quad \lambda > 0 \,. \tag{18}$$

Proof. Let $y \in S_{-b}^{n-1}$. Since

$$\lim_{t\to\infty} \frac{y(t)}{t^{n-1}/(n-1)!} = \dots = \lim_{t\to\infty} y^{(n-1)}(t) = \text{const} < 0 \,,$$

there exist constants $\lambda > 0$ and $t_0 > a$ such that

$$y(t) \geqq -\lambda t^{n-1} \quad \text{for} \quad t \geqq t_0 \,. \tag{19}$$

On the other hand, integration of (1) shows that

$$\int_a^\infty f(t, y(t))\, dt < \infty \,. \tag{20}$$

Combining (19) with (20) yields (18).

Conversely, suppose that (18) holds. Let $\eta \in \mathbb{R}$ be fixed. Since $\int_a^\infty f(t, \eta - ct^{n-1})\, dt$ is nonincreasing in $c$ for $c > \lambda$, a constant $c > \lambda$ can be chosen so that

$$\int_a^\infty f(t, \eta - ct^{n-1})\, dt \leqq c/2 \,.$$

Let $Y$ denote the set

$$Y = \{y \in C[a, \infty) : \eta - 2ct^{n-1} \leqq y(t) \leqq \eta - ct^{n-1},\ t \geqq a\}$$

and define the mapping $F: Y \to C[a, \infty)$ by

$$Fy(t) = \eta - 2ct^{n-1} + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \int_s^\infty f(r, y(r))\, dr\, ds \,, \quad t \geqq a \,.$$

It is easily verified that $F$ is continuous and maps $Y$ into a compact subset of $Y$, and so $F$ has a fixed point $y \in Y$ by the Schauder-Tychonoff fixed point theorem. Differentiation of the integral equation $y(t) = Fy(t)$ $n$ times shows that $y(t)$ is a solution of (1) on $[a, \infty)$. Since $\lim_{t\to\infty} y^{(n-1)}(t) = -2c(n-1)! < 0$, $y(t)$ is a member of $S_{-b}^{n-1}$.

**Theorem 3.** $S_{-u}^{n-1} \neq \emptyset$ *if and only if*

$$\int_a^\infty f(t, -\lambda t^{n-1})\, dt = \infty \quad \textit{for all} \quad \lambda > 0 \,. \tag{21}$$

Proof. Let $y \in S_{-u}^{n-1}$. Integrating (1), we get

$$\int_a^\infty f(t, y(t))\, dt = \infty \,. \tag{22}$$

Since

$$\lim_{t\to\infty} \frac{y(t)}{t^{n-1}/(n-1)!} = \dots = \lim_{t\to\infty} y^{(n-1)}(t) = -\infty \,,$$

for any given $\lambda > 0$ there is $t_0 > 0$ such that

$$y(t) \leqq -\lambda t^{n-1} \quad \text{for} \quad t \geqq t_0 \,. \tag{23}$$

The relation (21) follows from (22) and (23).

Conversely, if (21) holds, then Theorem 2 implies that $S_{-b}^{n-1} = \emptyset$, and hence $S_{-u}^{n-1} \neq \emptyset$ by Theorem 1.

**Theorem 4.** *Suppose that* (2) *holds. Then,* $S^{n-1}_{+b} \neq \emptyset$ *if and only if*

(24) $$\int_a^\infty f(t, \lambda t^{n-1})\,dt < \infty \quad \textit{for some} \quad \lambda > 0\,.$$

Proof. Let $y \in S^{n-1}_{+b}$. Clearly, (20) holds. There are constants $\lambda > 0$ and $t_0 > a$ such that

$$y(t) \geqq \lambda t^{n-1} \quad \text{for} \quad t \geqq t_0\,,$$

which, combined with (20), yields (24).

Suppose that (24) holds. In view of (2) and the Lebesgue dominated convergence theorem we have $\lim_{\eta \to -\infty} \int_a^\infty f(t, \eta + \lambda t^{n-1})\,dt = 0$ and hence

$$\int_a^\infty f(t, \eta + \lambda t^{n-1})\,dt \leqq \tfrac{1}{2}\lambda$$

for some $\eta < 0$. By means of the Schauder-Tychonoff theorem the mapping

$$Fy(t) = \eta + \frac{\lambda t^{n-1}}{2} + \int_a^t \frac{(t-s)^{n-2}}{(n-2)!} \int_s^\infty f(r, y(r))\,dr\,ds\,, \quad t \geqq a\,,$$

is shown to have a fixed element $y$ in the set

$$Y = \left\{ y \in C[a, \infty) \colon \eta + \frac{\lambda t^{n-1}}{2} \leqq y(t) \leqq \eta + \lambda t^{n-1},\ t \geqq a \right\}.$$

This fixed element $y$ is a member of $S^{n-1}_{+b}$, since $\lim_{t \to \infty} y^{(n-1)}(t) = \lambda(n-1)!/2 > 0$.

**B.** We now turn to the study of $S^k_+ \cup S^k_-$ for $1 \leqq k \leqq n-2$. Suppose that $y \in S^{k+1}_0$ for $1 \leqq k \leqq n-2$. Repeated integration of (1) over $[t, \infty)$ shows that

(25) $$\int_a^\infty t^{n-k-2} f(t, y(t))\,dt < \infty$$

and

(26) $$y^{(k+1)}(t) = (-1)^{n-k} \int_t^\infty \frac{(s-t)^{n-k-2}}{(n-k-2)!} f(s, y(s))\,ds\,, \quad t \geqq a\,,$$

which implies

(27) $$y^{(k)}(t) - y^{(k)}(a) = (-1)^{n-k} \int_a^t \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\,dr\,ds\,, \quad t \geqq a.$$

A direct consequence of (27) is that

(28) $$\int_a^\infty t^{n-k-1} f(t, y(t))\,dt < \infty \quad \text{for} \quad y \in S^k_{+b} \cup S^k_{-b}\,,$$

(29) $$\int_a^\infty t^{n-k-1} f(t, y(t))\,dt = \infty \quad \text{for} \quad y \in S^k_{+u} \cup S^k_{-u}\,,$$

where $S^k_{+u}$ [resp. $S^k_{-u}$] may have a member only if $n \equiv k \pmod 2$ [resp. $n \not\equiv k \pmod 2$] (see (10) and (11)).

The situations in which $S^k_{-b} \neq \emptyset$ and $S^k_{+b} \neq \emptyset$ can be characterized without difficulty.

**Theorem 5.** *Let* $1 \leqq k \leqq n-2$. *Then,* $S^k_{-b} \neq \emptyset$ *if and only if*

(30) $$\int_a^\infty t^{n-k-1} f(t, -\lambda t^k)\,dt < \infty \quad \textit{for some} \quad \lambda > 0\,.$$

**Theorem 6.** (i) *Let* $1 \leqq k \leqq n - 2$ *and* $n \equiv k \pmod 2$. *Then,* $S^k_{+b} \neq \emptyset$ *if and only if*

(31) $$\int_a^\infty t^{n-k-1} f(t, \lambda t^k)\, dt < \infty \quad \textit{for some} \quad \lambda > 0\,.$$

(ii) *Let* $1 \leqq k \leqq n - 2$ *and* $n \not\equiv k \pmod 2$. *Suppose that* (2) *holds. Then,* $S^k_{+b} \neq \emptyset$ *if and only if* (31) *is satisfied.*

Proof of Theorem 5. Let $y \in S^k_{-b}$. Then, we have $\lim_{t\to\infty} y(t)/t^k = \text{const} < 0$, which, combined with (28), implies (30).

Conversely, suppose that (30) holds. Let $p(t)$ be an arbitrary polynomial of degree $\leqq k - 1$ and define the mapping $F$ by

(32) $$Fy(t) = p(t) - ct^k + (-1)^{n-k-1} \int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, y(r))\, dr\, ds$$

for $t \geqq a$, where $c$ is a constant. If $n \equiv k \pmod 2$, then let $c > \lambda$ and define $Y$ to be the set of all $y \in C[a, \infty)$ satisfying

(33) $$p(t) - ct^k + (-1)^{n-k-1} \int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \,.$$
$$.\, f(r, p(r) - cr^k)\, dr\, ds \leqq y(t) \leqq p(t) - ct^k\,, \quad t \geqq a\,,$$

and if $n \not\equiv k \pmod 2$, then let $c > 2\lambda$ be such that

$$\int_a^\infty t^{n-k-1} f\left(t, p(t) - \frac{c}{2} t^k\right) dt \leqq \frac{c}{2}$$

and define $Y$ to be the set of all $y \in C[a, \infty)$ satisfying

$$p(t) - ct^k \leqq y(t) \leqq p(t) - \frac{c}{2} t^k\,, \quad t \geqq a\,.$$

It is easily checked that in both cases $F$ is a continuous mapping from $Y$ into a compact subset of $Y$. Therefore, $F$ has a fixed element $y \in Y$. Differentiating the integral equation $y(t) = Fy(t)$, we see that $y(t)$ is a solution of (1) on $[a, \infty)$ such that $\lim_{t\to\infty} y^{(k)}(t) = -ck! < 0$. This establishes the existence of a member of $S^k_{-b}$.

Proof of Theorem 6. In both (i) and (ii) the "only if" part follows from (28) and the relation $\lim_{t\to\infty} y(t)/t^k = \text{const} > 0$.

Suppose now that $n \equiv k \pmod 2$ and (31) holds. We define $F$ and $Y$, respectively, by (32) and (33) with $-c$ replaced by $c$ such that $0 < c < \lambda$. Then, it is easy to see that $F$ has a fixed element in $Y$ which gives a solution of (1) belonging to $S^k_{+b}$.

Suppose that $n \not\equiv k \pmod 2$ and (31) holds. Choose an $\eta < 0$ so that

$$\int_a^\infty t^{n-k-1} f(t, \eta + \lambda t^k)\, dt \leqq \tfrac{1}{2}\lambda\,,$$

which is possible since $\lim_{\eta\to-\infty} \int_a^\infty t^{n-k-1} f(t, \eta + \lambda t^k)\, dt = 0$ by (2) and (31). Applying the Schauder-Tychonoff theorem, we see that the mapping

$$Gy(t) = \eta + \frac{\lambda t^k}{2} + (-1)^{n-k-1} \int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, y(r))\, dr\, ds$$

has a fixed point $y$ in the set

$$Z = \left\{ y \in C[a, \infty) : \eta + \frac{\lambda t^k}{2} \leqq y(t) \leqq \eta + \lambda t^k,\ t \geqq a \right\}.$$

Obviously, $y = y(t)$ satisfies (1) and $\lim_{t\to\infty} y^{(k)}(t) = \lambda k!/2 > 0$, implying that $y \in S^k_{+b}$.

Theorems 7 and 8 below give necessary conditions and sufficient conditions for the existence of members of $S^k_{-u}$ and $S^k_{+u}$, $1 \leqq k \leqq n-2$. It would be of interest to bridge the gap between necessary conditions and sufficient conditions in each of these theorems.

**Theorem 7.** *Let* $1 \leqq k \leqq n-2$ *and* $n \not\equiv k \pmod 2$.

(i) *If* $S^k_{-u} \neq \emptyset$, *then*

(34) $$\int_a^\infty t^{n-k-2} f(t, -\lambda t^{k+1})\, dt < \infty \quad \textit{for all} \quad \lambda > 0$$

*and*

(35) $$\int_a^\infty t^{n-k-1} f(t, -\mu t^k)\, dt = \infty \quad \textit{for all} \quad \mu > 0 .$$

(ii) $S^k_{-u} \neq \emptyset$ *if*

(36) $$\int_a^\infty t^{n-k-2} f(t, -\lambda t^k)\, dt < \infty \quad \textit{for some} \quad \lambda > 0$$

*and*

(37) $$\int_a^\infty t^{n-k-1} f(t, -\mu t^{k+1})\, dt = \infty \quad \textit{for all} \quad \mu > 0 .$$

Proof. (i) Let $y \in S^k_{-u}$. Then, $\lim_{t\to\infty} y(t)/t^{k+1} = 0$ and $\lim_{t\to\infty} y(t)/t^k = -\infty$, and hence, for any $\lambda > 0$ and $\mu > 0$, there is $t_0 > a$ such that

$$y(t) \geqq -\lambda t^{k+1} \quad \text{and} \quad y(t) \leqq -\mu t^k \quad \text{for} \quad t \geqq t_0 .$$

Combining these inequalities with (25) and (29), respectively, we have the desired relations (34) and (35).

(ii) Suppose that (36) and (37) hold. Let $p(t)$ be any polynomial of degree $\leqq k-1$ and define

$$Fy(t) = p(t) - ct^k + (-1)^{n-k} \int_a^t \frac{(t-s)^k}{k!} \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\, dr\, ds , \quad t \geqq a ,$$

and

$$Y = \{ y \in C[a, \infty) : p(t) - c(t+1)^{k+1} \leqq y(t) \leqq p(t) - ct^k,\ t \geqq a \} ,$$

where $c > \lambda$ is chosen large enough so that

$$\int_a^\infty t^{n-k-2} f(t, p(t) - ct^k)\, dt \leqq c .$$

Such a choice of $c$ is possible because of (36) and the nondecreasing nature of $f(t, y)$

in $y$. Let $y \in Y$. Since

$$\int_a^t \frac{(t-s)^k}{k!} \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\, dr\, ds \leqq$$

$$\leqq \int_a^t \frac{(t-s)^k}{k!}\, ds \int_a^\infty \frac{(r-a)^{n-k-2}}{(n-k-2)!} f(r, p(r) - cr^k)\, dr \leqq$$

$$\leqq \frac{(t-a)^{k+1}}{(k+1)!} \frac{c}{(n-k-2)!} \leqq ct^{k+1}, \quad t \geqq a,$$

we have

$$p(t) - ct^k \geqq Fy(t) \geqq p(t) - ct^k - ct^{k+1} \geqq p(t) - c(t+1)^{k+1}, \quad t \geqq a,$$

implying that $Fy \in Y$. Thus $F$ maps $Y$ into itself. The continuity of $F$ and the relative compactness of $F(Y)$ are easily verified, and so there exists a fixed point $y$ of $F$ in $Y$. It is obvious that $y = y(t)$ gives a solution of (1) on $[a, \infty)$. The inclusion $y \in S^k_{-u}$ follows from the equations

$$y^{(k+1)}(t) = -\int_t^\infty \frac{(s-t)^{n-k-2}}{(n-k-2)!} f(s, y(s))\, ds, \tag{38}$$

$$y^{(k)}(t) = -ck! - \int_a^t \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\, dr\, ds. \tag{39}$$

In fact, (38) implies $\lim_{t\to\infty} y^{(k+1)}(t) = 0$. From (39) we have

$$y^{(k)}(t) \leqq -ck! - \int_a^t \frac{(r-a)^{n-k-1}}{(n-k-1)!} f(r, y(r))\, dr \leqq$$

$$\leqq -ck! - \int_a^t \frac{(r-a)^{n-k-1}}{(n-k-1)!} f(r, p(r) - c(r+1)^{k+1})\, dr,$$

whence, noting that $f(r, p(r) - c(r+1)^{k+1}) \geqq f(r, -2cr^{k+1})$ for sufficiently large $r$ and using (37), we conclude that $\lim_{t\to\infty} y^{(k)}(t) = -\infty$.

**Theorem 8.** *Let* $1 \leqq k \leqq n-2$ *and* $n \not\equiv k \pmod 2$.

(i) *If* $S^k_{+u} \neq \emptyset$, *then*

$$\int_a^\infty t^{n-k-2} f(t, \lambda t^k)\, dt < \infty \quad \textit{for all} \quad \lambda > 0 \tag{40}$$

*and*

$$\int_a^\infty t^{n-k-1} f(t, \mu t^{k+1})\, dt = \infty \quad \textit{for all} \quad \mu > 0. \tag{41}$$

(ii) *Suppose that* (2) *holds. Then,* $S^k_{+u} \neq \emptyset$ *if*

$$\int_a^\infty t^{n-k-2} f(t, \lambda t^{k+1})\, dt < \infty \quad \textit{for some} \quad \lambda > 0 \tag{42}$$

*and*

$$\int_a^\infty t^{n-k-1} f(t, \mu t^k)\, dt = \infty \quad \textit{for all} \quad \mu > 0. \tag{43}$$

Proof. (i) The desired conclusion follows readily from (25), (29) and the fact that $y \in S^k_{+u}$ satisfies

$$\lim_{t\to\infty} y(t)/t^k = \infty \quad \text{and} \quad \lim_{t\to\infty} y(t)/t^{k+1} = 0 .$$

(ii) Suppose that (42) and (43) hold. Let $c$, $0 < c < \lambda$, be fixed. Since

$$\lim_{\eta\to-\infty} \int_a^\infty t^{n-k-2} f(t, \eta + ct^{k+1})\, dt = 0$$

in view of (2) and (42), there exists an $\eta < 0$ such that

$$\int_a^\infty t^{n-k-2} f(t, \eta + c(t+1)^{k+1})\, dt \leqq c .$$

Then it can be shown that the mapping

$$Fy(t) = \eta + ct^k + (-1)^{n-k} \int_a^t \frac{(t-s)^k}{k!} \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\, dr\, ds$$

possesses a fixed point $y$ in the set

$$Y = \{y \in C[a, \infty) : \eta + ct^k \leqq y(t) \leqq \eta + c(t+1)^{k+1},\ t \geqq a\} .$$

This fixed point $y = y(t)$ is a solution of (1) on $[a, \infty)$ such that $\lim_{t\to\infty} y^{(k+1)}(t) = 0$. Since

$$y^{(k)}(t) = ck! + \int_a^t \int_s^\infty \frac{(r-s)^{n-k-2}}{(n-k-2)!} f(r, y(r))\, dr\, ds \geqq$$

$$\geqq ck! + \int_a^t \frac{(r-a)^{n-k-1}}{(n-k-1)!} f(r, \eta + cr^k)\, dr$$

and $f(r, \eta + cr^k) \geqq f(r, cr^k/2)$ for large $r$, using (43), we see that $\lim_{t\to\infty} y^{(k)}(t) = \infty$, showing that $y \in S^k_{+u}$.

**C.** It remains to examine the classes $S^0_b$, $S^0_+$ and $S^0_-$ (see (14)). Note that (25)–(29) also hold for $k = 0$; thus in particular

(44) $$\int_a^\infty t^{n-2} f(t, y(t))\, dt < \infty \quad \text{for} \quad y \in S^1_0 ;$$

(45) $$\int_a^\infty t^{n-1} f(t, y(t))\, dt < \infty \quad \text{for} \quad y \in S^0_b$$

and

(46) $$\int_a^\infty t^{n-1} f(t, y(t))\, dt = \infty \quad \text{for} \quad y \in S^0_+ \cup S^0_- .$$

**Theorem 9.** $S^0_b \neq \emptyset$ *if and only if*

(47) $$\int_a^\infty t^{n-1} f(t, \lambda)\, dt < \infty \quad \textit{for some} \quad \lambda \in \mathbb{R} .$$

Proof. Let $y \in S^0_b$. Then we have (45) which, in view of the boundedness of $y(t)$, yields (47).

Suppose that (47) holds. Let $n$ be even. Define

$$Fy(t) = \lambda + (-1)^{n+1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds , \quad t \geqq a$$

and

$$Y = \left\{ y \in C[a, \infty) \colon \lambda - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \lambda)\, ds \leqq y(t) \leqq \lambda,\ t \geqq a \right\}.$$

Then, there exists a fixed point of $F$ in $Y$, which provides a bounded solution of (1) on $[a, \infty)$.

Let $n$ be odd. We may suppose that $\lambda$ in (47) is negative. Choose a constant $c > 0$ large enough so that $-c < \lambda$ and

$$\int_a^\infty t^{n-1} f(t, -c)\, dt \leqq c,$$

and consider the mapping $G$ defined by

$$Gy(t) = -2c + (-1)^{n+1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds, \quad t \geqq a.$$

$G$ has a fixed point $y$ in the set

$$Z = \{ y \in C[a, \infty) \colon -2c \leqq y(t) \leqq -c,\ t \geqq a \}$$

and this $y(t)$ gives a bounded solution of (1).

**Theorem 10.** *Let $n$ be odd.*

(i) *If* $S_-^0 = S_{-u}^0 \neq \emptyset$, *then*

$$\int_a^\infty t^{n-2} f(t, -\lambda t)\, dt < \infty \quad \textit{for all} \quad \lambda > 0 \tag{48}$$

*and*

$$\int_a^\infty t^{n-1} f(t, -\mu)\, dt = \infty \quad \textit{for all} \quad \mu > 0. \tag{49}$$

(ii) $S_-^0 \neq \emptyset$ *if*

$$\int_a^\infty t^{n-2} f(t, -\lambda)\, dt < \infty \quad \textit{for some} \quad \lambda > 0 \tag{50}$$

*and*

$$\int_a^\infty t^{n-1} f(t, -\mu t)\, dt = \infty \quad \textit{for some} \quad \mu > 0. \tag{51}$$

Proof. (i) If $y \in S_-^0$, then

$$\lim_{t \to \infty} y(t)/t = 0 \quad \text{and} \quad \lim_{t \to \infty} y(t) = -\infty.$$

The first of the above and (44) give (48), while the second and (46) imply (49).

(ii) Suppose that (50) and (51) are satisfied. Let $\eta \in \mathbb{R}$ be fixed and take $c > \lambda$ so that

$$\int_a^\infty t^{n-2} f(t, \eta - c)\, dt \leqq c.$$

Then, proceeding as in the proof of (ii) of Theorem 7, the desired solution of class $S_-^0$ is obtained as a fixed element of the mapping

$$Fy(t) = \eta - c + (-1)^n \int_a^t \int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r, y(r))\, dr\, ds, \quad t \geqq a,$$

in the set

$$Y = \{ y \in C[a, \infty) \colon \eta - c(t+1) \leqq y(t) \leqq \eta - c,\ t \geqq a \}.$$

**Theorem 11.** *Let $n$ be even.*

(i) *If* $S^0_+ = S^0_{+u} \neq \emptyset$, *then*

$$\int_a^\infty t^{n-2} f(t, \lambda)\, dt < \infty \quad \textit{for all} \quad \lambda > 0 \tag{52}$$

*and*

$$\int_a^\infty t^{n-1} f(t, \mu t)\, dt = \infty \quad \textit{for all} \quad \mu > 0\,. \tag{53}$$

(ii) *Suppose that* (2) *holds. Then,* $S^0_+ \neq \emptyset$ *if*

$$\int_a^\infty t^{n-2} f(t, \lambda t)\, dt < \infty \quad \textit{for some} \quad \lambda > 0 \tag{54}$$

*and*

$$\int_a^\infty t^{n-1} f(t, \mu)\, dt = \infty \quad \textit{for all} \quad \mu > 0\,. \tag{55}$$

Proof. It suffices to repeat the arguments of the proof of Theorem 8 by letting $k = 0$.

Remarks. 1) For any $k$, $1 \leqq k \leqq n - 1$, $S^k_{+b} \neq \emptyset$ implies $S^k_{-b} \neq \emptyset$.

2) If $1 \leqq k \leqq n - 1$ and $n \not\equiv k \pmod 2$, then members of $S^k_{-b}$ and $S^k_{-u}$ cannot coexist, and the same is true for $S^0_b$ and $S^0_{-u}$.

3) If $S^{n-1}_{-u} \neq \emptyset$, then all the other classes are empty, so that $S = S^{n-1}_{-u}$.

4) None of the second parts of Theorems 7, 8, 10 and 11 is applicable to the prototype (3), since the two conditions therein guaranteeing $S^k_{+u} \neq \emptyset$ or $S^k_{-u} \neq \emptyset$ are not consistent for $f(t, y) = p(t) \exp(|y|^{\gamma-1} y)$, $\gamma > 0$. An example of equations of the form (1) having a solution of class $S^k_{+u}$ or $S^k_{-u}$ is given in Section 4.

5) All the existence theorems developed above are 'global' existence theorems in that the desired solutions are guaranteed to exist on the given interval $[a, \infty)$. In Theorems 4, 6, 8 and 11 condition (2) is required to ensure global existence of eventually positive solutions of (1) which grow to infinity as $t \to \infty$. We note that condition (2) can be deleted if we are content with 'local' existence of such solutions, that is, those existing on an interval of the form $[T, \infty)$, $T > 0$ being sufficiently large. The same remark applies to Theorems 14 and 15 given in the next section.

## 3. STRUCTURE OF THE SOLUTION SET

**A.** The theorems presented in the preceding section can be used to obtain useful information about the structure of the solution set of equation (1).

Consider first the conditions

$$\int_a^\infty t^{n-k-1} f(t, -\lambda t^k)\, dt < \infty \quad \text{for some} \quad \lambda > 0 \tag{$A_k$}$$

$$\int_a^\infty t^{n-k-1} f(t, -\lambda t^k)\, dt = \infty \quad \text{for all} \quad \lambda > 0\,. \tag{$B_k$}$$

Since for sufficiently large $t$

$$t^{n-k-1} f(t, -\lambda t^k) \geqq t^{n-k-2} f(t, -\lambda t^{k+1})\,,$$

($A_k$) implies ($A_{k+1}$), and ($B_{k+1}$) implies ($B_k$). This fact together with Theorems 2, 5 and 9 yields the following result.

**Theorem 12.** (i) *The condition*

$$(\mathrm{A}_0) \qquad \int_a^\infty t^{n-1} f(t, -\lambda)\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0$$

*ensures that* $S_b^0 \neq \emptyset$, $S_{-b}^1 \neq \emptyset, \ldots, S_{-b}^{n-1} \neq \emptyset$.

(ii) *The condition*

$$(\mathrm{B}_{n-1}) \qquad \int_a^\infty f(t, -\lambda t^{n-1})\, \mathrm{d}t = \infty \quad \text{for all} \quad \lambda > 0$$

*ensures that* $S_b^0 = S_{-b}^1 = \ldots = S_{-b}^{n-1} = \emptyset$.

If we consider the conditions

$$(\mathrm{C}_k) \qquad \int_a^\infty t^{n-k-2} f(t, -\lambda t^{k+1})\, \mathrm{d}t = \infty \quad \text{for some} \quad \lambda > 0\,,$$

$$(\mathrm{D}_k) \qquad \int_a^\infty t^{n-k-1} f(t, -\lambda t^{k})\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0\,,$$

then $(\mathrm{C}_k)$ implies $(C_{k-2})$, and $(\mathrm{D}_k)$ implies $(\mathrm{D}_{k+2})$, so that Theorems 7 and 10 give the following result.

**Theorem 13.** (i) *The condition*

$$(\mathrm{C}_{n-3}) \qquad \int_a^\infty t f(t, -\lambda t^{n-2})\, \mathrm{d}t = \infty \quad \text{for some} \quad \lambda > 0$$

*ensures that* $S_{-u}^k = \emptyset$ *for all* $k$, $0 \leqq k \leqq n-2$, *with* $n \not\equiv k \pmod 2$.

(ii) *One of the conditions*

$$(\mathrm{D}_1) \qquad \int_a^\infty t^{n-2} f(t, -\lambda t)\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0 \quad \text{and even} \quad n\,,$$

$$(\mathrm{D}_0) \qquad \int_a^\infty t^{n-1} f(t, -\lambda)\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0 \quad \text{and odd} \quad n\,,$$

*ensures that* $S_{-u}^k = \emptyset$ *for all* $k$, $0 \leqq k \leqq n-2$, *with* $n \not\equiv k \pmod 2$.

We now introduce the definition: Equation (1) is said to be *superlinear* [resp. *sublinear*] in $y > 0$ if, for each $t \geqq 0$, $f(t, y)/y$ is nondecreasing [resp. nonincreasing] in $y$ for $y > 0$.

Consider the conditions

$$(\mathrm{E}_k) \qquad \int_a^\infty t^{n-k-1} f(t, \lambda t^k)\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0\,,$$

$$(\mathrm{F}_k) \qquad \int_a^\infty t^{n-k-1} f(t, \lambda t^k)\, \mathrm{d}t = \infty \quad \text{for all} \quad \lambda > 0\,.$$

From the relation

$$\frac{t^{n-k-1} f(t, \lambda t^k)}{t^{n-k-2} f(t, \lambda t^{k+1})} = \frac{f(t, \lambda t^k)/\lambda t^k}{f(t, \lambda t^{k+1})/\lambda t^{k+1}}$$

it follows that $(\mathrm{E}_{k+1})$ implies $(\mathrm{E}_k)$, and $(\mathrm{F}_k)$ implies $(F_{k+1})$ if (1) is superlinear in $y > 0$, and conversely if (1) is sublinear in $y > 0$. From this fact and Theorems 4, 6 and 9 we obtain the following two theorems.

**Theorem 14.** *Let* (1) *be superlinear in* $y > 0$.

(i) $S_b^0 \neq \emptyset$, $S_{+b}^1 \neq \emptyset, \ldots, S_{+b}^{n-1} \neq \emptyset$ *if* (2) *holds and*

$$(\mathrm{E}_{n-1}) \qquad \int_a^\infty f(t, \lambda t^{n-1})\, \mathrm{d}t < \infty \quad \text{for some} \quad \lambda > 0\,.$$

(ii) $S^1_{+b} = \ldots = S^{n-1}_{+b} = \emptyset$ *if*

$$(F_1) \qquad \int_a^\infty t^{n-2} f(t, \lambda t)\, dt = \infty \quad \text{for all} \quad \lambda > 0\,.$$

**Theorem 15.** *Let* (1) *be sublinear in* $y > 0$.

(i) $S^0_b \neq \emptyset$, $S^1_{+b} \neq \emptyset, \ldots, S^{n-1}_{+b} \neq \emptyset$ *if* (2) *holds and*

$$(E_0) \qquad \int_a^\infty t^{n-1} f(t, \lambda)\, dt < \infty \quad \text{for some} \quad \lambda > 0\,.$$

(ii) $S^1_{+b} = \ldots = S^{n-1}_{+b} = \emptyset$ *if*

$$(F_{n-1}) \qquad \int_a^\infty f(t, \lambda t^{n-1})\, dt = \infty \quad \text{for all} \quad \lambda > 0\,.$$

Finally let us consider the conditions

$$(G_k) \qquad \int_a^\infty t^{n-k-2} f(t, \lambda t^k)\, dt = \infty \qquad \text{for some} \quad \lambda > 0\,,$$

$$(H_k) \qquad \int_a^\infty t^{n-k-1} f(t, \lambda t^{k+1})\, dt < \infty \quad \text{for some} \quad \lambda > 0\,.$$

Note that if (1) is superlinear in $y > 0$, then $(G_k)$ implies $(G_{k+2})$, and $(H_{k+2})$ implies $(H_k)$, and that if (1) is sublinear in $y > 0$, then the converse implications hold for $(G_k)$, $(G_{k+2})$, $(H_k)$ and $(H_{k+2})$. This observation combined with Theorems 8 and 11 leads to the next results.

**Theorem 16.** (i) *Let* (1) *be superlinear in* $y > 0$. *Then*, $S^k_{+u} = \emptyset$ *for* $0 \leqq k \leqq \leqq n - 2$ *with* $n \equiv k \pmod 2$, *provided*

$$(G_0) \qquad \int_a^\infty t^{n-2} f(t, \lambda)\, dt = \infty \quad \text{for some} \quad \lambda > 0$$

*in the case of even* $n$, *and*

$$(G_1) \qquad \int_a^\infty t^{n-3} f(t, \lambda t)\, dt = \infty \quad \text{for some} \quad \lambda > 0$$

*in the case of odd* $n$.

(ii) *Let* (1) *be sublinear in* $y > 0$. *Then*, $S^k_{+u} = \emptyset$ *for* $0 \leqq k \leqq n - 2$ *with* $n \equiv k \pmod 2$, *provided*

$$(G_{n-2}) \qquad \int_a^\infty f(t, \lambda t^{n-2})\, dt = \infty \quad \text{for some} \quad \lambda > 0\,.$$

**Theorem 17.** (i) *Let* (1) *be superlinear in* $y > 0$. *Then*, $S^k_{+u} = \emptyset$ *for* $0 \leqq k \leqq \leqq n - 2$ *with* $n \equiv k \pmod 2$, *provided*

$$(H_{n-2}) \qquad \int_a^\infty t f(t, \lambda t^{n-1})\, dt < \infty \quad \text{for some} \quad \lambda > 0\,.$$

(ii) *Let* (1) *be sublinear in* $y > 0$. *Then*, $S^k_{+u} = \emptyset$ *for* $0 \leqq k \leqq n - 2$ *with* $n \equiv k \pmod 2$, *provided*

$$(H_0) \qquad \int_a^\infty t^{n-1} f(t, \lambda t)\, dt < \infty \quad \text{for some} \quad \lambda > 0$$

*in the case of even* $n$, *and*

$$(H_1) \qquad \int_a^\infty t^{n-2} f(t, \lambda t^2)\, dt < \infty \quad \text{for some} \quad \lambda > 0$$

*in the case of odd* $n$.

B. We show that a characterization for the absence of increasing solutions can be obtained for a class of equations of the form

$$(56)\qquad y^{(n)} + \varphi(t)\, g(y) = 0,\ n \geqq 2\,,$$

where $\varphi: [0, \infty) \to (0, \infty)$ is continuous and $g: R \to (0, \infty)$ is continuous and nondecreasing. Our method is an adaptation of the techniques which were used in deriving standard oscillation criteria for Emden-Fowler type equations (see [1, 2, 7, 8]).

**Theorem 18.** *Suppose that*

$$(57)\qquad \int_\delta^\infty \frac{\mathrm{d}y}{g(y)} < \infty \quad \textit{for any} \quad \delta \in R\,.$$

*Then. every solution* $y(t)$ *of* (56) *has the property* $\lim_{t\to\infty} y(t) = -\infty$ *if and only if*

$$(58)\qquad \int_a^\infty t^{n-1}\, \varphi(t)\, \mathrm{d}t = \infty\,.$$

**Theorem 19.** *Suppose that* $g(y)/y$ *is nonincreasing for all sufficiently large* $y > 0$,

$$(59)\qquad h(z) = \inf_{x>0} \frac{g(xz)}{g(x)} > 0 \quad \textit{for} \quad z > 0$$

*and*

$$(60)\qquad \int_0^\delta \frac{\mathrm{d}z}{h(z)} < \infty \quad \textit{for some} \quad \delta > 0\,.$$

*Suppose moreover that* $\lim_{y\to-\infty} g(y) = 0$ *when* $n$ *is odd. Then, every solution* $y(t)$ *of* (56) *has the property* $\lim_{t\to\infty} y(t) = -\infty$ *if and only if*

$$(61)\qquad \int_a^\infty \varphi(t)\, g(\lambda t^{n-1})\, \mathrm{d}t = \infty \quad \textit{for all} \quad \lambda > 0\,.$$

**Definition.** We define $P_l$, $1 \leqq l \leqq n - 1$, to be the set of all solutions $y(t)$ of (56) on $[a, \infty)$ such that

$$(62)\qquad y^{(i)}(t) > 0\,, \quad 1 \leqq i \leqq l\,, \quad \text{and} \quad (-1)^{i-l}\, y^{(i)}(t) > 0\,, \quad l + 1 \leqq i \leqq n\,,$$

for all large $t$.

Proof of Theorem 18. It is easy to see that if $n$ is even, then

$$(63)\qquad S_+^{n-1} \cup S_+^{n-2} = P_{n-1}\,, \quad S_+^{n-3} \cup S_+^{n-4} = P_{n-3}, \ldots, S_+^3 \cup S_+^2 = P_3\,,$$
$$S_+^1 \cup S_0^1 = P_1\,,$$

and if $n$ is odd, then

$$(64)\qquad S_+^{n-1} \cup S_+^{n-2} = P_{n-1}\,, \quad S_+^{n-3} \cup S_+^{n-4} = P_{n-3}, \ldots, S_+^2 \cup S_+^1 = P_2\,.$$

It can be shown that if $y \in P_l$, then

$$(65)\qquad y'(t) \geqq \frac{(t - T)^{l-1}}{(l - 1)!} \int_t^\infty \frac{(s - t)^{n-l-1}}{(n - l - 1)!}\, \varphi(s)\, g(y(s))\, \mathrm{d}s\,, \quad t \geqq T,$$

provided $T$ is chosen large enough so that (62) holds for $t \geqq T$. In fact, putting

$z = y$, $i = l$, $k = n - 1$ in the equation

$$(66) \quad z^{(i)}(t) = \sum_{j=1}^{k} (-1)^{j-i} z^{(j)}(s) \frac{(s - t)^{j-i}}{(j - i)!} + (-1)^{k-i+1} \int_t^s \frac{(r - t)^{k-i}}{(k - i)!} z^{(k+1)}(r) \, dr$$

holding for $i \leqq k$, we have

$$y^{(l)}(t) = \sum_{j=l}^{n-1} (-1)^{j-l} \frac{(s - t)^{j-l}}{(j - l)!} y^{(j)}(s) + (-1)^{n-l} \int_t^s \frac{(r - t)^{n-l-1}}{(n - l - 1)!} y^{(n)}(r) \, dr \geqq$$

$$\geqq - \int_t^s \frac{(r - t)^{n-l-1}}{(n - l - 1)!} y^{(n)}(r) \, dr, \quad s \geqq t \geqq T.$$

Letting $s \to \infty$ in the above and noting that $y^{(n)}(r) = -\varphi(r)\, g(y(r))$, we find

$$(67) \qquad y^{(l)}(t) \geqq \int_t^\infty \frac{(s - t)^{n-l-1}}{(n - l - 1)!} \varphi(s)\, g(y(s)) \, ds, \quad t \geqq T.$$

If $l \geqq 2$, then from (66) with $z = y$, $s = T$, $i = 1$, $k = l - 1$ it follows that

$$(68) \qquad y'(t) = \sum_{j=1}^{l-1} \frac{(t - T)^{j-1}}{(j - 1)!} y^{(j)}(T) + (-1)^{l-1} \int_t^T \frac{(r - t)^{l-2}}{(l - 2)!} y^{(l)}(r) \, dr =$$

$$= \sum_{j=1}^{l-1} \frac{(t - T)^{j-1}}{(j - 1)!} y^{(j)}(T) + \int_T^t \frac{(t - r)^{l-2}}{(l - 2)!} y^{(l)}(r) \, dr \geqq$$

$$\geqq \int_T^t \frac{(t - r)^{l-2}}{(l - 2)!} y^{(l)}(r) \, dr \geqq y^{(l)}(t) \int_T^t \frac{(t - r)^{l-2}}{(l - 2)!} \, dr =$$

$$= \frac{(t - T)^{l-1}}{(l - 1)!} y^{(l)}(t), \quad t \geqq T.$$

Combining (67) with (68) yields (65). Clearly, (65) holds for $l = 1$. Since $g(y(t))$ is nondecreasing for $t \geqq T$, we see from (65) that

$$y'(t) \geqq g(y(t)) \frac{(t - T)^{l-1}}{(l - 1)!} \int_t^\infty \frac{(s - t)^{n-l-1}}{(n - l - 1)!} \varphi(s) \, ds, \quad t \geqq T,$$

which implies

$$\int_T^t \frac{y'(s)}{g(y(s))} \, ds \geqq \int_T^t \frac{(s - T)^{l-1}}{(l - 1)!} \int_s^\infty \frac{(r - s)^{n-l-1}}{(n - l - 1)!} \varphi(r) \, dr \, ds \geqq$$

$$\geqq \int_T^t \left( \int_T^r \frac{(s - T)^{l-1}}{(l - 1)!} \frac{(r - s)^{n-l-1}}{(n - l - 1)!} \, ds \right) \varphi(r) \, dr =$$

$$= \int_T^t \frac{(r - T)^{n-1}}{(n - 1)!} \varphi(r) \, dr, \quad t \geqq T.$$

It follows because of (57) that

$$\int_T^\infty \frac{(r-T)^{n-1}}{(n-1)!}\,\varphi(r)\,\mathrm{d}r \leqq \int_T^\infty \frac{y'(s)}{g(y(s))}\,\mathrm{d}s = \int_{y(T)}^{y(\infty)} \frac{\mathrm{d}u}{g(u)} < \infty\,,$$

implying that

$$(69) \qquad \int_a^\infty t^{n-1}\,\varphi(t)\,\mathrm{d}t < \infty\,.$$

Note that (69) is a necessary condition for the existence of a member of $S_b^0$ (see Theorem 9). The above observation shows that condition (58) implies that

$$S_+^{n-1} \cup S_+^{n-2} \cup \ldots \cup S_+^1 \cup S_0^1 = \emptyset \quad \text{for} \quad n \quad \text{even}\,,$$

$$S_+^{n-1} \cup S_+^{n-2} \cup \ldots \cup S_+^1 \cup S_b^0 = \emptyset \quad \text{for} \quad n \quad \text{odd}\,,$$

or equivalently

$$S = S_-^{n-1} \cup S_-^{n-2} \cup \ldots \cup S_-^1 \quad \text{for} \quad n \quad \text{even}\,,$$

$$S = S_-^{n-1} \cup S_-^{n-2} \cup \ldots \cup S_-^1 \cup S_-^0 \quad \text{for} \quad n \quad \text{odd}\,,$$

proving the "if" part of Theorem 18.

The "only if" part follows from the fact that (69) guarantees that $S_b^0 \neq \emptyset$ (see Theorem 9), that is, (56) has a solution not tending to $-\infty$ as $t \to \infty$.

Proof of Theorem 19. The "only if" part follows from Theorem 4: If (61) is violated, then $S_{+b}^{n-1} \neq \emptyset$ for (56), that is, (56) has a solution $y(t)$ such that $\lim_{t\to\infty} y(t)/t^{n-1} = \text{const} > 0$.

Suppose that (61) holds. Let $y \in P_l$ for some $l$, $1 \leqq l \leqq n-1$, with $n \not\equiv l \pmod 2$. Since $S_+^{n-1} = S_{+b}^{n-1} = \emptyset$ by Theorem 4, $y$ must be in $S_0^{n-1}$: $\lim_{t\to\infty} y^{(n-1)}(t) = 0$. We claim that

$$(70) \qquad y^{(l)}(r) \geqq \frac{(t-r)^{n-l-1}}{(n-l-1)!}\,y^{(n-1)}(t)\,, \quad t \geqq r \geqq T\,,$$

provided $T$ is sufficiently large. In fact, (70) with $l = n-1$ is clear, because $y^{(n)}(t) < 0$ for $t \geqq a$. Let $1 \leqq l < n-1$. Putting $z = y$, $t = r$, $s = t$, $i = l$, $k = n-2$ in (66), we obtain

$$(71) \qquad y^{(l)}(r) = \sum_{j=l}^{n-2} (-1)^{j-l}\,y^{(j)}(t)\,\frac{(t-r)^{j-l}}{(j-l)!} +$$

$$+ (-1)^{n-1-l}\int_r^t \frac{(s-r)^{n-2-l}}{(n-2-l)!}\,y^{(n-1)}(s)\,\mathrm{d}s \geqq \int_r^t \frac{(s-r)^{n-2-l}}{(n-2-l)!}\,y^{(n-1)}(s)\,\mathrm{d}s \geqq$$

$$\geqq \left(\int_r^t \frac{(s-r)^{n-2-l}}{(n-2-l)!}\,\mathrm{d}s\right) y^{(n-1)}(t) = \frac{(t-r)^{n-l-1}}{(n-l-1)!}\,y^{(n-1)}(t)\,, \quad t \geqq r \geqq T\,.$$

From (66) with $z = y$, $s = T$, $i = 0$, $k = l-1$ we have

$$y(t) = \sum_{j=0}^{l-1} y^{(j)}(T)\,\frac{(t-T)^j}{j!} + \int_T^t \frac{(t-r)^{l-1}}{(l-1)!}\,y^{(l)}(r)\,\mathrm{d}r \geqq$$

$$\geqq \int_T^t \frac{(t-r)^{l-1}}{(l-1)!}\, y^{(l)}(r)\, \mathrm{d}r\,, \quad t \geqq T\,,$$

which combined with (70) gives

$$y(t) \geqq \int_T^t \frac{(t-r)^{l-1}}{(l-1)!}\, \frac{(t-r)^{n-l-1}}{(n-l-1)!}\, y^{(n-1)}(t)\, \mathrm{d}r =$$

$$= \frac{(t-T)^{n-1}}{(n-1)(l-1)!\,(n-l-1)!}\, y^{(n-1)}(t)\,, \quad t \geqq T\,.$$

Therefore, there are constants $c > 0$ and $T' > T$ such that

$$y(t) \geqq ct^{n-1}\, y^{(n-1)}(t)\,, \quad t \geqq T'\,,$$

and hence in view of (59) we see that $y(t)$ satisfies the differential inequality

$$(72) \qquad y^{(n)}(t) + \varphi(t)\, g(ct^{n-1})\, h(y^{(n-1)}(t)) \leqq 0\,, \quad t \geqq T'\,.$$

Dividing (72) by $h(y^{(n-1)}(t))$ and integrating on $[T', t]$, we obtain

$$\int_{T'}^t \varphi(s)\, g(cs^{n-1})\, \mathrm{d}s \leqq -\int_{T'}^t \frac{y^{(n)}(s)}{h(y^{(n-1)}(s))}\, \mathrm{d}s = \int_{y^{(n-1)}(t)}^{y^{(n-1)}(T')} \frac{\mathrm{d}z}{h(z)}\,,$$

which implies

$$\int_{T'}^\infty \varphi(s)\, g(cs^{n-1})\, \mathrm{d}s \leqq \int_0^{y^{(n-1)}(T')} \frac{\mathrm{d}z}{h(z)} < \infty\,,$$

a contradiction to (61).

Finally suppose that $n$ is odd and $y \in S_b^0$. Then,

$$(73) \qquad \textstyle\int_a^\infty t^{n-1}\, \varphi(t)\, \mathrm{d}t < \infty$$

by Theorem 9. Since $g(y)/y$ is nonincreasing for sufficiently large $y$, say, $y \geqq y_0 > 0$, we have $g(t^{n-1})/t^{n-1} \leqq g(y_0)/y_0$ for all sufficiently large $t$, that is,

$$(74) \qquad g(t^{n-1}) \leqq c_0 t^{n-1} \quad (c_0 = g(y_0)/y_0)$$

for all large $t$. From (73) and (74) we have

$$\textstyle\int^\infty \varphi(t)\, g(t^{n-1})\, \mathrm{d}t < \infty\,,$$

which contradicts (61). Thus it follows that all solutions of (56) must tend to $-\infty$ as $t \to \infty$.

## 4. EXAMPLES AND APPLICATION

**A.** We present two examples which illustrate the results obtained in the preceding sections.

Example 1. We now consider the equation

$$(75) \qquad y^{(n)} + \exp(\varrho t^\sigma) \exp(|y|^{\gamma-1} y) = 0\,,$$

where $\varrho$, $\sigma > 0$ and $\gamma > 0$ are constants, which is a special case of (1) and (56) with

$$f(t, y) = \exp(\varrho t^\sigma) \exp(|y|^{\gamma - 1} y),$$
$$\varphi(t) = \exp(\varrho t^\sigma), \quad g(y) = \exp(|y|^{\gamma-1} y).$$

Note that (75) is superlinear in $y$ sufficiently large, and $g(y)$ satisfies condition (57).

(i) Let $\varrho > 0$ and $\sigma > (n-1)\gamma$. Then,

$$\int^\infty f(t, -\lambda t^{n-1})\, dt = \infty \quad \text{for all} \quad \lambda > 0,$$

so that $S^{n-1}_{-u} \neq \emptyset$ by Theorem 3, and hence $S = S^{n-1}_{-u}$ by Remark 3 in Section 2.

(ii) Let $\varrho < 0$ and $\sigma > (n-1)\gamma$. Then,

(76) $$\int^\infty t^{n-k-1} f(t, \lambda t^k)\, dt < \infty \quad \text{for all } k, \quad 0 \leqq k \leqq n-1, \quad \text{and} \quad \lambda \in \mathbb{R}$$

which, by Theorems 4, 5, 6, 9, guarantees the existence of members of $S^0_b$, $S^k_{-b}$, $S^k_{+b}$ for $1 \leqq k \leqq n-1$. From Theorems 3, 13 and 17 it follows that $S^{n-1}_{-u}$, $S^k_{-u}$, $S^k_{+u}$, $0 \leqq k \leqq n-2$, are empty. Therefore, we have

$$S = [S^{n-1}_{+b} \cup \ldots \cup S^1_{+b}] \cup S^0_b \cup [S^{n-1}_{-b} \cup \ldots \cup S^1_{-b}]$$

for the solutions of (75).

(iii) Let $\varrho = 0$. Then,

$$\int^\infty t^{n-1} \varphi(t)\, dt = \infty,$$

so that Theorem 18 implies that

(77) $$S = S^{n-1}_- \cup \ldots \cup S^1_- \quad \text{for} \quad n \quad \text{even},$$
$$S = S^{n-1}_- \cup \ldots \cup S^1_- \cup S^0_- \quad \text{for} \quad n \quad \text{odd}.$$

Since

$$\int^\infty t^{n-k-1} f(t, -\lambda t^k)\, dt < \infty \quad \text{for all } k, \quad 1 \leqq k \leqq n-1. \quad \text{and} \quad \lambda > 0,$$

we obtain $S^k_{-b} \neq \emptyset$ for $1 \leqq k \leqq n-1$ by Theorems 2 and 5, and this implies $S^k_{-u} = \emptyset$ for $1 \leqq k \leqq n-1$ (see Remark 2 at the end of Section 2). It follows that

$$S = S^{n-1}_{-b} \cup \ldots \cup S^1_{-b} \quad \text{for} \quad n \quad \text{even},$$
$$S = S^{n-1}_{-b} \cup \ldots \cup S^1_{-b} \cup S^0_{-u} \quad \text{for} \quad n \quad \text{odd}.$$

However, it is not known whether the class $S^0_{-u}$ actually has a member.

Example 2. Let $g: \mathbb{R} \to (0, \infty)$ be defined by

(78) $$g(y) = (1+y)^\alpha \quad \text{for} \quad y \geqq 0, \quad g(y) = (1-y)^{-\beta} \quad \text{for} \quad y \leqq 0,$$

where $\alpha$ and $\beta$ are constants such that $0 < \alpha < 1$ and $0 < \beta < 1$, and consider the equation

(79) $$y^{(n)} + (t+1)^\sigma g(y) = 0,$$

where $\sigma$ is a real constant. Equation (79) is sublinear in $y > 0$, and $g(y)$ satisfies the hypothesis of Theorem 19 with $h(z)$ taken to be

$$h(z) = z^\alpha \quad \text{for} \quad 0 < z \leqq 1, \quad h(z) = 1 \quad \text{for} \quad z \geqq 1.$$

(i) If $\sigma < -n$, then (i) of Theorem 15 shows that $S_b^0 \neq \emptyset, S_{+b}^k \neq \emptyset, 1 \leqq k \leqq \leqq n - 1$, which implies that $S_{-b}^k \neq \emptyset$, $1 \leqq k \leqq n - 1$, and $S_{-u}^k = \emptyset$, $0 \leqq k \leqq \leqq n - 1$ (see Remarks 1 and 2 in Section 2). If $\sigma < -n - \alpha$ with $n$ even or $\sigma < < -n - 2\alpha + 1$ with $n$ odd, then (ii) of Theorem 17 guarantees that $S_{+u}^k = \emptyset$ for $0 \leqq k \leqq n - 2$.

(ii) If $\sigma \geqq \beta(n - 1) - 1$, then we see that $S_{-u}^{n-1} \neq \emptyset$ by Theorem 3. and hence we have $S = S_{-u}^{n-1}$.

(iii) Applying (ii) of Theorems 7, 8, 10 and 11, we obtain $S_{-u}^k \neq \emptyset$ for $k$, $0 \leqq k \leqq \leqq n - 2$, such that $n \not\equiv k \pmod 2$ if

$$\beta - n + k(1 + \beta) \leqq \sigma < 1 - n + k(1 + \beta),$$

and $S_{+u}^k \neq \emptyset$ for $k$, $0 \leqq k \leqq n - 2$, such that $n \equiv k \pmod 2$ if

$$-n + k(1 - \alpha) \leqq \sigma < -n + (k + 1)(1 - \alpha).$$

(iv) We now apply Theorem 19 to conclude that if $\sigma \geqq -\alpha(n - 1) - 1$, then the set of solutions of (79) have the structure (77). Note that $-n < -\alpha(n - 1) - 1 < < \beta(n - 1) - 1$. Suppose in particular that $-\alpha(n - 1) - 1 < 1 + \beta - n$, i.e. $n - 1 < (1 + \beta)/(1 - \alpha)$, and let

$$-\alpha(n - 1) - 1 \leqq \sigma < 1 + \beta - n.$$

Then, $S_{-b}^k \neq \emptyset$, $1 \leqq k \leqq n - 1$, by Theorem 5, and so $S_{-u}^k = \emptyset$, $1 \leqq k \leqq n - 1$. It follows that if in addition $n$ is even, then

$$S = S_{-b}^{n-1} \cup \ldots \cup S_{-b}^1$$

for (79). Suppose that $-\alpha(n - 1) - 1 \leqq \beta - n$, i.e. $n - 1 \leqq \beta/(1 - \alpha)$, and let

$$\beta - n \leqq \sigma < 1 - n.$$

If in addition $n$ is odd, then $S_{-u}^0 \neq \emptyset$ (see (iii) above) and the solution set of (79) has a decomposition

$$S = S_{-b}^{n-1} \cup \ldots \cup S_{-b}^1 \cup S_{-u}^0.$$

**B.** The theory developed in Sections 2 and 3 can be applied to the elliptic partial differential equation

$$\Delta^m u + F(|x|, u) = 0, \quad x \in \Omega_a \subset \mathbb{R}^3, \quad m \geqq 2, \tag{80}$$

in an exterior domain $\Omega_a = \{x \in \mathbb{R}^3 : |x| \geqq a\}$, $a > 0$, where $\Delta$ denotes the three-dimensional Laplacian, and $F: [0, \infty) \times \mathbb{R} \to (0, \infty)$ is continuous and non-decreasing in the second variable.

We are interested in radially symmetric global solutions of (80) with various asymptotic behavior as $|x| \to \infty$. A radial function $u(x) = y(|x|)$ is a solution of (80) in $\Omega_a$ if and only if $y(t)$ satisfies the ordinary differential equation

$$(ty)^{(2m)} + t\,F(t, y) = 0, \quad t \geqq a,$$

i.e., if and only if $z(t) = t\, y(t)$ satisfies

$$z^{(2m)} + t\, F(t, t^{-1}z) = 0\,, \quad t \geqq a\,. \tag{81}$$

Observe that a solution of (81) of class $S^k_{-b}$ or $S^k_{+b}$ yields a solution $u(x)$ of (80) such that

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{k-1}} = \text{const} < 0 \tag{82}$$

or

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{k-1}} = \text{const} > 0\,, \tag{83}$$

and that a solution of (81) of class $S^k_{-u}$ or $S^k_{+u}$ gives rise to a solution of (80) such that

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{k}} = 0\,, \quad \lim_{|x| \to \infty} \frac{u(x)}{|x|^{k-1}} = -\infty \tag{84}$$

or

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{k}} = 0\,, \quad \lim_{|x| \to \infty} \frac{u(x)}{|x|^{k-1}} = \infty\,. \tag{85}$$

Criteria for the existence of these solutions of (80) are given below.

**Theorem 20.** (i) *Equation* (80) *has a radial solution* $u(x)$ *satisfying* (82) *for some* $k$, $1 \leqq k \leqq 2m - 1$, *if and only if*

$$\int^\infty t^{2m-k} F(t, -\lambda t^{k-1})\, \mathrm{d}t < \infty \quad \textit{for some} \quad \lambda > 0\,.$$

(ii) *Equation* (80) *has a radial solution* $u(x)$ *satisfying* (84) *for some odd* $k$, $0 \leqq k \leqq 2m - 2$, *if*

$$\int^\infty t^{2m-k-1} F(t, -\lambda t^{k-1})\, \mathrm{d}t < \infty \quad \textit{for some} \quad \lambda > 0$$

*and*

$$\int^\infty t^{2m-k} F(t, -\mu t^{k})\, \mathrm{d}t = \infty \quad \textit{for all} \quad \mu > 0\,.$$

**Theorem 21.** *Suppose that*

$$\lim_{u \to -\infty} F(t, u) = 0 \quad \textit{for fixed} \quad t \geqq 0\,.$$

(i) *Equation* (80) *has a radial solution* $u(x)$ *satisfying* (83) *if and only if*

$$\int^\infty t^{2m-k} F(t, \lambda t^{k-1})\, \mathrm{d}t < \infty \quad \textit{for some} \quad \lambda > 0\,.$$

(ii) *Equation* (80) *has a radial solution* $u(x)$ *satisfying* (85) *for some even* $k$, $0 \leqq k \leqq 2m - 2$, *if*

$$\int^\infty t^{2m-k-1} F(t, \lambda t^{k})\, \mathrm{d}t < \infty \quad \textit{for some} \quad \lambda > 0$$

*and*

$$\int^\infty t^{2m-k} F(t, \mu t^{k-1})\, \mathrm{d}t = \infty \quad \textit{for all} \quad \mu > 0\,.$$

We conclude with some remarks about the set of radial solutions of the elliptic equation

(86) $$\Delta^m u + \exp(\varrho|x|^{2m}) e^u = 0, \quad x \in \Omega_a \subset \mathbb{R}^3, \quad m \geqq 2,$$

where $\varrho$ is a constant.

(a) If $\varrho > 0$, then (86) has infinitely many radial solutions $u(x)$ satisfying

(87) $$\lim_{|x| \to \infty} \frac{u(x)}{|x|^{2m-2}} = -\infty,$$

and all radial solutions of (86) have the same asymptotic behavior (87).

(b) If $\varrho = 0$, then, for every $k$, $2 \leqq k \leqq 2m - 1$, (86) has infinitely many radial solutions $u(x)$ satisfying (82), and every radial solution $u(x)$ of (86) has asymptotic behavior (82) for some $k$, $2 \leqq k \leqq 2m - 1$, or (84) with $k = 1$:

(88) $$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = 0, \quad \lim_{|x| \to \infty} u(x) = -\infty.$$

It is difficult to construct a solution of (86) having the property (88).

(c) If $\varrho < 0$, then, for every $k$, $1 \leqq k \leqq 2m - 1$, there exist infinitely many radial solutions $u(x)$ of (86) satisfying (82) as well as those satisfying (83), Moreover, (86) possesses infinitely many radial solutions $u(x)$ such that

(89) $$\lim_{|x| \to \infty} |x| \, u(x) = \text{const} \in \mathbb{R}.$$

Every radial solution $u(x)$ of (86) satisfies either (89) or one of (82) and (83) for some $k$, $1 \leqq k \leqq 2m - 1$.

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