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ON CLOSED AND INDUCTIVELY CLOSED IMAGES OF PRODUCTS OF METRIC SPACES

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1. INTRODUCTION

Let C be a class of topological spaces. The class of all spaces which are continuous closed images of products of spaces of this class will be called the *φ-extension* of C. We shall be mainly interested in the φ-extension of the class of metric spaces. The spaces which belong to this class will be called *φ-spaces*. The problem of studying φ-spaces was raised by V. V. Filippov. The class of φ-spaces includes such important and well-studied types of spaces as dyadic spaces (i.e., continuous Hausdorff images of generalized Cantor cubes) and Lašnev spaces (i.e., continuous closed images of metric spaces). Dyadic and Lašnev spaces have been studied separately. The joint investigation of these two classes was stimulated by the well-known theorems stating that in them the property of a space of being first-countable is equivalent to its metrizability. So, the question naturally arises if a first-countable φ-space is metrizable. We have got a positive answer to the question. This fact, in our opinion, convincingly demonstrates the use of studying φ-spaces.

In the final section we investigate properties of spaces which are images of products of metric spaces under inductively closed mappings (these mappings generalize closed ones). The study provides us with further information on φ-spaces.

2. PRELIMINARIES

Throughout the paper all mappings are assumed to be continuous and *onto*, and all spaces are T_1. Products of spaces (i.e., Cartesian products endowed with the Tychonoff topology) are denoted by $\prod\{X_t: t \in T\}$ or shorter $\prod_t X_t$. The symbols p_t and p_S denote the projections of the product onto its factor X_t and subproduct $\prod\{X_t: t \in S\}$. For a family of sets γ $\bigcup\gamma$ stands for $\bigcup\{U: U \in \gamma\}$. The closure, the interior and the boundary of a set A are denoted by Cl A, Int A and Fr A respectively.

For references we use in the main [1]; original papers are quoted only if we can not refer to that book. The notation of cardinal functions follows [1].

In this note we want to offer, leaving out details, the main ideas used in the study of closed and inductively closed images of products of metric spaces. Striving for conciseness of the exposition we have omitted details in many proofs. Certain statements are given without proof.

3. CLOSED IMAGES OF PRODUCTS OF METRIC SPACES (φ-SPACES)

We recall that a subset of a T_1-space X is σ-discrete in X if it can be represented as a countable union of closed discrete subsets of X. Every metric space M has a dense subset that is σ-discrete in it, since M has a σ-discrete base.

Theorem 1. *Let $X = \prod\{X_t: t \in T\}$, where each X_t contains a dense subset which is σ-discrete in X_t (in particular, X_t is a metric space), and let $f: X \to Y$ be a closed map. Then $Y = Y_0 \cup Y_1$, where $\operatorname{Int} f^{-1}(y) = \emptyset$ for $y \in Y$ iff $y \in Y_0$ and Y_1 is σ-discrete in Y.*

Proof. Define $Y_0 = \{y \in Y: \operatorname{Int} f^{-1}(y) = \emptyset$ and $Y_1 = Y \setminus Y_0$. For each $y \in Y_1$ take a finite $T(y) \subseteq T$ and a non-empty open set $U(y) \subseteq \prod\{X_t: t \in T(y)\}$ such that $p_{T(y)}^{-1}(U(y)) \subseteq f^{-1}(y)$. Let B_t be a σ-discrete subset of X_t. The set $\prod\{B_t: t \in T(y)\}$ being dense in $\prod\{X_t: t \in T(y)\}$, there is a point $a_y \in U(y) \cap \prod\{B_t: t \in T(y)\}$. Put $A(y) = p_{T(y)}^{-1}(a_y)$ and $\gamma = \{A(y): y \in Y_1\}$.

We assert that the family γ is σ-discrete in X. Obviously, it suffices to show that for each $n \geqq 1$ the family $\gamma_n = \{A(y): y \in Y_1, |T(y)| = n\}$ is σ-discrete in X. The last can be proved by induction with respect to n.

Choose a point in every element of γ. Thus we obtain a set W which is σ-discrete in X. As f is closed, the set $f(W) = Y_1$ is σ-discrete in Y. This completes the proof.

Theorem 2. *Let $X = \prod_t X_t$, where each X_t is a metric space, and let $f: X \to Y$ be a closed map onto a q-space. Then $\operatorname{Fr} f^{-1}(y)$ is compact for each $y \in Y$.*

We shall remind that Y is called a q-space [2] provided for each $y \in Y$ there exists a countable set of neighbourhoods of y, $\{O_n(y)\}_{n\in N}$, such that, if $y_n \in O_n(y)$, then the sequence $\{y_n\}_{n\in N}$ has a convergent subsequence. Note that both countably compact spaces and spaces of pointwise countable type are q-spaces. Actually the last theorem is valid in the much more general case: one can suppose that all X_t's are isocompact (see [3]) θ-spaces (see [4]) (every normal weakly paracompact space is an isocompact θ-space). In this generalized formulation Theorem 2 is an extension of the corresponding result of Michael [2] (the subject goes back to Vaĭnšteĭn's lemma [1; 4.4.16]. The theorem has been proved also by Čertanov [5]. From Theorem 2 one readily deduces

Corollary 1. *Under the assumptions of Theorem 2 there exists a closed subspace*

X^* of X such that the restriction $f|X^*$ is a perfect map and $f(X^*) = Y$. In particular, f will be σ k-covering map.

The property of being a regular space is multiplicative, hereditary and preserved under perfect maps. Hence Corollary 1 yields

Corollary 2. *If a φ-space is a q-space, then it is regular.*

Corollary 3. *A countably compact φ-space is a dyadic space.*

Proof. Let f be a closed map of the product of metric spaces $X = \prod_t X_t$ onto a countably compact space Y. Since a countably compact space is a q-space, we can choose a subspace X^* of X as in Corollary 1. Therefore, by virtue of the well-known properties of countably compact spaces [1; 3.10.4, 3.10.10], the set $F = f^{-1}(Y) \cap X^*$ is countably compact. Consequently, each $p_t(F)$ is compact. Clearly, $f(\prod_t p_t(F)) = Y$, which is – by Corollary 2 – a Hausdorff space. It remains to notice that the continuous Hausdorff images of products of metric compact spaces are precisely the dyadic spaces.

The next theorem is one of our main results.

Theorem 3. *Let $X = \prod\{X_t: t \in T\}$, where each X_t is a metric space. If X admits a closed map onto a non-discrete q-space, then the family $\{X_t\}_{t\in T}$ contains at most countably many non-compact spaces.*

For the proof we shall need the following

Lemma 1. *Let $X = \prod\{X_t: t \in T\}$. If X has a non-empty compact G_δ-subset, then the family $\{X_t\}_{t\in T}$ contains at most countably many non-compacts spaces.*

Proof of Theorem 3. By virtue of Lemma 1 it suffices to find in X a non-empty compact G_δ-set. Let $f: X \to Y$ be a closed map onto a non-discrete q-space. Theorems 1 and 2 imply that $Y = Y_0 \cup Y_1$, where for $y \in Y$ $\operatorname{Int} f^{-1}(y) = \emptyset$ iff $y \in Y_0$, $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_1 is σ-discrete in Y. Two cases are possible.

1) Assume that $Y_0 \neq \emptyset$. Fix a $y_0 \in Y_0$ and a countable family $\{O_n(y_0)\}$ as in the definition of a q-space. By Corollary 2 we can suppose that $\operatorname{Cl} O_{n+1}(y_0) \subseteq O_n(y_0)$ for each n. Since Y is a q-space, $F = \bigcap_n O_n(y_0)$ is a closed countably compact G_δ-subset of Y. As Y_0 is a G_δ-set in Y, one can find, using the regularity of Y, a closed countably compact G_δ-subset F_1 of Y such that $y_0 \in F_1 \subseteq F \cap Y_0$. The restriction of f to $R = f^{-1}(F_1)$ is a perfect map. Therefore R is a closed countably compact G_δ-set in X. Since R is closed in the compact set $\prod\{p_t(R): t \in T\}$, it is also compact. Finally, $f^{-1}(y_0) \subseteq R$, so that $R \neq \emptyset$.

2) Let now $Y_0 = \emptyset$, i.e., $\operatorname{Int} f^{-1}(y) \neq \emptyset$ for each $y \in Y$. Since Y is a non-discrete q-space, there exists a countable set $C \subseteq Y$ having an accumulation point, say y^*. For every $y \in C$ choose a point $a_y \in \operatorname{Int} f^{-1}(y)$. Clearly, there exists a finite $T(y) \subseteq T$ such that the set $A(y) = p_{T(y)}^{-1} p_{T(y)}(a_y)$ is contained in $f^{-1}(y)$. As f is closed, there is

a point $b \in \mathrm{Fr}\, f^{-1}(y^*) \cap \mathrm{Cl} \bigcup\{A(y): y \in C\}$. Let $S = \bigcup\{T(y): y \in C\}$ and $R = p_S^{-1}\, p_S(b)$. One verifies that the inclusion $R \subseteq \mathrm{Fr}\, f^{-1}(y^*)$ holds. By Theorem 2 $\mathrm{Fr}\, f^{-1}(y^*)$ is compact, and so is its closed subset R. The set S being at most countable, $p_S(b)$ is a G_δ-set in $p_S(X)$, and hence R is a G_δ-set in X. The proof is finished.

Theorem 3 impels us to distinguish the following special case if a φ-space. A space is called a *ψ-space* if it is a closed image of the product of the from $\prod_t C_t \times M$, where each C_t is a metric compact space and M is a metric space. The ψ-space can be defined equivalently as a closed image of the product of a dyadic and a metric space. Notice that every ψ-space is paracompact. The almost metrizable spaces [6] are important examples of ψ-spaces.

The last theorem yields

Corollary 4. *In the realm of q-spaces the concepts of a φ-space and a ψ-space are equivalent.*

Theorem 4. *Let Y be a φ-space that is also a q-space. Then Y is a paracompact p-space.*

Proof. By Corollary 4 Y is a ψ-space. Applying Corollary 1, we conclude that Y is a perfect image of a closed subspace of the product of a compact space and a metric one. But the subspaces like this are exactly the paracompact p-spaces (see [7; Ch. V. 228, Ch. VI. 60]. To complete the proof it remains to notice that the property of being a paracompact p-space is preserved under prefect maps (see [8]).

Seeking to generalize theorems concerning φ-spaces, we have come to a consideration of the φ-extension of stratifiable [9] spaces. The class of stratifiable spaces contains metric spaces; however, the property of being stratifiable is more flexible: it is not only hereditary and countably-multiplicative, but also is invariant under closed maps. An analysis of our proofs shows that Theorems 1 – 3 remain true for the spaces belonging to the φ-extension of stratifiable spaces. Let us observe that Theorem 4 partially holds for the spaces of that kind, viz., one can assert that Y is paracompact, but not necessarily a p-space.

The following theorem plays the key role in the proofs of metrization theorems for φ-spaces.

Theorem 5. *Assume that each dyadic subspace of the ψ-space Y is metrizable. Then Y is a Lašnev space.*

The combined application of Theorems 3 and 5, together with criteria on the metrizability of dyadic spaces and Lašnev spaces, allows to get a series of theorems on the metrization of φ-spaces. In particular, the following important theorem holds.

Theorem 6. *A first-countable φ-space is metrizable.*

To prove Theorem 6 one should apply first Corollary 4 from Theorem 3, then [1; 3.12.12(e)] and Theorem 5, and finally [1; 4.4.17]. Let us mention that our paper [10] contains stronger than Theorem 6 results.

We conclude this section with the result which generalizes Vaĭnšteĭn's theorem on closed images of complete metric spaces [1; 4.5.13(e)].

Theorem 7. *Let X be a product of complete metric spaces. If a metric space Y is a closed image of X, then Y is completely metrizable.*

Proof. Suppose that Y is non-discrete (the contrary case is obvious). Then Theorem 3 is applicable. It is well-known that both compact and complete metric spaces are Čech-complete. Since compactness is multiplicative, we deduce that X can be considered as a countable product of Čech-complete spaces. Hence, X is Čech-complete [1; 3.9.8]. Applying Corollary 1 and the properties of Čech-completeness [1; 3.9.6, 3.9.10], we obtain that the metric space Y is Čech-complete, i.e., is metrizable in a complete manner.

4. INDUCTIVELY CLOSED IMAGES OF PRODUCTS OF METRIC SPACES

A map $f: X \to Y$ is called *inductively closed* if there exists a set $X^* \subseteq X$ such that the restriction $f|X^*$ is a closed map and $f(X^*) = Y$.

This definition is due to V. V. Filippov. Clearly, closed maps and retractions are inductively closed.

Following [1], for a space X by $e(X)$ we denote the *extent* of X, i.e., $\aleph_0 \sup\{|F|: F$ is a closed discrete subset of $X\}$. Observe that $e(X)$ does not exceed both the Lindelöf number and the hereditary Souslin number of X.

The fundamental result of the section is

Theorem 8. *Let $X_0 \subseteq \prod_t X_t$, where each X_t is a metric space, and let $f: X_0 \to Y$ be a closed map. Assume that one of the following holds:*

Case 1. $e(Y) \leqq \tau$,

Case 2. *Y is countably compact.*

Then for each t there exists a subspace Z_t of X_t – with $d(Z_t) \leqq \tau$ in Case 1 *and which is compact in Case* 2 *– such that $f(\prod_t Z_t \cap T_0) = Y$.*

(Likeness between Cases 1 and 2 is explainded by the fact that countable compactness of a space means finiteness of its closed discrete subsets).

For the proof we need

Lemma 2. *Let α be a disjoint open cover of a space X and $f: X \to Y$ be a closed map. If $e(Y) \leqq \tau$ (resp. Y is countably compact), then there exists a subfamily $\beta \subseteq \alpha$ such that:*

a) $|\beta| \leqq \tau$ *(resp. finite)*; b) $|f(U)| > \tau$ *(resp. infinite) for all $U \in \beta$*; c) $|f(\bigcup\alpha \setminus \beta)| \leqq \tau$ *(resp. finite).*

Proof of Theorem 8. Without loss of generality we can assume that $\dim X_t = 0$ for any X_t. Indeed, as shown by Morita [1; 4.4.J], one can find a metric space X_t'

with $\dim X'_t = 0$ which admits a perfect map, say h_t, onto X_t. Since $h = \prod_t h_t$ is a perfect map [1; 3.7.7], so is the map g defined as the restriction of h to $X'_0 = h^{-1}(X_0)$. Therefore the map $f \circ g: X'_0 \to Y$ is closed. If we have found in each X'_t a subspace Z'_t with the required properties, then $Z_t = h_t(Z'_t)$ will suit our purposes. Further we shall suppose that $\dim X_t = 0$.

We shall give first the proof for Case 1, so that let $e(Y) \leqq \tau$. Since $\dim X_t = 0$, each open cover of any subset of X_t has a refinement consisting of disjoint open sets of arbitrarily small diameter [1; 7.3.1]. Using this fact and Lemma 2, for each X_t and $n = 1, 2, \ldots$ one can construct by recursion with respect to n a family λ_{nt} and its subfamily μ_{nt} such that: (1) λ_{nt} consists of disjoint open subsets of $p_t(X_0)$, each having diameter $<1/n$; (2) $\bigcup \lambda_{1t} = p_t(X_0)$ and $\bigcup \lambda_{n+1,t} = \bigcup \mu_{nt}$; (3) $|\mu_{nt}| \leqq \tau$; (4) $|f(p_t^{-1}(W))| > \tau$ for all $W \in \mu_{nt}$; (5) $|f(\bigcup\{p_t^{-1}(W): W \in \lambda_{nt} \setminus \mu_{nt}\})| \leqq \tau$.

Put $M_{nt} = \bigcup \mu_{nt}$ and $M_t = \bigcap_{n \geqq 1} M_{nt}$. One can show that $d(M_t) \leqq \tau$. If $f(\prod_t M_t \cap X_0) = Y$, the proof is completed. Otherwise, for every point $y \in Y \setminus (\prod_t M_t \cap X_0)$ choose a point in $f^{-1}(y)$. Let R be the set of all so chosen points. It turns out that $|p_t(R) \setminus M_t| \leqq \tau$ for each t. Let $Z_t = M_t \cup p_t(R)$. The preceding inequalities imply that $d(Z_t) \leqq \tau$. Evidently, $f(\prod_t Z_t \cap X_0) = Y$.

The proof for Case 2 is similar to that for Case 1. Making use of the equality $\dim X_t = 0$ and Lemma 2, one can construct, for each X_t and $n \geqq 1$, a family λ_{nt} and its subfamily μ_{nt} such as follows. They satisfy conditions (1) and (2) indicated above, as well as conditions (3)–(5), in which one should replace "$\leqq \tau$" by "finite" and "$> \tau$" by "infinite". The set M_t is defined as earlier. It appears that M_t is compact. Let us consider the case when $Y \setminus f(\prod_t M_t \cap X_0) \neq \emptyset$. Define R as above. One can prove that if the set $p_t(R) \setminus M_t$ is infinite, then any its infinite subset has an accumulation point in M_t. Thus the set $Z_t = M_t \cup p_t(R)$ is compact. Besides, $f(\prod_t Z_t \cap X_0) = Y$.

From Theorem 8 we infer

Theorem 9. *Let f be an inductively closed map of $\prod_t X_t$, where each X_t is a metric space, and let $P \subseteq f(X)$. Assume that one of the following holds:*

Case 1. *$e(P) \leqq \tau$,*

Case 2. *P is countably compact.*

Then for each t there exists a subspace Z_t of X_t – with $d(Z_t) \leqq \tau$ in Case 1 *and which is compact in Case* 2 *– such that $f(\prod_t Z_t) \supseteq P$.*

Let us proceed to applications of Theorem 9.

Theorem 10. *Let Y be an inductively closed image of a product of metric spaces and $e(Y) \leqq \tau$. Then we have:*

(a) *the Šanin number of $Y \leqq \tau$;*

(b) *if Y is a Hausdorff space and* $\psi(Y) \leqq 2^{\tau}$, *then* $d(Y) \leqq \tau$;

(c) *if Y is a Hausdorff space and* $\psi(Y) \leqq \tau$, *then* $nw(Y) \leqq \tau$.

Proof. Theorem 9 implies that Y can be considered as a continuous image of $\prod\{Z_t: t \in T\}$, where Z_t is metric and $d(Z_t) \leqq \tau$ for each $t \in T$. By virtue of the Šanin theorem [1; 2.7.11], $\check{s}(\prod\{Z_t: t \in T\}) \leqq \tau$. Continuous mappings do not increase the Šanin number, thus (a) is proved. In case (b) Gleason's factoring theorem (cf. [1; 2.7.13]) implies that there exists an $S \subseteq T$ with $|S| \leqq 2^{\tau}$ such that $\prod\{Z_t: t \in S\}$ can be mapped onto Y. By [1; 2.3.15] $d(\prod\{Z_t: t \in S\}) \leqq \tau$, therefore $d(Y) \leqq \tau$. Finally, to prove (c), we apply again Gleason's theorem and find an $S' \subseteq T$ with $|S'| \leqq \tau$ such that $\prod\{Z_t: t \in S'\}$ admits a map onto Y. Each Z_t being metric, $d(Z_t) = w(Z_t)$. Hence $w(\prod\{Z_t: t \in S'\}) \leqq \tau$ [1; 2.3.13], so that $nw(Y) \leqq \tau$.

As it is seen from the proof, Theorem 10(c) for $\tau = \aleph_0$ can be strengthened as follows.

Theorem 11. *Let a Hausdorff space Y be an inductively closed image of a product of metric spaces. If* $e(Y) = \aleph_0$ *and* $\psi(Y) = \aleph_0$, *then Y is a closed image of a metric space of weight* $\aleph_0$.

Applying first the assertion of Theorem 9 for Case 1 and then a technique of Arhangel'skiĭ [11], one can obtain

Theorem 12. *Let X be a product of metric spaces,* $f: X \to Y$ *be a closed map,* $e(Y) \leqq \tau$, *and the tightness of* $Y \leqq \tau$. *Then there exists a closed set* $R \subseteq X$ *such that* $w(R) \leqq \tau$ *and* $f(R) = Y$.

In particular, if $\tau = \aleph_0$ in Theorem 12, then Y is a closed image of a metric space of weight $\aleph_0$.

The statement of Theorem 9 for Case 2 yields

Corollary 5. *Let a Hausdorff space Y be an inductively closed image of a product of metric spaces. If a set* $P \subseteq Y$ *is countably compact, then P is contained in a dyadic subspace of Y.*

Notice that the above corollary generalizes Corollary 3. Since dyadicity is hereditary with respect to non-empty closed G_δ-sets [1; 4.5.10], Corollary 5 implies that if the set P is, moreover, a non-empty closed G_δ-set, then it is a dyadic space.

Let us note in conclusion that a more detailed exposition of the problems considered in this note can be found in [12].

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