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STRUCTURAL STABILITY OF LINEAR DISCRETE SYSTEMS VIA THE EXPONENTIAL DICHOTOMY

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INTRODUCTION

Consider the difference equation

$$x(n+1) = A(n)\,x(n)\,, \quad n \in N = \{0, 1, \ldots\} \tag{1}$$

where $A(n)$ is an invertible $k \times k$-matrix for $n \in N$ such that

$$|A(n)| \leqq M\,, \quad |A^{-1}(n)| \leqq M\,, \quad M > 0\,, \quad n \in N\,.$$

We denote by W the space of the systems of the form (1) and by $|\cdot|$ the Euclidean norm.

Equation (1) is said to *possess an exponential dichotomy* if there exist a projection P ($P^2 = P$) and constants $K > 0$, $0 < p < 1$, such that

$$|X(n)\,PX^{-1}(m)| \leqq Kp^{n-m}\,, \qquad n \geqq m$$

$$|X(n)\,(I - P)\,X^{-1}(m)| \leqq Kp^{m-n}\,, \quad m \geqq n$$

where $X(n)$ is the fundamental matrix solution of (1) such that $X(0) = I$.

Consider a system in W

$$y(n+1) = B(n)\,y(n)\,. \tag{2}$$

According to [5, p. 17], (1) and (2) are said to be topologically equivalent if there exists a function $h: N \times R^k \to R^k$ with the following properties:

(i) if $|x| \to \infty$, then $|h(n, x)| \to \infty$ uniformly with respect to n,

(ii) the map $h_n(\cdot) = h(n, \cdot)$ from R^k to R^k is a homeomorphism for each n,

(iii) the map $g_n(\cdot) = h_n^{-1}(\cdot)$ from R^k to R^k also has property (i),

(iv) if $x(n)$ is a solution of (1) then $h(n, x(n))$ is a solution of (2).

Equation (1) is called structurally stable if there exists $\delta > 0$ such that if (2) belongs to W and $|B(n) - A(n)| < \delta$ then (2) is topologically equivalent to (1).

The results of this paper are:

(i) If equation (1) has an exponential dichotomy then it is topologically equivalent to the system

$$(3) \qquad x_\iota(n+1) = e_\iota\, x_\iota(n)\,, \quad \iota = 1, 2, \ldots, k$$

where $e_\iota = 1/\mathrm{e}$ or $e_\iota = \mathrm{e}$.

(ii) System (1) is structurally stable if and only if it has an exponential dichotomy.

The above results are the discrete analogues of those of Palmer [4] and [5]. We denote that the first result is not derived directly form the continuous case.

We also note that some results on exponential dichotomy and structural stability of discrete systems are included in the papers [6], [7], [8], [9], [10], [11].

MAIN RESULTS

Proposition 1. *If equation* (1) *has an exponential dichotomy then it is topologically equivalent to* (3).

Proof. Suppose that (1) has an exponential dichotomy. Let the rank of the corresponding projection P be l. Using the same method as in [2, pp. 39–41] we find an invertible bounded matrix $S(n)$ with bounded inverse such that the transformation $x = S(n)\, y$ transforms (1) into the system

$$(4) \qquad y(n+1) = \operatorname{diag}\left(A_1(n), A_2(n)\right) y(n)$$

where $A_1(n)$ is an $l \times l$ matrix and $A_2(n)$ is a $(k-l) \times (k-l)$ matrix.

Moreover, the system

$$(5) \qquad y(n+1) = A_1(n)\, y(n)$$

has an exponential dichotomy with a projection of rank equal to l and the system

$$(6) \qquad y(n+1) = A_2(n)\, y(n)$$

has an exponential dichotomy with a projection of rank equal to 0.

Consider equation (5). Let $Y_1(n)$ be a fundamental matrix solution of (5). By Gram-Schmidt orthogonalization of the columns of $Y_1(n)$, [2, p. 87], starting with the first column, we obtain a unitary matrix $U_1(n)$ and an upper triangular matrix $\bar{U}(n)$ in which the diagonal elements are real and positive functions for all $n \in N$ such that $U_1(n) = Y_1(n)\, \bar{U}(n)$. The change of variables $y = U_1(n)\, z$ transforms (5) into the system

$$(7) \qquad z(n+1) = U_1^{-1}(n+1)\, A_1(n)\, U_1(n)\, z(n) = B_1(n)\, z(n)\,.$$

The matrix $\bar{U}^{-1}(n)$ is a matrix solution of (7) since $U_1(n) = Y_1(n)\, \bar{U}(n)$. So $B_1(n) = \bar{U}^{-1}(n+1)\, \bar{U}(n)$. Therefore $B_1(n)$ is an upper triangular matrix in which the diagonal are real and positive functions on N.

Consider the differential equation

$$(8) \qquad z' = B(t)\, z\,, \quad B(t) = \begin{cases} \log B_1(0)\,, & 0 \leqq t < 1 \\ \log B_1(1)\,, & 1 \leqq t < 2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \log B_1(n)\,, & n \leqq t < n+1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{cases}$$

From [1, *p.* 39] we have

$$\log B_1(n) = \frac{1}{2\pi i} \int_\gamma (zI - B_1(n))^{-1} \log z \, dz$$

where γ is any simple closed curve which contains in its interior every characteristic root of $B_1(n)$ but not the origin.

We claim that $\log B_1(n)$ is a real valued bounded matrix for $n \in N$. It is obvious that it is a real matrix since $B_1(n)$ is an upper triangular matrix with real positive diagonal elements. Since the matrices $B_1(n), B_1^{-1}(n)$ are bounded for $n \in N$ there exist constants $\lambda, \mu > 0$ such that $\lambda \leqq \lambda_\iota(n) \leqq \mu$, $n \in N$, $\iota = 1, \dots, l$ and $\lambda_\iota(n)$ are the eigenvalues of $B_1(n)$. We choose $r > \max\{\frac{1}{2}(\varepsilon + \mu - \lambda), \frac{1}{2}(\varepsilon + \mu)\}$, $\lambda > \varepsilon > 0$. So $-r + \varepsilon + \mu < r < r - \varepsilon + \lambda$. Consider $z_0 \in R^+$: $r < z_0 < r - \varepsilon + \lambda$. It is obvious that $|z_0 - \lambda| < r - \varepsilon$ and $|z_0 - \mu| < r - \varepsilon$. Let γ be the sphere $|z - z_0| = r$. Then we have $|z - \lambda_\iota(n)| = |z - z_0 + z_0 - \lambda_\iota(n)| \geqq |z - z_0| - |z_0 - \lambda_\iota(n)| \geqq r - (r - \varepsilon) = \varepsilon$, $\iota = 1, \dots, l$. So if $|B_1(n)| \leqq L$, $L > 0$, $n \in N$ then [2, p. 47] implies

$$|(zI - B_1(n))^{-1}| \leqq \frac{c|zI - B_1(n)|^{l-1}}{|z - \lambda_1(n)| \dots |z - \lambda_1(n)|} \leqq \frac{c(z_0 + r + L)^{l-1}}{\varepsilon^l}$$

c a constant.

Therefore $\log B_1(n)$ is a bounded matrix for $n \in N$. Hence our claim is proved.

Let $Z_1(t)$ be the fundamental matrix of (8) such that $Z_1(0) = I$. Then

$$Z_1(t) = e^{\int_n^t \log B_1(n) \, ds} Z_1(n) = e^{(t-n) \log B_1(n)} Z_1(n), \quad n \leqq t < n + 1,$$

$$Z_1(t) = e^{(t-n-1) \log B_1(n+1)} Z_1(n + 1), \quad n + 1 \leqq t < n + 2.$$

The above relations yield

$$Z_1(t) = e^{(t-n-1) \log B_1(n+1)} e^{\log B_1(n)} Z_1(n), \quad n + 1 \leqq t < n + 2.$$

Take $t = n + 1$. Then we obtain

$$Z_1(n + 1) = B_1(n) Z_1(n), \quad n \in N.$$

Therefore $Z_1(n)$ is a fundamental matrix solution of (7).

We prove that equation (8) has an exponential dichotomy. If $z(t)$ is a solution of (8) then for $t \in R^+$ we have

$$z(t) = Z_1(t) Z_1^{-1}([t]) z[t]), \quad [t] \text{ being the integral part of } t.$$

If $|B(t)| \leqq \bar{L}$ then for $t \in R^+$ we have $|Z_1(t) Z_1^{-1}([t])| \leqq e^{\bar{L}(t-[t])} \leqq e^{\bar{L}}$. So for $t \geqq s \geqq 0$ and provided e^{-a}, $a > 0$, $\bar{K} > 0$ are the constants of the exponential dichotomy of (7) we have

$$|z(t)| \leqq e^{\bar{L}} |z([t])| \leqq \bar{K} e^{\bar{L}} e^{-a([t]-[s])} |z([s])| \leqq$$
$$\leqq \bar{K} e^{\bar{L}} e^{-a(t-1-s-1)} |z([s])| \leqq \bar{K} e^{2\bar{L}} e^{2a} e^{-a(t-s)} |z(s)|, \quad t \geqq s \geqq 0.$$

Then by Palmer's Theorem in [4, p. 9] and arguing as in [5, p. 20] we prove that (8) and $x_1' = -x_1$ are topologically equivalent (c.f. [5, p. 17]). Let $h_1: R^+ \times R^1 \to R^1$ be the corresponding homeomorphism. If $z(t, 0, x)$ is a solution of (8) such that $z(0) = x$ we get

$$h_1(t, z(t, 0, x)) = e^{-t} h_1(0, x), \quad t \in R^+, \quad x \in R^1 .$$

Take $t = n$, $n \in N$. So (7) and $x_1(n + 1) = e^{-1} x_1(n)$ are topologically equivalent.

Now consider system (6). Proceeding as in (5) we obtain a unitary matrix $U_2(n)$ such that the change of variables $y = U_2(n) z$ transforms (6) into a system

$$z(n + 1) = B_2(n) z(n) \tag{9}$$

where $B_2(n)$ is a $(k - l) \times (k - l)$ upper triangular matrix in which the diagonal elements are real and positive functions on N. Arguing as in (7) we prove that (9) and $x_2(n + 1) = e x_2(x)$ are topologically equivalent. Let $h_2: N \times R^{k-1} \to R^{k-1}$ be the corresponding homeomorphism. Therefore the system

$$z(n + 1) = \operatorname{diag}(B_1(n), B_2(n)) z(n)$$

is topologically equivalent to (3). Hence (4) is topologically equivalent to (3). The corresponding homeomorphism is $\bar{h}(n, x) = (h_1(n, \{U^{-1}(n) x\}_1), h_2(n, \{U^{-1}(n) x\}_2))$ where $\{U^{-1}(n) x\}_1$ and $\{U^{-1}(n) x\}_2$ are the components of $U^{-1}(n) x$ in R^1 and R^{k-1}, respectively, $U(n) = \operatorname{diag}(U_1(n), U_2(n))$. So (1) and (3) are topologically equivalent. The corresponding homeomorphism is $h(n, x) = \bar{h}(n, S^{-1}(n) x)$ and the proof of the proposition is complete.

Proposition 2. *Equation* (1) *is structurally stable if and only if it has an exponential dichotomy.*

Proof. The proof of necessity is given in [10, Proposition 2]. Using Proposition 1, the roughness of the exponential dichotomy [3, p. 232] and the same argument as in [5, p. 20] we can easily prove sufficiency.

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