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SPECTRUM OF 2-DIMENSIONAL MANIFOLDS

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I get results in the following two directions:

(i) It is well known, see [2], that the interval $(-\infty, 2\min_M K)$ does not belong to $\operatorname{Spec}^1(M^2)$, K being the Gauss curvature. Here, I prove that even $(2\max_M K, 6\min_M K) \not\subset \operatorname{Spec}^1(M^2)$. To accomplish this, I study the more general equation $(\Delta + l)\,\omega = 0$ of the Schrödinger type with $l: M^2 \to \mathbb{R}$ a function; compare with [1].

(ii) According to [3], the eigenvalues of Δ on $\bigwedge^1(S^2)$, $S^2 \equiv S^2(1)$ a unit sphere, are ${}^1\lambda_k = (k+1)(k+2)$; $k = 0, 1, 2, \ldots$, and the multiplicity of ${}^1\lambda_k$ is $2(2k+3)/(k+1)$. I get the multiplicity of ${}^1\lambda_1 = 6$ equal to 10 in contradiction to the just quoted formula.

1. Be given a Riemannian manifold $(M, \mathrm{d}s^2)$, $\dim M = 2$. In a coordinate neighbourhood $U \subset M$, we may write

$$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,, \tag{1.1}$$

ω^1, ω^2 being linearly independent 1-forms. It is well known that there exists a unique 1-form ω_1^2 such that

$$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2\,, \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2\,; \tag{1.2}$$

the Gauss curvature K of $(M, \mathrm{d}s^2)$ is then defined by

$$\mathrm{d}\omega_1^2 = -K\omega^1 \wedge \omega^2\,. \tag{1.3}$$

On M, be given a 1-form ω; in U, let us write

$$\omega = a_1\omega^1 + a_2\omega^2\,. \tag{1.4}$$

The covariant derivatives a_{ij} (we write a_{ij} instead of $a_{i;j}$) are defined by

$$\mathrm{d}a_1 - a_2\omega_1^2 = a_{11}\omega^1 + a_{12}\omega^2\,, \quad \mathrm{d}a_2 + a_1\omega_1^2 = a_{21}\omega^1 + a_{22}\omega^2\,. \tag{1.5}$$

The exterior differentiation and the Cartan's lemma yield

$$\begin{aligned} \mathrm{d}a_{11} - (a_{12} + a_{21})\,\omega_1^2 &= a_{111}\omega^1 + a_{112}\omega^2\,, \\ \mathrm{d}a_{12} + (a_{11} - a_{22})\,\omega_1^2 &= a_{121}\omega^1 + a_{122}\omega^2\,, \\ \mathrm{d}a_{21} + (a_{11} - a_{22})\,\omega_1^2 &= a_{211}\omega^1 + a_{212}\omega^2\,, \\ \mathrm{d}a_{22} + (a_{12} + a_{21})\,\omega_1^2 &= a_{221}\omega^1 + a_{222}\omega^2\,, \end{aligned} \tag{1.6}$$

the second covariant derivatives satisfy

(1.7) $$a_{121} - a_{112} = Ka_2\,, \quad a_{212} - a_{221} = Ka_1\,.$$

Using a further prolongation, we get

(1.8)
$$\begin{aligned}
da_{111} - (a_{112} + a_{121} + a_{211})\,\omega_1^2 &= a_{1111}\omega^1 + a_{1112}\omega^2\,,\\
da_{112} + (a_{111} - a_{122} - a_{212})\,\omega_1^2 &= a_{1121}\omega^1 + a_{1122}\omega^2\,,\\
da_{121} + (a_{111} - a_{122} - a_{221})\,\omega_1^2 &= a_{1211}\omega^1 + a_{1212}\omega^2\,,\\
da_{122} + (a_{112} + a_{121} - a_{222})\,\omega_1^2 &= a_{1221}\omega^1 + a_{1222}\omega^2\,,\\
da_{211} + (a_{111} - a_{212} - a_{221})\,\omega_1^2 &= a_{2111}\omega^1 + a_{2112}\omega^2\,,\\
da_{212} + (a_{112} + a_{211} - a_{222})\,\omega_1^2 &= a_{2121}\omega^1 + a_{2122}\omega^2\,,\\
da_{221} + (a_{121} + a_{211} - a_{222})\,\omega_1^2 &= a_{2211}\omega^1 + a_{2212}\omega^2\,,\\
da_{222} + (a_{122} + a_{212} + a_{221})\,\omega_1^2 &= a_{2221}\omega^1 + a_{2222}\omega^2
\end{aligned}$$

with

(1.9)
$$\begin{aligned}
&a_{1121} - a_{1112} = K(a_{12} + a_{21})\,, \quad a_{1221} - a_{1212} = K(a_{22} - a_{11})\,,\\
&a_{2121} - a_{2112} = K(a_{22} - a_{11})\,, \quad a_{2212} - a_{2221} = K(a_{12} + a_{21})\,.
\end{aligned}$$

The differential consequences of (1.7) are

(1.10)
$$\begin{aligned}
&a_{1211} - a_{1121} = K_1a_2 + Ka_{21}\,, \quad a_{1212} - a_{1122} = K_2a_2 + Ka_{22}\,,\\
&a_{2121} - a_{2211} = K_1a_1 + Ka_{11}\,, \quad a_{2122} - a_{2212} = K_2a_1 + Ka_{12}\,,
\end{aligned}$$

the covariant derivatives $K_i \equiv K_{;i}$ of K being defined by

(1.11) $$dK = K_1\omega^1 + K_2\omega^2\,.$$

The Hodge $*$-operator is defined by

(1.12) $$*1 = \omega^1 \wedge \omega^2\,, \quad *\omega^1 = \omega^2\,, \quad *\omega^2 = -\omega^1\,, \quad *\omega^1 \wedge \omega^2 = 1\,,$$

the codifferential δ and the Laplace operator Δ by

(1.13) $$\delta\Omega = (-1)^p *^{-1} d * \Omega \quad \text{for} \quad \Omega \in \bigwedge^p(M)\,, \quad \Delta = -(d\delta + \delta d)$$

resp. For our 1-form (1.4), we get

(1.14) $$d\omega = (a_{21} - a_{12})\,\omega^1 \wedge \omega^2\,, \quad \delta\omega = -(a_{11} + a_{22})\,,$$

(1.15) $$\Delta\omega = (a_{111} + a_{122} - Ka_1)\,\omega^1 + (a_{211} + a_{222} - Ka_2)\,\omega^2\,.$$

Theorem 1. *Let (M, ds^2) be a compact manifold without boundary,* dim $M = 2$. *Let $l: M \to \mathbb{R}$ be a function and let the 1-form ω satisfy*

(1.16) $$(\Delta + l)\,\omega = 0\,.$$

If

(1.17) $$\max_M l < 2 \min_M K\,,$$

then $\omega \equiv 0$.

Proof. Consider the 1-forms

$$(1.18)\qquad \begin{aligned}\tau_1 &= \delta^{ij}a_i a_{jk}\omega^k = (a_1a_{11} + a_2a_{21})\,\omega^1 + (a_1a_{12} + a_2a_{22})\,\omega^2\,,\\ \tau_2 &= \delta^{ij}a_i a_{kj}\omega^k = (a_1a_{11} + a_2a_{12})\,\omega^1 + (a_1a_{21} + a_2a_{22})\,\omega^2\,,\\ \tau_3 &= \delta^{ij}a_k a_{ij}\omega^k = (a_1a_{11} + a_1a_{22})\,\omega^1 + (a_2a_{11} + a_2a_{22})\,\omega^2\,.\end{aligned}$$

Then

$$(1.19)\qquad \begin{aligned}&\mathrm{d} * (\tau_1 + \tau_2 - \tau_3) = \{(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2 +\\ &+ a_1(a_{111} + a_{122}) + a_2(a_{211} + a_{222}) + K(a_1^2 + a_2^2)\}\,\omega^1 \wedge \omega^2\,.\end{aligned}$$

From (1.16) and (1.15)

$$(1.20)\qquad a_{111} + a_{122} + (l - K)\,a_1 = 0\,,\quad a_{211} + a_{222} + (l - K)\,a_2 = 0\,;$$

this and the Stokes theorem applied to (1.19) yield

$$(1.21)\qquad \int_M \{(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2 + (2K - l)\,(a_1^2 + a_2^2)\}\,\omega^1 \wedge \omega^2 = 0\,,$$

and the Theorem follows easily.

Theorem 2. *Let $(M, \mathrm{d}s^2)$ be a compact manifold without boundary*, $\dim M = 2$. *Let $l\colon M \to \mathbb{R}$ be a function and let the 1-form ω satisfy (1.16). If*

$$(1.22)\qquad 2\max_M K < \min_M l \leqq \max_M l < 6\min_M K\,,$$

we have $\omega \equiv 0$.

Proof. From (1.20), we get

$$(1.23)\qquad \begin{aligned}a_{1111} + a_{1221} + (l_1 - K_1)\,a_1 + (l - K)\,a_{11} &= 0\,,\\ a_{2111} + a_{2221} + (l_1 - K_1)\,a_2 + (l - K)\,a_{21} &= 0\,,\\ a_{1112} + a_{1222} + (l_2 - K_2)\,a_1 + (l - K)\,a_{12} &= 0\,,\\ a_{2112} + a_{2222} + (l_2 - K_2)\,a_2 + (l - K)\,a_{22} &= 0\,,\end{aligned}$$

l_i being defined by $\mathrm{d}l = l_1\omega^1 + l_2\omega^2$; see (1.11). We easily get

$$(1.24)\qquad \mathrm{d} * \mathrm{d}\{(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2\} = (J + J')\,\omega^1 \wedge \omega^2$$

with

$$(1.25)\qquad \begin{aligned}J &= 2(a_{111} - a_{221})^2 + 2(a_{121} + a_{211})^2 +\\ &\quad + 2(a_{112} - a_{222})^2 + 2(a_{122} + a_{212})^2\,,\\ J' &= 2(a_{11} - a_{22})\,(a_{1111} - a_{2211} + a_{1122} - a_{2222}) +\\ &\quad + 2(a_{12} + a_{21})\,(a_{1211} + a_{2111} + a_{1222} + a_{2122})\,.\end{aligned}$$

We may write

$$\begin{aligned}J &= (a_{111} - a_{221} - a_{122} - a_{212})^2 + (a_{121} + a_{211} + a_{112} - a_{222})^2 +\\ &\quad + (a_{111} - a_{221} + a_{122} + a_{212})^2 + (a_{121} + a_{211} - a_{112} + a_{222})^2\,;\end{aligned}$$

using (1.7) and (1.20) in the last two terms, we get

$$(1.26)\qquad \begin{aligned}J &= (a_{111} - a_{221} - a_{122} - a_{212})^2 + (a_{121} + a_{211} + a_{112} - a_{222})^2 +\\ &\quad + (2K - l)^2\,(a_1^2 + a_2^2)\,.\end{aligned}$$

Using (1.9), (1.10) and (1.23), we get after elementary calculations

$$(1.27)\qquad J' = 2(4K - l)\{(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2\} + \\ + 2(2K_1 - l_1)\{(a_{11} - a_{22})\,a_1 + (a_{12} + a_{21})\,a_2\} + \\ + 2(2K_2 - l_2)\{(a_{12} + a_{21})\,a_1 - (a_{11} - a_{22})\,a_2\}\,.$$

For $f: M \to \mathbb{R}$, f_i be defined by $\mathrm{d}f = f_1\omega^1 + f_2\omega^2$. From (1.18), (1.19) and (1.20),

$$(1.28)\qquad \mathrm{d} * f(\tau_1 + \tau_2 - \tau_3) = \{f[(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2] + \\ + f(2K - l)(a_1^2 + a_2^2) + f_1[(a_{11} - a_{22})\,a_1 + (a_{12} + a_{21})\,a_2] + \\ + f_2[(a_{12} + a_{21})\,a_1 - (a_{11} - a_{22})\,a_2]\}\,\omega^1 \wedge \omega^2\,.$$

Now, from (1.24), (1.26), (1.27) and (1.28) for $f = 2(l - 2K) - r$, $r \in \mathbb{R}$, we get the final formula

$$(1.29)\qquad \mathrm{d} * \{\mathrm{d}[(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2] + (2l - 4K - r)(\tau_1 + \tau_2 - \tau_3)\} = \\ = \{(a_{111} - a_{221} - a_{122} - a_{212})^2 + (a_{121} + a_{211} + a_{112} - a_{222})^2 + \\ + (4K - r)[(a_{11} - a_{22})^2 + (a_{12} + a_{21})^2] + \\ + (l - 2K)(2K - l + r)(a_1^2 + a_2^2)\}\,\omega^1 \wedge \omega^2\,.$$

Let us take

$$(1.30)\qquad r = \min_M K + \tfrac{1}{2}\max_M l\,.$$

Then (1.22) implies

$$(1.31)\qquad 4K - r = 4(K - \min_M K) + \tfrac{1}{2}(6\min_M K - \max_M l) > 0\,, \\ l - 2K = (l - \min_M l) + 2(\max_M K - K) + \\ + (\min_M l - 2\max_M K) > 0\,, \\ 2K - l + r = 2(K - \min_M K) + (\max_M l - l) + \\ + \tfrac{1}{2}(6\min_M K - \max_M l) > 0\,,$$

and our Theorem follows from the Stokes theorem applied to (1.29).

2. Let $(M, \mathrm{d}s^2) = (S^2(1), \mathrm{d}s_0^2)$ be a unit sphere in the Euclidean 3-space E^3 with the induced metric $\mathrm{d}s_0^2$. To each point $m \in S^2$ (in a coordinate neighbourhood) let us associate an orthonormal frame $\{m; v_1, v_2, v_3\}$ such that $m + v_3$ is the center of S^2. Then we may write

$$(2.1)\qquad \mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,,\quad \mathrm{d}v_1 = \omega_1^2 v_2 + \omega^1 v_3\,, \\ \mathrm{d}v_2 = -\omega_1^2 v_1 + \omega^2 v_3\,,\quad \mathrm{d}v_3 = -\omega^1 v_1 - \omega^2 v_2\,.$$

Let V^3 be the vector space of E^3, and let $\Omega: V^3 \to \mathbb{R}$ be a 1-form. At the point $m \in S^2$, let $v = v^\alpha e_\alpha(m) \in V$; $\alpha, \beta, \ldots = 1, 2, 3$; be a vector and $\Omega(v) = \Omega_\alpha v^\alpha$. The 1-form Ω being fixed, we easily find $\mathrm{d}\Omega_\alpha - \Omega_\beta\omega_\alpha^\beta = 0$, i.e.,

$$(2.2)\qquad \mathrm{d}\Omega_1 - \Omega_2\omega_1^2 = \Omega_3\omega^1\,,\quad \mathrm{d}\Omega_2 + \Omega_1\omega_1^2 = \Omega_3\omega^2\,,\quad \mathrm{d}\Omega_3 = -\Omega_1\omega^1 - \Omega_2\omega^2\,.$$

Denote by ω_Ω the restriction of Ω to S^2, i.e.,

$$\omega_\Omega = \Omega_1\omega^1 + \Omega_2\omega^2 . \tag{2.3}$$

Using (2.2), we get the covariant derivatives $\Omega_{ij} \equiv \Omega_{i;j}$, $\Omega_{ijk} \equiv \Omega_{i;jk}$ as follows:

$$\Omega_{11} = \Omega_3 , \quad \Omega_{12} = \Omega_{21} = 0 , \quad \Omega_{22} = \Omega_3 ; \tag{2.4}$$
$$\Omega_{111} = -\Omega_1 , \quad \Omega_{122} = \Omega_{211} = 0 , \quad \Omega_{222} = -\Omega_2 .$$

From (1.15),

$$(\Delta + 2)\,\omega_\Omega = 0 . \tag{2.5}$$

Further,

$$*\omega_\Omega = -\Omega_2\omega^1 + \Omega_1\omega^2 \tag{2.6}$$

and, similarly,

$$(\Delta + 2) * \omega_\Omega = 0 . \tag{2.7}$$

Theorem 3. *Let $S^2 \subset E^3$ be a unit sphere and ω a 1-form on S^2 satisfying $(\Delta + 2)\,\omega = 0$. Then there are 1-forms $\Omega, \Omega' \colon V^3 \to \mathbb{R}$ such that*

$$\omega = \omega_\Omega + *\omega_{\Omega'} . \tag{2.8}$$

The multiplicity of the eigenvalue $2 \in \mathrm{Spec}^1(S^2)$ is thus 6.

Proof. From the formula (1.21) for $K = 1$, $l = 2$, we get $a_{11} - a_{22} = a_{12} + a_{21} = 0$. Thus the equations (1.5) take the from

$$da_1 - a_2\omega_1^2 = a\omega^1 - a'\omega^2 , \quad da_2 + a_1\omega_1^2 = a'\omega^1 + a\omega^2 . \tag{2.9}$$

Using the usual prolongation procedure, we get the existence of functions $b, b', \ldots \ldots, f, f'$ such that

$$da = (b - \tfrac{1}{2}a_1)\,\omega^1 - (b' + \tfrac{1}{2}a_2)\,\omega^2 , \tag{2.10}$$
$$da' = (b' - \tfrac{1}{2}a_2)\,\omega^1 + (b + \tfrac{1}{2}a_1)\,\omega^2 ,$$
$$db + b'\omega_1^2 = (c - \tfrac{1}{2}a)\,\omega^1 - (c' + \tfrac{1}{2}a')\,\omega^2 , \tag{2.11}$$
$$db' - b\omega_1^2 = (c' - \tfrac{1}{2}a')\,\omega^1 + (c + \tfrac{1}{2}a)\,\omega^2 ,$$
$$dc + 2c'\omega_1^2 = e\omega^1 - e'\omega^2 , \quad dc' - 2c\omega_1^2 = e'\omega^1 + e\omega^2 ,$$
$$de + 3e'\omega_1^2 = (f + c)\,\omega^1 - (f' - c')\,\omega^2 ,$$
$$de' - 3e\omega_1^2 = (f' + c')\,\omega^1 + (f - c)\,\omega^2 .$$

Further,

$$0 = \int_{\partial S^2} *d(c^2 + c'^2) = 4\int_{S^2} (e^2 + e'^2 + c^2 + c'^2)\,\omega^1 \wedge \omega^2 , \tag{2.12}$$

i.e., $e = e' = c = c' = 0$. The equations $(2.11_{1,2})$ reduce to

$$db + b'\omega_1^2 = -\tfrac{1}{2}(a\omega^1 + a'\omega^2) , \quad db' - b\omega_1^2 = -\tfrac{1}{2}(a'\omega^1 - a\omega^2) , \tag{2.13}$$

and the system (2.9 + 10 + 13) is completely integrable. Now, take

$$\Omega_1 = \tfrac{1}{2}a_1 - b , \quad \Omega_2 = \tfrac{1}{2}a_2 + b' , \quad \Omega_3 = a , \tag{2.14}$$
$$\Omega'_1 = \tfrac{1}{2}a_2 - b' , \quad \Omega'_2 = -\tfrac{1}{2}a_1 - b , \quad \Omega'_3 = a' ;$$

it is easy to see that Ω_α, Ω'_α satisfy the equations of the type (2.2). Thus we get two 1-forms $\Omega, \Omega': V^3 \to \mathbb{R}$ with

$$
\begin{aligned}
\omega_\Omega &= \left(\tfrac{1}{2}a_1 - b\right)\omega^1 + \left(\tfrac{1}{2}a_2 + b'\right)\omega^2\,, \\
\omega_{\Omega'} &= \left(\tfrac{1}{2}a_2 - b'\right)\omega^1 - \left(\tfrac{1}{2}a_1 + b\right)\omega^2\,,
\end{aligned}
\tag{2.15}
$$

and (2.8) follows. QED.

Let $\Psi: V \times V \to \mathbb{R}$ be a bilinear symmetric form. Its coordinates at the point $m \in S^2$ be $\Psi_{\alpha\beta} = \Psi(v_\alpha(m), v_\beta(m))$; we have $\Psi_{\alpha\beta} = \Psi_{\beta\alpha}$ and

$$
\mathrm{d}\Psi_{\alpha\beta} - \Psi_{\gamma\beta}\omega_\alpha^\gamma - \Psi_{\alpha\gamma}\omega_\beta^\gamma = 0\,, \tag{2.16}
$$

i.e.,

$$
\begin{aligned}
& d\Psi_{11} - 2\Psi_{12}\omega_1^2 = 2\Psi_{13}\omega^1\,, \\
& \mathrm{d}\Psi_{22} + 2\Psi_{12}\omega_1^2 = 2\Psi_{23}\omega^2\,, \\
& \mathrm{d}\Psi_{33} = -2\Psi_{13}\omega^1 - 2\Psi_{23}\omega^2\,, \\
& \mathrm{d}\Psi_{12} + (\Psi_{11} - \Psi_{22})\,\omega_1^2 = \Psi_{23}\omega^1 + \Psi_{13}\omega^2\,, \\
& \mathrm{d}\Psi_{13} - \Psi_{23}\omega_1^2 = (\Psi_{33} - \Psi_{11})\,\omega^1 - \Psi_{12}\omega^2\,, \\
& \mathrm{d}\Psi_{23} + \Psi_{13}\omega_1^2 = -\Psi_{12}\omega^1 + (\Psi_{33} - \Psi_{22})\,\omega^2\,.
\end{aligned}
\tag{2.17}
$$

The form Ψ generates the 1-form ω_Ψ on S^2 as follows: Let $m \in S^2$, $t \in T_m(S^2)$, $v_3(m)$ the normal unit vector at m; then

$$
\omega_\Psi(t) = \Psi(v_3(m), t)\,. \tag{2.18}
$$

In the coordinates,

$$
\omega_\Psi = \Psi_{13}\omega^1 + \Psi_{23}\omega^2\,. \tag{2.19}
$$

From (2.17), we get the covariant derivatives

$$
\begin{aligned}
\Psi_{13;1} &= \Psi_{33} - \Psi_{11}\,, & \Psi_{13;2} &= -\Psi_{12}\,, \\
\Psi_{23;1} &= -\Psi_{12}\,, & \Psi_{23;2} &= \Psi_{33} - \Psi_{22}\,; \\
\Psi_{13;11} &= -4\Psi_{13}\,, & \Psi_{13;22} &= -\Psi_{13}\,, \\
\Psi_{23;11} &= -\Psi_{23}\,, & \Psi_{23;22} &= -4\Psi_{23}\,.
\end{aligned}
\tag{2.20}
$$

Thus, see (1.15),

$$
(\Delta + 6)\,\omega_\Psi = 0\,. \tag{2.21}
$$

Analoguously, we get

$$
(\Delta + 6) * \omega_\Psi = 0\,. \tag{2.22}
$$

Theorem 4. *Let $S^2 \subset E^3$ be a unit sphere and ω a solution of $(\Delta + 6)\,\omega = 0$ on S^2. Then there are bilinear symmetric forms $\Psi, \Psi': V \times V \to \mathbb{R}$ with vanishing trace such that*

$$
\omega = \omega_\Psi + *\omega_{\Psi'}\,. \tag{2.23}
$$

The multiplicity of the eigenvalue $6 \in \mathrm{Spec}^1(S^2)$ is thus 10.

Proof. In the integral formula based on (1.29), take $K = 1$, $l = 6$, $r = 4$. Then

$$(2.24)\qquad a_{111} - a_{221} - a_{122} - a_{212} = 0\,,\quad a_{121} + a_{211} + a_{112} - a_{222} = 0\,;$$

from this, (1.10) and (1.7), we get the existence of functions A, A' such that (1.6) become

$$(6.25)\qquad \begin{aligned} da_{11} - (a_{12} + a_{21})\,\omega_1^2 &= (A' - \tfrac{5}{2}a_1)\,\omega^1 + (A - \tfrac{1}{2}a_2)\,\omega^2\,,\\ da_{12} + (a_{11} - a_{22})\,\omega_1^2 &= (A + \tfrac{1}{2}a_2)\,\omega^1 - (A' + \tfrac{5}{2}a_1)\,\omega^2\,,\\ da_{21} + (a_{11} - a_{22})\,\omega_1^2 &= -(A + \tfrac{5}{2}a_2)\,\omega^1 + (A' + \tfrac{1}{2}a_1)\,\omega^2\,,\\ da_{22} + (a_{12} + a_{21})\,\omega_1^2 &= (A' - \tfrac{1}{2}a_1)\,\omega^1 + (A - \tfrac{5}{2}a_2)\,\omega^2\,. \end{aligned}$$

The prolongations yield the existence of functions $B, B', \ldots, E, E'$ such that

$$(6.26)\qquad \begin{aligned} dA + A'\omega_1^2 &= \{B + \tfrac{3}{4}(a_{21} - a_{12})\}\,\omega^1 - \{B' + \tfrac{3}{4}(a_{11} + a_{22})\}\,\omega^2\,,\\ dA' - A\omega_1^2 &= \{B' - \tfrac{3}{4}(a_{11} + a_{22})\}\,\omega^1 + \{B - \tfrac{3}{4}(a_{21} - a_{12})\}\,\omega^2\,; \end{aligned}$$

$$(6.27)\qquad \begin{aligned} dB + 2B'\omega_1^2 &= (C - A)\,\omega^1 - (C' + A')\,\omega^2\,,\\ dB' - 2B\omega_1^2 &= (C' - A')\,\omega^1 + (C + A)\,\omega^2\,; \end{aligned}$$

$$(6.28)\qquad \begin{aligned} dC + 3C'\omega_1^2 &= D\omega^1 - D'\omega^2\,,\\ dC' - 3C\omega_1^2 &= D'\omega^1 + D\omega^2\,,\\ dD + 4D'\omega_1^2 &= (E + \tfrac{3}{2}C)\,\omega^1 - (E' - \tfrac{3}{2}C')\,\omega^2\,,\\ dD' - 4D\omega_1^2 &= (E' + \tfrac{3}{2}C')\,\omega^1 + (E - \tfrac{3}{2}C)\,\omega^2\,. \end{aligned}$$

From this,

$$(6.29)\qquad 0 = \int_{\partial S^2} *d(C^2 + C'^2) = 2\int_{S^2}\left[2(D^2 + D'^2) + 3(C^2 + C'^2)\right]\omega^1 \wedge \omega^2\,,$$

and (6.27) reduce to

$$(6.30)\qquad dB + 2B'\omega_1^2 = -A\omega^1 - A'\omega^2\,,\quad dB' - 2B\omega_1^2 = -A'\omega^1 + A\omega^2\,.$$

The system (1.5) + (2.25 + 26 + 30) is completely integrable. Now, define

$$(6.31)\qquad \begin{aligned} &\Psi_{11} = \tfrac{1}{12}(4B' - 5a_{11} + a_{22})\,,\quad \Psi_{22} = -\tfrac{1}{12}(4B' - a_{11} + 5a_{22})\,,\\ &\Psi_{33} = \tfrac{1}{3}(a_{11} + a_{22})\,,\quad \Psi_{12} = \Psi_{21} = \tfrac{1}{12}(4B - 3a_{12} - 3a_{21})\,,\\ &\Psi_{13} = \Psi_{31} = -\tfrac{1}{6}(2A' - 3a_1)\,,\quad \Psi_{23} = \Psi_{32} = -\tfrac{1}{6}(2A - 3a_2)\,;\\ &\Psi'_{11} = -\tfrac{1}{12}(4B + a_{12} + 5a_{21})\,,\quad \Psi'_{22} = \tfrac{1}{12}(4B + 5a_{12} + a_{21})\,,\\ &\Psi'_{33} = \tfrac{1}{3}(a_{21} - a_{12})\,,\quad \Psi'_{12} = \Psi'_{21} = \tfrac{1}{12}(4B' + 3a_{11} - 3a_{22})\,,\\ &\Psi'_{13} = \Psi'_{31} = \tfrac{1}{6}(2A + 3a_2)\,,\quad \Psi'_{23} = \Psi'_{32} = -\tfrac{1}{6}(2A' + 3a_1)\,. \end{aligned}$$

By definition, they satisfy (2.17) and generate two symmetric bilinear forms $\Psi, \Psi'\colon V^3 \times V^3 \to \mathbb{R}$ with zero trace. Now,

$$(6.32)\qquad \begin{aligned} \omega_\Psi &= -\tfrac{1}{6}(2A' - 3a_1)\,\omega^1 - \tfrac{1}{6}(2A - 3a_2)\,\omega^2\,,\\ \omega_{\Psi'} &= \tfrac{1}{6}(2A + 3a_2)\,\omega^1 - \tfrac{1}{6}(2A' + 3a_1)\,\omega^2\,, \end{aligned}$$

and (2.23) follows.

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