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ON CONJUGATE POINTS OF SOLUTIONS OF NON-SELFADJOINT DIFFERENTIAL SYSTEMS

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1. INTRODUCTION

The aim of the presentpaper is to study the oscillation properties of solutions of linear differential systems

$$y' = B(x)\,z\,, \quad z' = C(x)\,y\,, \tag{1.1}$$

where $B(x)$, $C(x)$ are $n \times n$ matrices of continuous real-valued functions and $y(x)$, $z(x)$ are n-dimensional vectors of real-valued functions. The principal method used here is the method of transformations of systems (1.1), which is described in Section 2. In Section 3 it is shown that the oscillation behaviour of solutions of (1.1) can be studied by means of oscillation behaviour of solutions of certain selfadjoint differential systems. Section 4 involves application of the results of the preceding sections to investigation of scalar differential equations of the even order.

The following notation is used. Identity matrices are denoted by the symbol E, the symbol 0 is used for the zero matrix of any dimension. The transpose of a matrix A is denoted by A^T, and A is said to be *symmetric* whenever $A^T = A$. If A is a symmetric $n \times n$ matrix, we write $A > 0$ ($\geqq 0$) to indicate that the matrix A is positive (non-negative) definite. $C^m(I)$ denotes the space of real-valued functions having continuous m-th derivatives on the interval I, C^0 means continuity. If $A(x)$ is matrix of real-valued functions we write $A(x) \in C^m(I)$ if all entries of this matrix are of the class $C^m(I)$. Throughout the paper I denotes a subinterval of the real line of an arbitrary kind. A pair of n-dimensional vectors $(y(x), z(x))$ is a solution of (1.1) on I if $y(x), z(x) \in C^1(I)$ and equations (1.1) are indentically satisfied on I. If $Y(x), Z(x) \in C^1(I)$ are $n \times n$ matrices for which $Y'(x) = B(x)\,Z(x)$, $Z'(x) = C(x)\,Y(x)$ on I, we refer to the pair of matrices $(Y(x), Z(x))$ as to a $2n \times n$ solution of (1.1).

2. TRANSFORMATIONS OF NON-SELFADJOINT SYSTEMS

Consider the following transformation of system (1.1)

$$(2.1) \qquad \begin{aligned} y &= H(x)\, u\,, \\ z &= K(x)\, u + L(x)\, v\,, \end{aligned}$$

where $H(x), K(x), L(x) \in C^1(I)$ are $n \times n$ matrices of real-valued functions, $H(x)$, $L(x)$ being nonsingular, for which

$$(2.2) \qquad \begin{aligned} H'(x) - B(x)\, K(x) &= 0\,, \\ L'(x) + K(x)\, H^{-1}(x)\, B(x)\, L(x) &= 0\,. \end{aligned}$$

Transformation (2.1) transforms system (1.1) to the system

$$(2.3) \qquad u' = B_1(x)\, v\,, \quad v' = C_1(x)\, u\,,$$

where

$$(2.4) \qquad \begin{aligned} B_1(x) &= H^{-1}(x)\, B(x)\, L(x)\,, \\ C_1(x) &= L^{-1}(x)\,(-K'(x) + C(x)\, H(x))\,. \end{aligned}$$

Indeed, from (1.1) and (2.1) it follows $y' = H'u + Hu' = BKu + BLv$ and hence $u' = H^{-1}(-H' + BK)\, u + H^{-1}BLv = B_1 v$. Further, $z' = K'u + Ku' + L'v + Lv' = CHu$. This implies $v' = L^{-1}[(-K' + CH)\, u - Ku' - L'v] = L^{-1}(-K' + CH)\, u - L^{-1}(KH^{-1}BL + L')\, v = C_1 v$.

System (1.1) is uniquely determined by the pair of matrices $B(x), C(x)$. For this reason we shall sometimes denote this system by (B, C). If (B, C) and (B_1, C_1) are two differential systems of the form (1.1), there exist $n \times n$ matrices $H(x), K(x), L(x)$ of the class $C^1(I)$, $H(x), L(x)$ being nonsingular, satisfying (2.2), and the matrices $B(x), C(x)$ and $B_1(x), C_1(x)$ are connected by relations (2.4), we say that transformation (2.1) transforms system (B, C) to system (B_1, C_1). This fact will be denoted by $(B, C) \to^{H,K,L} (B_1, C_1)$. It would perhaps be more suitable to say that transformation (2.1) transforms (B_1, C_1) to (B, C) since a solution $(y(x), z(x))$ of (B, C) is in (2.1) expressed by means of a solution $(u(x), v(x))$ of (B_1, C_1). But for our purpose it is more convenient the former terminology and the following statement shows that this minor inaccuracy is immaterial.

Lemma 1. *Let* $(B, C) \to^{H,K,L} (B_1, C_1) \to^{M,N,P} (B_2. C_2)$. *Then* $(B_1, C_1) \to^{H^{-1}, -L^{-1}KH^{-1}, L^{-1}} (B, C) \to^{HM, KM+LN, LP} (B_2, C_2)$.

Proof. Let $(B, C) \to^{H,K,L} (B_1, C_1)$, i.e. the matrices $H(x), K(x), L(x)$ satisfy (2.2), (2.4) and (u, v) is a solution of (B_1, C_1) if and only if $(y, z) = (H(x)\, u, K(x)\, u + L(x)\, v)$ is a solution of (B, C). As

$$\begin{bmatrix} H & 0 \\ K & L \end{bmatrix} \begin{bmatrix} H^{-1} & 0 \\ -L^{-1}KH^{-1} & L^{-1} \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix},$$

we have $u = H^{-1}y$, $v = -L^{-1}KH^{-1}y + L^{-1}z$. Further, $(H^{-1})' - B_1(-L^{-1}KH^{-1}) = -H^{-1}H'H^{-1} + H^{-1}BLL^{-1}KH^{-1} =$

$= -H^{-1}(H' - BK)H^{-1} = 0$, $(L^{-1})' + (-L^{-1}KH^{-1})HB_1L^{-1} = -L^{-1}L'L^{-1} - L^{-1}KH^{-1}BLL^{-1} = -L^{-1}(L' + KH^{-1}BL)L^{-1} = 0$, $B = HB_1L^{-1}$ and $C = LC_1H^{-1} + K'H^{-1} = LC_1H^{-1} + K'H^{-1} - (L' + KH^{-1}BL)L^{-1}KH^{-1} = LC_1H^{-1} - (L'L^{-1}K + KH^{-1}BK - K')H^{-1} = LC_1H^{-1} - L(L^{-1}L'L^{-1}KH^{-1} + L^{-1}KH^{-1}H'H^{-1} - L^{-1}K'H^{-1}) = L(-(-L^{-1}KH^{-1})' + C_1H^{-1})$, hence $(B_1, C_1) \to^{H^{-1},-L^{-1}KH^{-1},L^{-1}} (B, C)$. Now let $(B, C) \to^{H,K,L} (B_1, C_1) \to^{M,N,P} (B_2, C_2)$. Then $(HM)' - B(KM + LN) = H'M + HM' - BKM - BLN = (H' - BK)M + H(M' - H^{-1}BLN) = H(M' - B_1N) = 0$, $(LP)' + (KM + LN)(HM)^{-1}B(LP) = L'P + LP' + KH^{-1}BLP + LNM^{-1}H^{-1}BLP = (L' + KH^{-1}BL)P + L(P' + NM^{-1}B_1P) = 0$, $B_2 = M^{-1}B_1P = M^{-1}H^{-1}BLP = (HM)^{-1}B(LP)$ and $C_2 = P^{-1}(-N' + C_1M) = -P^{-1}N' + P^{-1}(L^{-1}CH - L^{-1}K')M = (LP)^{-1}C(HM) - (LP)^{-1}(K'M + LN' + KB_1N - KH^{-1}BLN) = (LP)^{-1}[C(HM) - (K'M + LN' + KM' + L'N)] = (LP)^{-1}[-(KM + LN)' + C(HM)]$. The proof is complete.

Directly can be proved this statement.

Lemma 2. *If transformation* (2.1) *transforms* (1.1) *to* (2.3) *then the transformation* $y = L^{T-1}(x)u$, $z = -(L^{-1}(x)K(x)H^{-1}(x))^T u + H^{T-1}(x)v$ *transforms the system*

$$y' = B^T(x)z, \quad z' = C^T(x)y \tag{2.5}$$

to the system

$$u' = B_1^T(x)v, \quad v' = C_1^T(x)u. \tag{2.6}$$

In the sequel the principal role is played by the following statement.

Theorem 1. *There exist* $n \times n$ *matrices* $H(x), K(x), L(x) \in C^1(I)$, $H(x)$, $L(x)$ *being nonsingular, satisfying* (2.2) *and such that transformation* (2.1) *transforms system* (1.1) *to the system*

$$u' = Q(x)v, \quad v' = -Q^T(x)u. \tag{2.7}$$

The matrix $Q(x)$ *is given by the relation*

$$Q(x) = H^{-1}(x)B(x)L(x). \tag{2.8}$$

Proof. Let $(U_i(x), V_i(x))$, $(Y_i(x), Z_i(x))$, $i = 1, 2$, be the $2n \times n$ solutions of (1.1) and (2.5), respectively, for which $(U_1(a), V_1(a)) = (Y_1(a), Z_1(a)) = (0, E)$, $(U_2(a), V_2(a)) = (Y_2(a), Z_2(a)) = (E, 0)$, where $a \in I$. Denote

$$\mathfrak{U}_1 = \begin{bmatrix} 0 & U_1 \\ Y_1 & 0 \end{bmatrix}, \quad \mathfrak{U}_2 = \begin{bmatrix} U_2, & 0 \\ 0 & Y_2 \end{bmatrix}, \quad \mathfrak{B}_1 = \begin{bmatrix} Z_1 & 0 \\ 0 & V_1 \end{bmatrix}, \quad \mathfrak{B}_2 = \begin{bmatrix} 0 & Z_2 \\ V_2 & 0 \end{bmatrix}. \tag{2.9}$$

$$B_0 = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix}. \tag{2.10}$$

Then $(\mathfrak{U}_i, \mathfrak{B}_i)$ are solutions of the 4n-dimensional selfadjoint system

$$\mathfrak{U}' = B_0\mathfrak{B}, \quad \mathfrak{B}' = C_0\mathfrak{U} \tag{2.11}$$

and by differentiation we can verify that

$$(2.12)\qquad \begin{aligned} \mathfrak{U}_1^T\mathfrak{V}_2 - \mathfrak{V}_1^T\mathfrak{U}_2 &= -E\,,\\ \mathfrak{U}_1^T\mathfrak{V}_1 - \mathfrak{V}_1^T\mathfrak{U}_1 &= \ \ 0\,,\\ \mathfrak{V}_2^T\mathfrak{U}_2 - \mathfrak{U}_2^T\mathfrak{V}_2 &= \ \ 0\,. \end{aligned}$$

Consequently

$$(2.13)\qquad \begin{aligned} \mathfrak{V}_2\mathfrak{U}_1^T - \mathfrak{V}_1\mathfrak{U}_2^T &= -E\,,\\ \mathfrak{U}_1\mathfrak{U}_2^T - \mathfrak{U}_2\mathfrak{U}_1^T &= \ \ 0\,,\\ \mathfrak{V}_1\mathfrak{V}_2^T - \mathfrak{V}_2\mathfrak{V}_1^T &= \ \ 0\,, \end{aligned}$$

and using (2.9) we obtain

$$(2.14)\qquad \begin{aligned} U_1^TZ_1 - V_1^TY_1 &= \ \ 0\,, & Y_2U_1^T - Y_1U_1^T &= \ \ 0\,,\\ Y_2^TV_2 - Z_2^TU_2 &= \ \ 0\,, & V_2Z_1^T - V_1Z_2^T &= \ \ 0\,,\\ Y_1^TV_2 - Z_1^TU_2 &= -E\,, & Z_2U_1^T - Z_1U_2^T &= -E\,,\\ U_1^TZ_2 - V_1^TY_2 &= -E\,, & V_2Y_1^T - V_1Y_2^T &= -E\,. \end{aligned}$$

Now let $A = \mathfrak{U}_1\mathfrak{U}_1^T + \mathfrak{U}_2\mathfrak{U}_2^T = \operatorname{diag}\{A_1, A_2\}$, where $A_1 = U_1U_1^T + U_2U_2^T$, $A_2 = Y_1Y_1^T + Y_2Y_2^T$. By means of identities (2.13) it can be proved, see e.g. [2], that the matrix A is positive definite, and hence the matrices A_1, A_2 are also positive definite. Let $D_i > 0$, $i = 1, 2$, be the symmetric $n\times n$ matrices for which $D_i^2 = A_i$ and let $T_1(x)$, $T_2(x)$ be the solutions of the differential systems

$$(2.15)_1\qquad T_1' = [D_1^{-1}B(V_1U_1^T + V_2U_2^T)\,D_1^{-1} - D_1^{-1}D_1']\,T_1\,,\qquad T_1(a) = E\,,$$

$$(2.15)_2\qquad T_2' = [D_2^{-1}(B^T(Z_1Y_1^T + Z_2Y_2^T)\,D_2^{-1} - D_2^{-1}D_2')]\,T_2\,,\quad T_2(a) = E\,.$$

Further, let

$$D = \begin{bmatrix} D_1 & 0\\ 0 & D_2\end{bmatrix},\quad T = \begin{bmatrix} T_1 & 0\\ 0 & T_2\end{bmatrix}.$$

Then $T = [D^{-1}(B_0\mathfrak{V}_1\mathfrak{U}_1^T + B_0\mathfrak{V}_2\mathfrak{U}_2^T)\,D^{-1} - D^{-1}D')]\,T$, $T(a) = E$, and by a direct computation we can verify that $[D^{-1}B_0(\mathfrak{V}_1\mathfrak{U}_1^T + \mathfrak{V}_2\mathfrak{U}_2^T)\,D^{-1} - D^{-1}D']^T + D^{-1}B_0(\mathfrak{V}_1\mathfrak{U}_1^T + \mathfrak{V}_2\mathfrak{U}_2^T - D^{-1}D') = 0$, hence the matrix $T(x)$ is orthogonal (i.e. $T^{-1}(x) = T^T(x)$) which implies that the matrices $T_1(x)$, $T_1(x)$ are also orthogonal. Let us set

$$(2.16)\qquad \begin{aligned} H(x) &= D_1(x)\,T_1(x)\,,\\ K(x) &= (V_1(x)\,U_1^T(x) + V_2(x)\,U_2^T(x))\,H^{T-1}(x)\,,\\ L(x) &= D_2^{-1}(x)\,T_2(x)\,. \end{aligned}$$

Then $HH^T = D_1T_1T_1^TD_1 = D_1^2 = A_1 = U_1U_1^T + U_2U_2^T$, $LL^T = D_2^{-1}T_2T_2^TD_2^{-1} = D_2^{-2} = A_2^{-1} = (Y_1Y_1^T + Y_2Y_2^T)^{-1}$, $H' - BK = D_1'T_1 + D_1T_1' - B(V_1U_1^T + V_2U_2^T)\,H^{T-1} = D_1'T_1 + D_1[D_1^{-1}B(V_1U_1^T + V_2U_2^T)\,D_1^{-1} - D_1^{-1}D_1']\,T_1 - B(V_1U_1^T + V_2U_2^T)\,H^{T-1} = B(V_1U_1^T + V_2U_2^T)\,D_1^{-1}\,T_1 - B(V_1U_1^T + V_2U_2^T)\,D_1^{-1}T_1 = 0$, $L' + KH^{-1}BL = -D_2^{-1}D_2'D_2^{-1}T_2 + D_2^{-1}T_2' +$

$+ (V_1U_1^T + V_2U_2^T) H^{T-1}H^{-1}BD_2^{-1}T_2 = -D_2^{-1}D_2'D_2^{-1}T_2 + D_2^{-1}T_2'T_2^TT_2 +$
$+ (V_1U_1^T + V_2U_2^T) H^{T-1}H^{-1}BD_2^{-1}T_2 = -D_2^{-1}D_2'D_2^{-1}T_2 - D_2^{-1}T_2T_2^{T\prime}D_2D_2^{-1}T_2 +$
$+ (V_1U_1^T + V_2U_2^T)(HH^T)^{-1} BL = -D_2^{-1}D_2'D_2^{-1}T_2 - D_2^{-1}T_2T_2^T(-D_2'D_2^{-1} +$
$+ D_2^{-1}(Y_1Z_1^T + Y_2Z_2^T) BD_2^{-1}) D_2L + (V_1U_1^T + V_2U_2^T)(HH^T)^{-1} BL =$
$= -D_2^{-1}D_2'L + D_2^{-1}D_2'L - D_2^{-2}(Y_1Z_1^T + Y_2Z_2^T) BL + (V_1U_1^T + V_2U_2^T) D_1^{-2}BL =$
$= D_2^{-2}[-(Y_1Z_1^T + Y_2Z_2^T)(U_1U_1^T + U_2U_2^T) + (Y_1Y_1^T + Y_2Y_2^T)(V_1U_1^T +$
$+ V_2U_2^T)] D_1^{-2}BL = D_2^{-2}[Y_1(-Z_1^TU_1 + Y_1^TV_1) U_1^T + Y_1(-Z_1^TU_2 + Y_1^TV_2) U_2^T +$
$+ Y_2(-Z_2^TU_1 + Y_2^TV_1) U_1^T + Y_2(-Z_2^TU_2 + Y_2^TV_2) U_2^T] D_1^{-2}BL = D_2^{-2}(Y_2U_1^T -$
$- Y_1U_2^T) D_1^{-2}BL = 0$. Finally, let $Q(x) = H^{-1}(x) B(x) L(x)$. According to (2.4) it remains to prove that $L^{-1}(-K' + CH) = -Q^T$. $L^{-1}(-K' + CH) =$
$= L^{-1}[-(V_1'U_1^T + V_1U_1^{T\prime} + V_2'U_2^T + V_2U_2^{T\prime}) H^{T-1} + (V_1U_1^T +$
$+ V_2U_2^T) H^{T-1}H^{T\prime}H^{T-1} + CH] = L^{-1}[-C(U_1U_1^T + U_2U_2^T) H^{T-1} -$
$- (V_1V_1^T + V_1V_2^T) B^TH^{T-1} + (V_1U_1^T + V_2U_2^T) H^{T-1}K^TB^TH^{T-1} + CH] =$
$= L^{-1}[-CHH^TH^{T-1} - (V_1V_1^T + V_2V_2^T) B^TH^{T-1} + (V_1U_1^T +$
$+ V_2U_2^T)(HH^T)^{-1}(U_1V_1^T + U_2V_2^T) B^TH^{T-1} + CH] = L^T[-L^{T-1}L^{-1}(V_1V_1^T +$
$+ V_2V_2^T) + L^{T-1}L^{-1}(V_1U_1^T + V_2U_2^T)(HH^T)^{-1}(U_1V_1^T + U_2V_2^T)] B^TH^{T-1} =$
$= L^T[-(Y_1Y_1^T + Y_2Y_2^T)(V_1V_1^T + V_2V_2^T) + (Y_1Y_1^T + Y_2Y_2^T)(V_1U_1^T +$
$+ V_2U_2^T)(HH^T)^{-1}(U_1V_1^T + U_2V_2^T)] B^TH^{T-1} = L^T[-(Y_1Y_1^T + Y_2Y_2^T)(V_1V_1^T +$
$+ V_2V_2^T) + (Y_1Z_1^T + Y_2Z_2^T)(U_1U_1^T + U_2U_2^T)(HH^T)^{-1}(U_1V_1^T + U_2V_2^T)] B^TH^{T-1} =$
$= L^T[Y_1(-Y_1^TV_1 + Z_1^TU_1) V_1^T + Y_1(-Y_1^TV_2 + Z_1^TU_2) V_2^T + Y_2(-Y_2^TV_1 +$
$+ Z_2^TU_1) V_1 + Y_2(-Y_2^TV_2 + Z_2^TU_2) V_2^T] B^TH^{T-1} = L^T(Y_1V_2^T - Y_2V_1^T) B^TH^{T-1} =$
$= -L^TB^TH^{T-1} = -Q^T$. In the last computations the identities (2.13) have been used. The proof is complete.

Remark 1. Let $L(x) = H^{T-1}(x)$ in (2.1) and let the matrices $H(x), K(x)$ satisfy the identities $H'(x) = B(x) K(x)$, $H^T(x) K(x) = K^T(x) H(x)$. It can be verified that in this case the matrices $B_1(x), C_1(x)$ are symmetric if and only if the matrices $B(x)$, $C(x)$ are symmetric. Consequently, in this case the transformation form Theorem 1 transforms selfadjoint systems to selfadjoint system and Theorem 1 generalizes the main result of [3]. This theorem also generalizes the main result of [4], where transformations of differential systems of the second order $(F(x) Y')' + G(x) Y = 0$ are investigated.

3. CONJUGATE POINTS

Let $(Y(x), Z(x))$ be the $2n \times n$ solution of (1.1) satisfying $(Y(a), Z(a)) = (0, E)$ for some $a \in I$. A point $b > a$ $(b < a)$ is said to be a *right (left) conjugate point of a with respect to* (1.1) if there exists an n-dimensional constant vector c such that $Y(b) c = 0$ and the vector function $y(x) = Y(x) c$ is not identically zero between a and b. Further, we say that b is a *conjugate point of multiplicity* k $(1 \leqq k \leqq n)$ if there exist linearly independent vectors $c_1, \ldots, c_k$ for which $Y(b) c_i = 0$ and $y_i(x) = Y(x) c_i$, $i = 1, \ldots, k$, are not identically zero between a and b. It is obvious that b is a right conjugate point of a if and only if a is a left conjugate point of b.

Lemma 3. *Let $(S_1(x), C_1(x))$, $(S_2(x), C_2(x))$ be $2n \times n$ solutions of the differential systems*

$$(3.1)_1 \qquad y' = Q(x)\, z\,, \quad z' = -Q^T(x)\, x\,,$$

$$(3.1)_2 \qquad y' = Q^T(x)\, z\,, \quad z' = -Q(x)\, y\,,$$

respectively, where $Q(x) \in C^0(I)$ is an $n \times n$ matrix of real-valued functions. If for some $a \in I$, $(S_i(a), C_i(a)) = (0, E)$, $i = 1, 2$, then the following identities hold.

$$(3.2) \qquad \begin{aligned} S_2^T S_2 + C_1^T C_1 &= E\,, \quad S_1 S_1^T + C_1 C_1^T = E\,, \\ S_1^T S_1 + C_2^T C_2 &= E\,, \quad S_2 S_2^T + C_1 C_1^T = E\,. \end{aligned}$$

Proof. See Kreith [6].

Theorem 2. *Let $a \in I$ and let $a < r_1 \leqq r_2 \leqq \dots$, $a < \bar{r}_1 \leqq \bar{r}_2 \leqq \dots$ $(a > l_1 \geqq \geqq l_2 \geqq \dots$, $a > l_1 \geqq l_2 \geqq \dots)$ be the sequences of its right (left) conjugate points with respect to (1.1) and (2.5) respectively, every point repeated a number of times equal to its multiplicity. Then $r_i = \bar{r}_i$ $(l_i = l_i)$.*

Proof. Let $(U(x), V(x))$, $(Y(x), Z(x))$ be the $2n \times n$ solutions of (1.1) and (2.5) respectively, for which $(U(a), V(a)) = (Y(a), Z(a)) = (0, E)$. By Theorem 1 and Lemma 2 there exist $n \times n$ matrices $H(x)$, $K(x)$, $L(x) \in C^1(I)$, $H(x)$, $L(x)$ being nonsingular, and an $n \times n$ matrix $Q(x)$ such that

$$(3.3)_1 \qquad (U(x), V(x)) = (H(x)\, S_1(x), K(x)\, S_1(x) + L(x)\, C_1(x))\,,$$

$$(3.3)_2 \qquad (Y(x), Z(x)) = (L^{T-1}(x)\, S_2(x), (-L^{-1}(x)\, K(x)\, H^{-1}(x))^T S_2(x) + + H^{T-1}(x)\, C_2(x))\,,$$

where $(S_1(x), C_1(x))$, $(S_2(x), C_2(x))$ are $2n \times n$ solutions of $(3.1)_1$ and $(3.1)_2$ respectively. Let b be a k-multiple (left or right) conjugate point of a with respect to (1.1). By $(3.3)_1$ there exist k linearly independent unit vectors $c_1, \dots, c_k$ such that $S_1(b)\, c_i = 0$, $i = 1, \dots, k$. By (3.2) $c_i^T C_2^T(b)\, C_2(b)\, c_i = 1$, which implies the existence of unit vectors $d_1, \dots, d_k$ for which $d_i^T\, C_2(b)\, C_2^T(b)\, d_i = 1$ and by (3.2) $d_i^T\, S_2(b)\, S_2^T(b)\, d_i = 0$, hence $S_2^T(b)\, d_i = 0$. It follows $S_2(b) f_i = 0$, $i = 1, \dots, k$, where f_i are linearly i dependent unit vectors. Consequently, by $(3.3)_2$ b is a k-multiple conjugate point of a with respect to (2.5). By the same method we prove that every k-multiplied conjugate point of a with respect to (2.5) is also the k-multiplied conjugate point of a with respect to (1.1). The proof is complete.

Theorem 3. *Two points $a, b \in I$ are conjugate with respect to (1.1) if and only if they are conjugate with respect to (2.11).*

Proof. Let $a, b \in I$ be conjugate with respect to (1.1). Then by Theorem 2 these points are also conjugate with respect to (2.5). This implies that for the $2n \times n$ solutions $(U(x), V(x))$, $(Y(x), Z(x))$ of (1.1) and (2.5) respectively, satisfying $(U(a), V(a)) = (Y(a), Z(a)) = (0, E)$, we have $U(b)\, c_1 = 0$, $Y(b)\, c_2 = 0$, c_1, c_2 being nonzero

n-dimensional vectors. Set

$$\mathfrak{U}(x) = \begin{bmatrix} 0 & U(x) \\ Y(x) & 0 \end{bmatrix}, \quad \mathfrak{V}(x) = \begin{bmatrix} Z(x) & 0 \\ 0 & V(x) \end{bmatrix}.$$

$(\mathfrak{U}(x), \mathfrak{V}(x))$ is the $4n \times 2n$ solution of (2.11) satisfying $(\mathfrak{U}(a), \mathfrak{V}(a)) = (0, E)$ and $\mathfrak{U}(b)\, c = 0$, where $c = (c_2^T, c_1^T)^T$. Hence a, b are conjugate with respect to (2.11). As all arguments can be reversed, the proof is complete.

Remark 2. There exists relatively large theory concerning oscillation properties of solutions of selfadjoint differential systems of the form (1.1), see e.g. [1], [4]. Theorem 3 enables us to use some results of this theory for the investigation of non-selfadjoint systems.

4. SCALAR DIFFERENTIAL EQUATIONS OF THE EVEN ORDER

Let $p_{i,j}(x) \in C^i(I)$, $1 \leqq i, j \leqq n$, be real-valued functions, $p_{n,n}(x) \neq 0$ and consider a scalar differential equation of the $2n$-th order

$$\sum_{i=0}^{n} (-1)^i \left[\sum_{j=0}^{n} p_{i,j}(x)\, y^{(j)} \right]^{(i)} = 0 . \tag{4.1}$$

Setting $y = u_1, \ldots, y^{(n-1)} = u_n$, $v_n = \sum_{j=0}^{n} p_{n,j}(x)\, y^{(j)}$, $v_{n-k} = -v'_{n-k+1} + \sum_{j=0}^{n} p_{n-k,j}(x)\, y^{(j)}$, $k = 1, \ldots, n-1$, we have

$$u' = A(x)\, u + B(x)\, v\,, \quad v' = C(x)\, u + D(x)\, v\,, \tag{4.2}$$

where $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T$ and

$$A_{i,j} = \begin{cases} 1 \quad \text{for} \quad i = j+1\,, \quad 1 \leqq i, j \leqq n-1\,, \\ 0 \quad \text{for} \quad i \neq j+1\,, \quad 1 \leqq i, j \leqq n-1\,, \\ -p_{n,n}^{-1}(x)\, p_{n,j-1}(x)\,, \quad \text{for} \quad i = n,\ 1 \leqq j \leqq n\,, \end{cases}$$

$$B_{i,j} = \begin{cases} p_{n,n}^{-1}(x) \quad \text{for} \quad i, j = n\,, \\ 0 \quad \text{otherwise}\,, \end{cases}$$

$$C_{i,j} = p_{i-1,j-1}(x) - p_{i-1,n}(x)\, p_{n,n}^{-1}(x)\, p_{n,j-1}(x), \quad 1 \leqq i, j \leqq n,$$

$$D_{i,j} = \begin{cases} -1 \quad \text{for} \quad j = i+1\,, \quad 1 \leqq i, j \leqq n-1\,, \\ 0 \quad \text{for} \quad j \neq i+1\,, \quad 1 \leqq i, j \leqq n-1\,, \\ p_{i-1,n}(x)\, p_{n,n}^{-1}(x) \quad \text{for} \quad j = n,\ 1 \leqq i \leqq n\,. \end{cases}$$

Similarly, the adjoint equation

$$\sum_{i=0}^{n} (-1)^i \left[\sum_{j=0}^{n} p_{j,i}(x)\, y^{(j)} \right]^{(i)} = 0 \tag{4.3}$$

can be written in the form

$$w' = -D^T(x)\, w + B^T(x)\, z\,, \quad z' = C^T(x)\, w - A^T(x)\, z\,,$$

where $w = (w_1, \dots, w_n)^T$, $z = (z_1, \dots, z_n)$ and $w_1 = y, \dots, w_n = y^{(n-1)}$, $z_n = \sum_{j=0}^{n} p_{j,n}(x)\, y^{(j)}$, $z_{n-k} = -z'_{n-k+1} + \sum_{j=0}^{n} p_{j,n-k}(x)\, y^{(j)}$, $1 \leqq k \leqq n-1$.

Now let us set

$$P_{i,j}(x) = \begin{bmatrix} 0 & p_{j,i}(x) \\ p_{i,j}(x) & 0 \end{bmatrix},$$

$0 \leqq i, j \leqq n$, and consider the 2-dimensional selfadjoint differential system

$$\sum_{i=0}^{n} (-1)^i \left[\sum_{j=0}^{n} P_{i,j}(x)\, y^{(j)}\right]^{(i)} = 0 . \tag{4.4}$$

Recall that two points $a, b \in I$ are said to be conjugate with respect to (4.4) if there exists a nontrivial solution $y(x) = (y_1(x), y_2(x))^T$ of (4.4) for which $y_i^{(j)}(a) = 0 = y_i^{(j)}(b)$, $0 \leqq j \leqq n-1$, $i = 1, 2$. The conjugate points with respect to equation (4.1) are defined analogously.

Theorem 4. *Two points $a, b \in I$ are conjugate with respect to equation* (4.1) *if and only if they are conjugate with respect to system* (4.4).

Proof. Let $a, b \in I$ be conjugate with respect to (4.1). By the definition of conjugate points with respect to (4.1) and according to transformation which enables to write equation (4.1) in the form (4.2), these points are also conjugate with respect to (4.2). Set $u = R_1(x)\, u_1$, $v = R_2(x)\, v_1$, where $R_1' = A(x)\, R_1$, $R_2' = D(x)\, R_2$. By a direct computation we obtain

$$\begin{aligned} u_1' &= R_1^{-1}(x)\, B(x)\, R_2(x)\, v_1 , \\ v_1' &= R_2^{-1}(x)\, C(x)\, R_1(x)\, u_1 . \end{aligned} \tag{4.5}$$

Similarly, if we set $w = R_1^{T-1}(x)\, w_1$, $z = R_2^{T-1}(x)\, z_1$, we have

$$\begin{aligned} w_1' &= R_2^T(x)\, B^T(x)\, R_1^{T-1}(x)\, z_1 , \\ z_1' &= R_1^T(x)\, C^T(x)\, R_2^{T-1}(x)\, w_1 . \end{aligned} \tag{4.6}$$

Using Theorem 2 and the nonsingularity of the matrices $R_1(x)$, $R_2(x)$ we see that a, b are conjugate with respect to (4.5) and (4.6), and thus they are also conjugate with respect to (4.3), i.e. there exist nontrivial solutions $y_1(x), y_2(x)$ of (4.1) and (4.3) respectively, for which $y_i^{(j)}(a) = 0 = y_i^{(j)}(b)$, $i = 1, 2$, $0 \leqq j \leqq n-1$. If we set $y(x) = (y_1(x), y_2(x))^T$, we can verify that $y(x)$ is a solution of (4.4), hence a, b are also conjugate points with respect to (4.4). As all arguments can be reversed, the proof is complete.

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