Czechoslovak Mathematical Journal

Francisco Gallego Lupiañez

α-paracompact subsets and well-situated subsets

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 2, 191–197

Persistent URL: http://dml.cz/dmlcz/102212

CZECHOSLOVAK MATHEMATICAL JOURNAL

Mathematical Institute of the Czechoslovak Academy of Sciences

V. *38 (113)*, PRAHA 30. 6. 1988, No 2

α-PARACOMPACT SUBSETS AND WELL-SITUATED SUBSETS

FRANCISCO GALLEGO LUPIAÑEZ, Madrid*)

(Received November 20, 1985)

INTRODUCTION

In Section 1 we study α-paracompact subsets, defined by C. E. Aull. We obtain some covering properties of α-paracompact subsets which are similar to the properties of paracompact spaces. In particular, we characterize α-paracompact subsets in regular spaces. Moreover, we study the behaviour of α-paracompact subsets under perfect mappings.

In Section 2 we consider R. Telgársky's well-situated subsets. The properties of α-paracompact subsets of Section 1 yield properties of well-situated subsets. Well-situated subsets are related to Tamano's problem (i.e.: to give an intrinsic description of T_2 spaces X such that $X \times Y$ is paracompact for each paracompact T_2 space Y) which remains open.

In Section 3 we solve a problem of Telgársky. We establish that in the realm of T_2 spaces, the class Π^* is perfect.

1. α-PARACOMPACT SUBSETS

C. E. Aull in [1] defined the notion of an α-paracompact subset. A subset E of a topological space X is said to be *α-paracompact in X* if every covering of E by open subsets of X has a refinement by open subsets of X, locally finite in X, which covers E. We continue in this paper the study of α-paracompact subsets. We omit the proofs in this section.

1.1. Proposition. *Let X be a topological space. Then:*

1) *If X is T_2, E is an α-paracompact subset in X and F is a closed subset oj E, then F is α-paracompact in X.*

2) *If $\{E_j\}_{j\in J}$ is a set of subsets of X, locally finite in X and such that E_j is α-paracompact in X for every $j \in J$ and there exists a locally finite family of open*

*) Part of this paper is contained in the author's Doctoral Thesis written under the supervision of Professor E. Outerelo. This paper has been published in a shorted version in Quest. & Ans. Gen. Topology *5* (1987), 293—302.

subsets $\{U_j\}_{j\in J}$ of X such that $E_j \subset U_j$ for every $j \in J$, then $\bigcup_{j\in J} E_j$ is α-paracompact in X. In particular, every finite union of α-paracompact subsets is α-paracompact.

1.2. Remark. In Proposition 1.1, point 2), the hypothesis "$\{U_j\}_{j\in J}$ is a locally finite family" cannot be replaced by the hypothesis "$\{U_j\}_{j\in J}$ is a locally finite set". Indeed, in the Niemytski plane $X = \{(x, y) \in \mathbf{R}^2/y \geqq 0\}$ (in which for $y_0 > 0$ the neighbourhoods of (x_0, y_0) are the usual neighbourhoods in the plane relativized with respect to X, while for $y_0 = 0$ the neighbourhoods of $(x_0, 0)$ consist of open circles with center (x_0, y) and radius y with the point $(x_0, 0)$ for each $y > 0$), $\{E_q\}_{q\in\mathbf{Q}}$ where $E_q = \{(q, 0)\}$ is a locally finite set of α-paracompact subsets of X such that $\bigcup_{q\in\mathbf{Q}} E_q$ is not α-paracompact ([1] p. 50), and $\{U_q\}_{q\in\mathbf{Q}}$ where $U_q = X$ for each $q \in \mathbf{Q}$ is a locally finite set such that $E_q \subset U_q$ for each $q \in \mathbf{Q}$.

1.3. Theorem. *Let X be a regular space and E a subset of X. The following conditions are equivalent*:

a) *E is an α-paracompact subset in X.*

b) 1) *Every covering $\mathscr{U}$ of E by open subsets of X has a refinement $\mathscr{V}$ by open subsets of X, σ-locally finite in X, which covers E, and* 2) *Every open subset U of X such that $E \subset U$ has an open subset V such that $E \subset V \subset \overline{V} \subset U$.*

c) *Every covering $\mathscr{U}$ of E by open subsets of X has a refinement $\mathscr{A} = \{A_s\}_{s\in S}$ by arbitrary sets of X, locally finite in X, such that $E \subset (\bigcup_{s\in S} A_s)^0$.*

d) *Every covering $\mathscr{U}$ of E by open subsets of X has a refinement $\mathscr{F} = \{F_j\}_{j\in J}$ by closed subsets of X, locally finite in X, such that $E \subset (\bigcup_{j\in J} F_j)^0$.*

Remark. Theorem 1.3 implies Corollary 3 and Theorem 4 of [1].

1.4. Proposition. *Let X be a regular space and E an α-paracompact subset in X. Then*:

1) *Every covering $\mathscr{U}$ of E by open subsets of X has a refinement by open subsets of X, barycentric in X, which covers E.*

2) *Every covering $\mathscr{U}$ of E by open subsets of X has a star refinement by open subsets of X, which covers E.*

1.5. Proposition. *Let X be a regular space and E an α-paracompact subset in X. Then for every family $\{F_s\}_{s\in S}$ of subsets of E, locally finite (discrete) in X, there is a family $\{U_s\}_{s\in S}$ of open subsets of X, locally finite (discrete) in X and such that $F_s \subset U_s$ for every $s \in S$.*

We pass now to the study of the behaviour of the α-paracompact subsets under perfect mappings.

1.6. Proposition. *Let X and X' be topological spaces and $f: X \to X'$ a perfect mapping. If E' is an α-paracompact subset in X' then $f^{-1}(E')$ is an α-paracompact subset in X.*

1.7. Remark. Proposition 1.6 implies that if X and Y are topological spaces, E is an α-paracompact subset in X and Y is compact, then $E \times Y$ is an α-paracompact subset in $X \times Y$.

However, if X and Y are topological spaces, E is an α-paracompact subset in X and F is an α-paracompact subset in Y, $E \times F$ is not necessarily an α-paracompact subset in $X \times Y$. Indeed, $\boldsymbol{Q}$ is an α-paracompact subset in the Michael line $(\boldsymbol{R}, T)$ $\boldsymbol{R} \setminus \boldsymbol{Q}$ is an α-paracompact subset in $\boldsymbol{R} \setminus \boldsymbol{Q}$, but $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is not an α-paracompact subset in $(\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$. (Since the sets $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ and $C = \{(x, x)/x \in \boldsymbol{R} \setminus \boldsymbol{Q}\}$ are disjoint closed sets which are not strongly separated, it follows from Theorem 5 in [1] that $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is not an α-paracompact subset.)

1.8. Proposition. *Let X and X' be topological spaces, where X is regular, let f be a perfect mapping from X onto X' and E' a subset of X'. If $f^{-1}(E')$ is an α-paracompact subset in X then E' is an α-paracompact subset in X'.*

1.9. Remark. In Proposition 1.8 the hypothesis "f is a mapping onto" cannot be omitted. Indeed, let $(\boldsymbol{R}, T)$ be the Michael line. Then the mapping $i: \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) \to\to (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is a perfect mapping but is not onto, $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is an α-paracompact subset in $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ with the usual topology, and $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is not an α-paracompact subset in $(\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ (see 1.7).

1.10. Proposition. *Let X and X' be topological spaces and f a perfect and open mapping from X onto X'; if E is an α-paracompact subset in X then $f(E)$ is an α-paracompact subset in X'.*

1.11. Remark. Let X and X' be topological spaces and f a perfect mapping from X onto X'; if E is an α-paracompact subset in X, $f(E)$ is not necessarily α-paracompact in X'. Indeed, let $(\boldsymbol{R}, T)$ be the Michael line, $j_1: \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) \to\to \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) + (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ and $j_2: (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q}) \to\to \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) + (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$. Then the mapping onto $f: \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) + (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q}) \to (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ such that

$$f(j_1(x, y)) = (x, y) \quad \text{if} \quad (x, y) \in \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$$
$$f(j_2(x, y)) = (x, y) \quad \text{if} \quad (x, y) \in \boldsymbol{R} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$$

is a perfect mapping, $f(j_1(\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}))) = \boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$, $j_1(\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}))$ is an α-paracompact subset in $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q}) + (\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ and $\boldsymbol{Q} \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ is not an α-paracompact subset in $(\boldsymbol{R}, T) \times (\boldsymbol{R} \setminus \boldsymbol{Q})$ (1.7).

2. WELL-SITUATED SUBSETS

The concept of a well-situated subset was introduced by R. Telgársky in [4]. Using the notion of an α-paracompact subset, H. W. Martin phrased the definition of a well-situated subset of a space X as follows: a subset E of a space X is said to be *well-situated in* X if for every paracompact T_2 space Y, $E \times Y$ is an α-paracompact subset in $X \times Y$([3]).

If E is a well-situated subset of a space X then E is an α-paracompact subset in X, but $\boldsymbol{Q}$ is α-paracompact in $(\boldsymbol{R}, T)$, the Michael line, and $\boldsymbol{Q}$ is not a well-situated subset in $(\boldsymbol{R}, T)$ (cf. 1.7).

From Section 1 we easily obtain the following theorems.

2.1. Proposition. *Let X be a topological space. Then*:

1) *If X is T_2, E is a well-situated subset in X and F is a closed subset of E, then F is a well-situated subset in X.*

2) *If $\{E_j\}_{j \in J}$ is a set of subsets of X, locally finite in X and such that E_j is a well-situated subset in X for every $j \in J$, and there exists a locally finite family of open subsets $\{U_j\}_{j \in J}$ of X such that $F_j \subset U_j$ for every $j \in J$, then $\bigcup_{j \in J} E_j$ is a well-situated subset in X. In particular, every finite union of well-situated subsets is well-situated.*

2.2. Theorem. *Let X be a regular space and E a subset of X. The following conditions are equivalent*:

a) *E is a well-situated subset in X.*

b) *For every paracompact T_2 space Y*: 1) *Every covering $\mathscr{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathscr{V}$ by open subsets of $X \times Y$, σ-locally finite in $X \times Y$, which covers $E \times Y$, and* 2) *Every open subset U of $X \times Y$ such that $E \times Y \subset U$ has an open subset V such that $E \times Y \subset V \subset \overline{V} \subset U$.*

c) *For every paracompact T_2 space Y, every covering $\mathscr{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathscr{A} = \{A_s\}_{s \in S}$ by arbitrary sets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset (\bigcup_{s \in S} A_s)^0$.*

d) *For every paracompact T_2 space Y, every covering of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathscr{F} = \{F_j\}_{j \in J}$ by closed subsets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset (\bigcup_{j \in J} F_j)^0$.*

2.3. Proposition. *Let X be a regular space and E a well-situated subset in X. Then*:

1) *For every paracompact T_2 space Y, every covering $\mathscr{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement by open subsets of $X \times Y$, barycentric in $X \times Y$, which covers $E \times Y$.*

2) *For every paracompact T_2 space Y, every covering of $E \times Y$ by open subsets of $X \times Y$ has a star refinement by open subsets of $X \times Y$ which covers $E \times Y$.*

2.4. Proposition. *Let X be a regular space and E a well-situated subset in X. Then for every paracompact T_2 space Y, for every family $\{F_s\}_{s \in S}$ of subsets of $E \times Y$, locally finite (discrete) in $X \times Y$, there is a family $\{U_s\}_{s \in S}$ of open subsets of $X \times Y$, locally finite (discrete) in $X \times Y$ and such that $F_s \subset U_s$ for any $s \in S$.*

2.5. Proposition. *Let X and X' be topological spaces and $f: X \to X'$ a perfect* *mapping. If E' is a well-situated subset in X' then $f^{-1}(E')$ is a well-situated subset in X.*

Proof. For every paracompact T_2 space Y, $fx\,1Y\colon X \times Y \to X' \times Y$ is a perfect mapping. Now 1.6 implies that $f^{-1}(E')$ is well-situated.

2.6. Proposition. *Let X and X' be T_2 topological spaces where X' is paracompact, let E be a well-situated subset in X and F a closed subset of X'. Then $E \times F$ is an α-paracompact subset in $X \times X'$.*

Proof follows from 1.1.

Remark. In [4] R. Telgársky denoted by Π the class of all T_2 spaces X such that $X \times Y$ is paracompact for each paracompact T_2 space Y. Let X and Y be T_2 topological spaces, E a well-situated subset in X and $Y \in \Pi$. Then $E \times Y$ is well-situated in $X \times Y$. (Indeed, for each paracompact T_2 space Z, $Y \times Z$ is paracompact and T_2, hence $(E \times Y) \times Z$ is α-paracompact in $(X \times Y) \times Z$.)

2.7. Proposition. *Let X and X' be topological spaces, where X is regular, let f be a perfect mapping from X onto X' and E' a subset of X'. If $f^{-1}(E')$ is a well-situated subset in X then E' is a well-situated subset in X'.*

Proof. For every paracompact T_2 space Y, $f \times 1Y$ is a perfect mapping from $X \times Y$ onto $X' \times Y$. It follows from 1.8 that E' is well-situated.

2.8. Remark. In Proposition 2.7 the hypothesis "f is a mapping onto" cannot be omitted. In deed, let $(\boldsymbol{R}, T)$ be the Michael line. The mapping $i\colon \boldsymbol{Q} \to\to (\boldsymbol{R}, T)$ is perfect but is not onto, $\boldsymbol{Q} \in \Pi$ ([4] p. 66) but $\boldsymbol{Q}$ is not well-situated in $(\boldsymbol{R}, T)$ (cf. 1.7).

2.9. Proposition. *Let X and X' be topological spaces and f a perfect and open mapping from X onto X': If E is a well-situated subset in X then $f(E)$ is a well-situated subset in X'.*

Proof. For every paracompact T_2 space Y, $f \times 1Y$ is a perfect and open mapping from $X \times Y$ onto $X' \times Y$. Now 1.10 implies that $f(E)$ is well-situated.

2.10. Remark. Let X and X' be topological spaces and f a perfect mapping from X onto X'. If E is a well-situated subset in X, $f(E)$ is not necessarily a well-situated subset in X'. (See 1.11).

3. THE CLASS Π^*

In [4] R. Telgársky denoted by Π^* the class of all paracompact T_2 spaces which are well-situated in every paracompact T_2 space in which they are embedded as closed subsets.

R. Telgársky showed that Π^* is a very wide class contained in the class Π, and raised the following questions:

1. Is the class Π^* perfect? ([4], Problem 2.1.)

2. Does the class of all paracompact C-scattered spaces coincide with the class Π^*? ([4], Problem 2.3.)

In the present paper, we shall give an affirmative answer to question 1.

3.1. Proposition. *Let E and E' be topological spaces where E is T_2, and let f be a perfect mapping from E onto E'. If $E' \in \Pi^*$ then $E \in \Pi^*$.*

Proof. Let X be a paracompact T_2 space such that E is embedded in X as a closed subset. Let $j_1: X \to X + E'$, $j_2: E' \to X + E'$ be the embeddings of the subspaces X and E' in the sum $X + E'$, let $X \cup_f E'$ be the adjunction space determined by X, E' and f and let $q: X + E' \to X \cup_f E'$ be the natural quotient mapping. As the mapping f is closed, q is a continuous and closed mapping; since $X + E'$ is paracompact and T_2, $X \cup_f E'$ is paracompact (this follows from the Michael Theorem) and T_2. Further, $q \circ j_2: E' \to X \cup_f E'$ is a homeomorphic embedding and $(q \circ j_2)(E')$ is closed in $X \cup_f E'$. Since $E' \in \Pi^*$ and $X \cup_f E'$ is paracompact and T_2, $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$.

Let $\hat{f} = q \circ j_1: X \to X \cup_f E'$. Clearly $\hat{f}$ is a perfect mapping.

$$\begin{tikzcd} X \arrow[r, "\hat{f}"] & X \cup_f E' \\ E \arrow[u, hook] \arrow[r, "f"] & E' \arrow[u, hook, "q \circ j_2"'] \end{tikzcd}$$

Since $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$, Proposition 2.5 implies that $\hat{f}^{-1}((q \circ j_2)(E'))$ is a well-situated subset in X. However,

$$\hat{f}^{-1}((q \circ j_2)(E')) = j_1^{-1}(q^{-1}(q(j_2(E')))) = E.$$

Thus E is a well-situated subset in X. Hence $E \in \Pi^*$.

3.2. Proposition. *Let E and E' be topological spaces and f a perfect mapping from E onto E'. If $E \in \Pi^*$ then $E' \in \Pi^*$.*

Proof. Since f is continuous, $G_f = \{(x, f(x)) \in E \times E' / x \in E\}$ is homeomorphic to E, hence $G_f \in \Pi^*$.

Since E is T_{3a} and f is a perfect mapping from E onto E', G_f is a closed subset of $\beta E \times E'$ (see [5], proof of Theorem 3.10).

Let X' be a paracompact T_2 space such that E' is embedded in X' as a closed subset. Then G_f is a closed subset of $\beta E \times X'$ which is paracompact and T_2. Thus G_f is a well-situated subset in $\beta E \times X'$.

The projection $p_2: \beta E \times X' \to X'$ is perfect and open, and $p_2(G_f) = E'$. It follows from 2.9 that E' is a well-situated subset in X'.

3.3. Theorem. *In the realm of T_2 spaces, the class Π^* is perfect (i.e., if E and E' are topological spaces where E is T_2, and f is a perfect mapping from E onto E' then $E \in \Pi^*$ if and only if $E' \in \Pi^*$).*

Proof follows from 3.1 and 3.2.

Added in proofs. The author learned, after writing this paper, that J. D. Wine [in: Locally paracompact spaces, Glasnik Mat., *10* (*30*) (1975), 351–357] has obtained Proposition 1.1.2), and that I. Kovacević [in: Subsets and paracompactness, Zbornik Radova PMF Univ. u Novom Sadu, ser Mat. *14* (*1984*), 79–87] has obtained also the implication a) $\Rightarrow$ b) of the Theorem 1.3. The author thanks to Professors J. D. Wine and I. Kovacević for making available their papers to him.

References

[1] *C. E. Aull:* Paracompact subsets. Proc. Second Prague. Topological Symposium (1966) 45–51.
[2] *R. Engelking:* General Topology. Polish Scientific Publishers, Warszawa, 1977.
[3] *H. W. Martin:* Linearly ordered covers, normality and paracompactness. Top. and its Appl. *12* (1981) 305–313.
[4] *R. Telgársky:* *C*-scattered and paracompact spaces. Fund. Math. *73* (1971) 59–74.
[5] *R. Telgársky:* Corcerning product of paracompact spaces. Fund. Math. *74* (1972) 153–159.

Author's address: Departamento de Geometria y Topologia, Facultad de Matematicas, Universidad Complutense de Madrid, Spain.