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ALGEBRAIC SPECTRAL SUBSPACES AND AUTOMATIC CONTINUITY

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1. INTRODUCTION

In introducing the super-decomposable operators, [14] also gave an algebraic description of the spectral maximal subspaces. This algebraic characterization, which has later been extended to the larger class of well-decomposable operators [1], has found use in the applications to automatic continuity [1, 14].

However, the defining algebraic description of these subspaces makes sense for an arbitrary linear operator and this paper attempts to gain further insight into what may be said about an operator in terms of these algebraic spectral subspaces.

We begin by establishing, in Section 2, some of the basic properties of the class $\{E_T(F)\}$ (definitions are given there), among them $\bigcap$-stability. Then Section 3 studies what more might be said if the $E_T(F)$-spaces are assumed closed whenever $F \subset \mathbb{C}$ is closed. It turns out that for many closed sets F, the closedness of $E_T(F)$ ensures that this space be spectral maximal (Proposition 3.3). Thus, in these cases the $E_T(F)$ space coincides with the space $X_T(F)$ of all elements of X with local spectrum in F (Proposition 3.4). The generality of these notions is tempered by the observation that if all $E_T(F)$, F closed, are closed, then the operator T must have the single valued extension property (Corollary 3.6). This section concludes with some remarks on the closedness of $E_T(F)$ when T is well-decomposable.

In Section 4 is proved a general continuity result for linear maps that intertwine, in a certain general sense, two given decomposable (+ a bit more) operators. This result (Theorem 4.1) follows a rich tradition of work on intertwining maps and their continuity, going back to [10], but the applicability of this result is considerable. Notably we obtain, in Section 5, a necessary and sufficient condition for the existence of discontinuous module derivations.

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2. ALGEBRAICALLY SPEAKING

We begin by establishing some of the basic properties of the T-invariant subspaces that this paper deals with. Here X is a vector space, $T: X \rightsquigarrow X$ a linear mapping and $A \subseteq \boldsymbol{C}$ any proper subset of the complex plane.

Definition. Consider the class of linear subspaces Y of X with the property that $(T - \lambda) Y = Y$ for every $\lambda \in \boldsymbol{C} \setminus A$. Let

$$E_T(A) := \operatorname{span} Y.$$

Evidently $E_T(A)$ is the largest linear subspace of X on which all the restrictions of $T - \lambda$, $\lambda \in \boldsymbol{C} \setminus A$, are surjective.

Thus, $E_T(\emptyset)$ is the largest T-divisible subspace of X.

Remarks. a) These spaces were introduced in [11] via a transfinite sequence of spaces: with X, T and A as above, let $X(A, 0) := X$; for a non-limit ordinal $\alpha + 1$

$$X(A, \alpha + 1) := \bigcap_{\lambda \in \boldsymbol{C} \setminus A} (T - \lambda) X(A, \alpha)$$

and for a limit ordinal α

$$X(A, \alpha) := \bigcap_{\beta < \alpha} X(A, \beta).$$

Standard cardinality arguments show that this decreasing sequence is eventually constant; on its constant eventual value $T - \lambda$ must be surjective, for each $\lambda \in \boldsymbol{C} \setminus A$. Since $E_T(A) \subset X(A, \alpha)$ for every α, the eventually constant value of $\{X(A, \alpha)\}$ is $E_T(A)$.

b) If X is a Hilbert space and T is a bounded normal operator then it is a consequence of [15] that for every closed set $F \subset \boldsymbol{C}$ the eventually constant value of $\{X(F, \alpha)\}$ is reached at $X(F, 1)$; in other words

$$E_T(F) = \bigcap_{\lambda \in \boldsymbol{C} \setminus F} (T - \lambda) X.$$

First some elementary and basic facts.

Lemma 2.1. *$E_T(A)$ is hyperinvariant, so that $E_T(A)$ is S-invariant, for any linear map S commuting with T.*

Proof. For $\lambda \notin A$, $(T - \lambda) S E_T(A) = S(T - \lambda) E_T(A) = S E_T(A)$, so that $S E_T(A) \subset E_T(A)$, by maximality.

Corollary. *For every proper $A \subset \boldsymbol{C}$,*

$$E_T(A) = E_T(A \cap \sigma(T)),$$

where $\sigma(T)$ is the complement of the set of points $\lambda \in \boldsymbol{C}$ for which $T - \lambda$ is a bijection onto X.

Proof. If $\lambda \in A \setminus \sigma(T)$ then $(T - \lambda)^{-1}$ is a well defined linear map which commutes with T. Hence, $(T - \lambda)^{-1} E_T(A) \subseteq E_T(A)$ and so $E_T(A) = (T - \lambda) E_T(A)$ for all $\lambda \in A \setminus \sigma(T)$, as well as for all $\lambda \notin A$. Hence $E_T(A) \subset E_T(A \cap \sigma(T))$. The other inclusion is immediate.

Corollary. *If X is a Banach space and T is bounded, then for*

$$Y := \overline{E_T(A)}$$

we have $\sigma(T \mid Y) \subset \sigma(T)$.

Proof. If $\lambda \notin \sigma(T)$ then $(T - \lambda) E_T(A) = E_T(A)$ and consequently, $(T - \lambda) Y = Y$; since $T - \lambda$ is 1-1, this shows $\lambda \in \varrho(T \mid Y)$. □

The next result is basic to much of the following discussion. Its proof is related to the proof of [7, Proposition 1.3.5]. Note that the statement is purely algebraic; no topological assumptions.

Proposition 2.2. *With X, T and A as above, let $\lambda_0 \in A$ and let*

$$E_{\lambda_0} := \{x \in X \mid (T - \lambda_0) x \in E_T(A)\} .$$

Then

$$E_{\lambda_0} = E_T(A) .$$

Proof. Since $E_T(A) = (T - \lambda) E_T(A)$ for every $\lambda \notin A$, we see that for $x \in E_T(A)$, $(T - \lambda_0) x = (T - \lambda) x + (\lambda - \lambda_0) x \in E_T(A)$ so that $E_T(A) \subset E_{\lambda_0}$. Let $y_0 \in E_{\lambda_0}$, let

$$Y := E_T(A) + \boldsymbol{C} y_0$$

and let $z = y + \alpha y_0 \in Y$. With $\lambda \notin A$, consider

$$y - \frac{\alpha}{\lambda_0 - \lambda} (T - \lambda_0) y_0 \in E_T(A)$$

and choose $y_1 \in E_T(A)$ so that

$$(T - \lambda) y_1 = y - \frac{\alpha}{\lambda_0 - \lambda} (T - \lambda_0) y_0 .$$

Now let

$$z_0 := y_1 + \frac{\alpha}{\lambda_0 - \lambda} y_0 \in Y$$

and compute

$$(T - \lambda) z_0 = (T - \lambda) y_1 + \frac{\alpha}{\lambda_0 - \lambda} (T - \lambda) y_0$$

$$= y - \frac{\alpha}{\lambda_0 - \lambda} (T - \lambda_0) y_0 + \frac{\alpha}{\lambda_0 - \lambda} (T - \lambda_0) y_0 + \alpha y_0 = y + \alpha y_0 = z .$$

Thus $Y \subset (T - \lambda) Y \subset Y$, and since $\lambda \in \boldsymbol{C} \setminus A$ is arbitrary, $E_{\lambda_0} \subset E_T(A)$ follows.

Remark. This shows that $E_T(A)$ is T-absorbent [17, Definition IV, 4.1].

Corollary 2.3. $\ker (T - \lambda_0) \subset E_T(A)$, *for every* $\lambda_0 \in A$.

Also from Proposition 2.2 we may prove $E_T(A_1 \cap A_2) = E_T(A_1) \cap E_T(A_2)$ for any proper subsets A_1, A_2 of $\boldsymbol{C}$:

Proposition 2.4. $E_T(A_1 \cap \ldots \cap A_n) = E_T(A_1) \cap \ldots \cap E_T(A_n)$, *for any finite collection* $\{A_1, \ldots, A_n\}$ *of proper subsets of* $\mathbf{C}$.

Proof. By finite induction only the case of two sets A_1 and A_2 needs consideration. Also, by the Corollary of Lemma 2.1 we may assume that A_1 and A_2 are subsets of $\sigma(T)$.

Suppose first that we have established our claim whenever $A_1 \cup A_2 = \sigma(T)$. For arbitrary A_1, A_2 in $\sigma(T)$, let $A_3 := A_2 \cup (\sigma(T) \setminus A_1)$.

$$E_T(A_1) \cap E_T(A_2) \subset E_T(A_1) \cap E_T(A_3) =$$
$$= E_T(A_1 \cap A_3) = E_T(A_1 \cap A_2) \subset E_T(A_1) \cap E_T(A_2)\,.$$

Thus there is no loss in assuming that $A_1 \cup A_2 = \sigma(T)$. Let $Y := E_T(A_1) \cap E_T(A_2)$ and let $\lambda \notin A_1 \cap A_2$. As we saw (in the proof of the first corollary of Lemma 2.1) $E_T(A) = (T - \lambda)\, E_T(A)$, for any A, if $\lambda \notin \sigma(T)$ so we may assume $\lambda \in \sigma(T)$, hence $\lambda \in A_1 \setminus A_2$, say. If $x_0 \in Y$ then there is $y \in E_T(A_2)$ so that $x_0 = (T - \lambda)\, y$. But since $\lambda \in A_1$ and $x_0 \in E_T(A_1)$, Proposition 2.2 implies that $y \in E_T(A_1)$. Hence $(T - \lambda)\, Y = Y$ for every $\lambda \notin A_1 \cap A_2$ and by maximality $Y \subset E_T(A_1 \cap A_2)$. As the other inclusion is immediate, the proof is complete.

Before proving $\bigcap$-stability for arbitrary families of subsets of $\mathbf{C}$ we also need this next observation.

Lemma 2.5. *Let* $F \subset G$ *be proper subsets of* $\mathbf{C}$, *let* $Q: X \curvearrowright X/E_T(F)$ *be the quotient map and let* $\tilde{T}: X/E_T(F) \curvearrowright X/E_T(F)$ *be the map induced by* T (*thus* $\tilde{T}Q = QT$). *Then*

$$E_{\tilde{T}}(G) = Q\, E_T(G)\,.$$

Proof. If $\lambda \notin G$ then $(\tilde{T} - \lambda)\, Q\, E_T(G) = Q(T - \lambda)\, E_T(G) = Q\, E_T(G)$, so $Q\, E_T(G) \subset E_{\tilde{T}}(G)$. For the reverse inclusion, suppose $Qx \in E_{\tilde{T}}(G)$ and choose $Qy \in E_{\tilde{T}}(G)$ so that $(\tilde{T} - \lambda)\, Qy = Qx$. Thus $Q(T - \lambda)\, y = Qx$ so that $x - (T - \lambda)\, y \in E_T(F)$ and $x, y \in Q^{-1}(E_{\tilde{T}}(G))$. Since $\lambda \notin F$ there is $z \in E_T(F) \subset E_T(G)$ so that $x = (T - \lambda)\, y + (T - \lambda)\, z = (T - \lambda)\,(y + z)$. It follows that $(T - \lambda) \,.\, Q^{-1}(E_{\tilde{T}}(G)) = Q^{-1}(E_{\tilde{T}}(G))$, hence $Q^{-1}\, E_{\tilde{T}}(G) \subset E_T(G)$.

Theorem 2.6. *Suppose* $\{F_\delta\}_{\delta \in D}$ *is a family of proper subsets of* $\mathbf{C}$. *Then*

$$E_T\big(\bigcap_{\delta \in D} F_\delta\big) = \bigcap_{\delta \in D} E_T(F_\delta)\,.$$

Proof. Let $Y := \bigcap_{\delta \in D} E_T(F_\delta)$. Evidently $E_T(\bigcap F_\delta) \subset Y$. Since stability under finite intersections has been established there is no harm in replacing the index set D by the set of finite subsets of D, ordered by inclusion. Thus we may assume, with no loss of generality, that D is a directed set and that $\delta_1 \geqq \delta_2$ implies $F_{\delta_1} \subseteq F_{\delta_2}$.

Let $\lambda \notin \bigcap F_\delta$ and choose δ_0 so that $\lambda \notin F_{\delta_0}$. Let

$$Z := \bigcap_{\delta \geqq \delta_0} E_T(F_\delta)\,;$$

then $Y \subseteq Z$ is trivial. On the other hand if $z \in Z$ and if $E_T(F_\delta)$ is given, then there is $\delta' \geqq \delta_0$ for which $\delta' \geqq \delta$ also holds. Consequently $z \in E_T(F_{\delta'}) \subset E_T(F_\delta)$, so that $z \in Y$. To show that $(T - \lambda)\, Y = Y$ for every $\lambda \notin \bigcap F_\delta$ it is enough, therefore, to show that $(T - \lambda)\, Z = Z$ for the λ chosen above.

Suppose first that $E_T(\emptyset) = \{0\}$. Since $\lambda \notin \bigcap F_\delta$ we may, to our given $z \in Z$, choose $z_\delta \in E_T(F_\delta)$ such that $(T - \lambda)\, z_\delta = z$, for each $\delta \geqq \delta_0$. Consider $\delta', \delta'' \geqq \delta_0$ and select $\delta''' \geqq \delta', \delta''' \geqq \delta''$. If $z_{\delta'} \neq z_{\delta''}$ then $z_{\delta'''} \neq z_{\delta'}$, say. Hence $z_{\delta'''} - z_{\delta'} \in$ $\in \ker(T - \lambda) \subseteq E_T(\{\lambda\})$. Hence $z_{\delta'''} - z_{\delta'} \in E_T(F_{\delta'}) \cap E_T(\{\lambda\}) = E_T(\emptyset) = \{0\}$. It follows that if $(T - \lambda)\, z_\delta = z$ then $z_\delta \in E_T(F_\delta)$ is independent of $\delta \geqq \delta_0$. Thus $z_\delta \in Z$ and surjectivity of $T - \lambda$ on Y has been established for every $\lambda \notin \bigcap F_\delta$.

This completes the proof of Theorem 2.6 if $E_T(\emptyset) = \{0\}$. To remove this additional assumption invoke Lemma 2.5. In $X/E_T(\emptyset)$ we have that $E_{\hat{T}}(\emptyset) = Q\, E_T(\emptyset) = \{0\}$. Hence by the work already done

$$E_{\hat{T}}(\bigcap F_\delta) = \bigcap E_{\hat{T}}(F_\delta)$$

and so

$$E_T(\bigcap F_\delta) = Q^{-1}(E_{\hat{T}}(\bigcap F_\delta)) = \bigcap Q^{-1} E_{\hat{T}}(F_\delta) = \bigcap E_T(F_\delta)\,. \qquad \square$$

3. CLOSED $E_T(F)$

In this section we shall consider the implications of assuming $E_T(F)$ to be closed, when F is a closed subset of $\boldsymbol{C}$. Throughout this section we assume X to be a Banach space and T to be a bounded linear operator on X.

First let us remark that if $E_T(F)$ is given then there is a smallest closed subset F_0, say, of $\boldsymbol{C}$ which we may think of as the support of the space. Specifically

$$F_0 := \operatorname{supp}(E_T(F)) := \bigcap \{F' \mid F' \text{ closed and } E_T(F') = E_T(F)\}\ ;$$

For the support we have the following observation:

Lemma 3.1. *Let F_0 be the support of $E_T(F)$ and suppose $E_T(F)$ is closed. Then $\delta\, \sigma(T \mid E_T(F)) \subset F_0 \subset \sigma(T \mid E_T(F))$ (where δ denotes the topological boundary).*

Proof. If $\lambda \in \delta\sigma(T \mid E_T(F))$ then $(T - \lambda)\, E_T(F) \neq E_T(F)$ (cf. e.g. [11, Lemma 2.2]). Consequently λ cannot belong to $\boldsymbol{C} \setminus F_0$. For the second inclusion simply observe that

$$(T - \lambda)\, E_T(F) = E_T(F)$$

for every $\lambda \in \boldsymbol{C} \setminus \sigma(T \mid E_T(F))$, hence $F_0 \subset \sigma(T \mid E_T(F))$, by minimality of F_0.

For a compact subset F of $\boldsymbol{C}$ let $\hat{F}$ denote the polynomially convex hull of F; it is a standard fact that $\hat{F}$ equals the union of F and all bounded components of $\boldsymbol{C} \setminus F$. Thus $\hat{F}$ is F together with all the holes in F. We then have

Corollary 3.2. $\sigma(T \mid E_T(F)) \subset \hat{F}_0$.

Proof. $\delta\sigma(T \mid E_T(F)) \subset F_0 \subset \sigma(T \mid E_T(F)) \subset \hat{F}_0$.

We now recall the notion of spectral maximality. [7, p. 18].

Definition. Let Lat (T) denote the closed T-invariant subspaces of X. A subspace $Z \in \operatorname{Lat}(T)$ is then said to be *spectral maximal* if $Y \in \operatorname{Lat}(T)$ and $\sigma(T \mid Y) \subset \sigma(T \mid Z)$ imply that $Y \subset Z$.

In light of the maximality condition in the very definition of $E_T(F)$ the next result is not surprising.

Proposition 3.3. *If F is compact and $F = \hat{F}$ and if $E_T(F)$ is closed then $E_T(F)$ is spectral maximal. Moreover, if $E_T(\emptyset)$ is zero then $E_T(F)$ is spectral maximal whenever $E_T(F)$ is closed and F closed. Conversely, if $E_T(F)$ is closed for all closed F and if $Z \in \operatorname{Lat}(T)$ is spectral maximal then $Z = E_T(\sigma(T \mid Z))$.*

Proof. From Corollary 3.2 it follows that $\sigma(T \mid E_T(F)) \subset \hat{F}_0 \subset \hat{F} = F$, hence if $Y \in \operatorname{Lat}(T)$ and $\sigma(T \mid Y) \subset \sigma(T \mid E_T(F))$, then $\sigma(T \mid Y) \subset F$. But then $Y \subset E_T(F)$, since $(T - \lambda)\, Y = Y$ for all $\lambda \notin F$.

To show spectral maximality of $E_T(F)$ for arbitrary closed F, when $E_T(\emptyset) = \{0\}$, it is enough to show that $\sigma(T \mid E_T(F)) \subset F$. If $\lambda \in \sigma(T \mid E_T(F)) \setminus F$ then $(T - \lambda)\, E_T(F) = E_T(F)$, so $\ker((T - \lambda)|_{E_T(F)}) \neq \{0\}$ (otherwise $\lambda \in \varrho(T \mid E_T(F))$, by the open mapping theorem). Since $\ker(T - \lambda) \subset E_T(\{\lambda\})$, by Corollary 2.3, we conclude that

$$\{0\} \neq E_T(F) \cap E_T(\{\lambda\}) = E_T(\emptyset)\,.$$

This contradicts our assumption about $E_T(\emptyset)$.

Finally, suppose $Z \in \operatorname{Lat}(T)$ is spectral maximal. Since $(T - \lambda) Z = Z$ for all $\lambda \notin \sigma(T \mid Z)$ we get that

$$Z \subset E_T(\sigma(T \mid Z))\,.$$

Since $\sigma(T \mid E_T(\sigma(T \mid Z)) \subset \sigma(T \mid Z)$, spectral maximality of Z gives the other inclusion.

The notion of local spectrum allows another description of the closed $E_T(F)$-spaces.

Recall [17, p. 185] that if $x \in X$ then the local resolvent set $\varrho_T(x)$ is defined as the union of all open subsets of $\boldsymbol{C}$ on which the equation

$$(T - \lambda)\, x(\lambda) = x$$

has an analytic solution $x(\lambda)$. The local spectrum $\sigma_T(x)$ of x is then

$$\sigma_T(x) := \boldsymbol{C} \setminus \varrho_T(x)\,.$$

It is easy to see that if $F \subset \boldsymbol{C}$ and if

$$X_T(F) := \{x \in X \mid \sigma_T(x) \subset F\}$$

then $X_T(F)$ is a linear subspace of X. Moreover, if

$$(T - \lambda)\, x(\lambda) = x$$

for all $\lambda \in \varrho_T(x)$ then $\sigma_T(x(\lambda)) = \sigma_T(x)$ for each $\lambda \in \varrho_T(x)$. Hence if $x \in X_T(F)$, then $x(\lambda) \in X_T(F)$ for every $\lambda \notin F$. In other words, since $X_T(F)$ is easily seen to be T-invariant,

$$(T - \lambda)\, X_T(F) = X_T(F)\,.$$

It follows that

$$X_T(F) \subset E_T(F)$$

for any $F \subset \boldsymbol{C}$. There is a converse.

Proposition 3.4. *If F is compact and $F = \hat{F}$ and if $E_T(F)$ is closed, then $E_T(F) = X_T(F)$. If $E_T(\emptyset) = \{0\}$ then closedness of F and of $E_T(F)$ are sufficient that $E_T(F) = X_T(F)$.*

Proof. As we saw in the proof of Proposition 3.3 under either set of assumptions we have that $\sigma(T \mid E_T(F)) \subset F$; hence, if $x \in E_T(F)$, $(T - \lambda\,|_{E_T(F)})^{-1}\, x$ is an analytic solution on $\boldsymbol{C} \setminus F$ of the equation $(T - \lambda)\, x(\lambda) = x$. Consequently, $\sigma_T(x) \subset F$ for every $x \in E_T(F)$. This gives $E_T(F) \subset X_T(F)$ and the reverse inclusion always holds.

Proposition 3.5. *Suppose $G \subset \boldsymbol{C}$ is an open and connected set on which the equation $(T - \lambda) f(\lambda) \equiv 0$ has an analytic solution $f(\lambda) \not\equiv 0$. If $F \subset \boldsymbol{C}$ is closed and $E_T(F)$ is closed and if $F \cap G \neq \emptyset$ then $G \subset \sigma(T \mid E_T(F))$.*

Proof. Let $\lambda_0 \in F \cap G$. By T-absorbency (Proposition 2.2) $f(\lambda_0) \in E_T(F)$. Moreover, since $Tf(\lambda) = \lambda f(\lambda)$ for all $\lambda \in G$, differentiation yields $Tf^{(n)}(\lambda) = \lambda f^{(n)}(\lambda) + n f^{(-1)}(\lambda)$ for all $\lambda \in G$, $n = 1, 2, \dots$, so that $(T - \lambda) f^{(n)}(\lambda) = n f^{(n-1)}(\lambda)$ for all $\lambda \in G$ and $n = 1, 2, \dots$. In particular $(T - \lambda) f^{(n)}(\lambda_0) = n f^{(n-1)}(\lambda_0)$, so, again by T-absorbency, $f^{(n)}(\lambda_0) \in E_T(F)$, $n = 0, 1, \dots$. If $\{\lambda \mid |\lambda - \lambda_0| < \delta\} \subset G$, then $f(\lambda) = \sum_{n=0}^{\infty} f^{(n)}(\lambda_0)\, (\lambda - \lambda_0)^n/n! \in E_T(F)$ (since $E_T(F)$ is closed) for all λ with $|\lambda - \lambda_0| < \delta$. By analytic continuation it follows that $f(\lambda) \in E_T(F)$ for all $\lambda \in G$ and since $(T - \lambda) f(\lambda) = 0$ we conclude that if $f(\lambda) \neq 0$ then $\lambda \in \sigma(T \mid E_T(F))$. Consequently, since $f \not\equiv 0$ it follows that $G \subset \sigma(T \mid E_T(F))$.

The operator T is said to have the *single valued extension property* (SVEP) if the only analytic solution f to the equation $(T - \lambda) f(\lambda) \equiv 0$, where λ ranges over some open set $G \subset \boldsymbol{C}$, is the zero solution. The previous Proposition then yields this next result.

Corollary 3.6. *If $E_T(F)$ is closed for every closed $F \subset \boldsymbol{C}$, then T has the single valued extension property.*

Proof. Actually, it will be enough to make the formally weaker assumption that $E_T(D)$ is closed whenever D is a closed disc in $\boldsymbol{C}$: suppose $(T - \lambda) f(\lambda) = 0$ for every $\lambda \in G$, where G is open and connected. If $f(\lambda_0) \neq 0$, choose $\delta > 0$ so that $f(\lambda) \neq 0$ for all λ with $|\lambda - \lambda_0| < \delta$. Let $D = \{\lambda \in \boldsymbol{C} \mid |\lambda - \lambda_0| \leqq \delta/2\}$. Then (Proposition 3.3):

$$\sigma(T \mid E_T(D)) \subset D\,.$$

But $D \neq G$ and $G \subset \sigma(T \mid E_T(D))$, by Proposition 3.5. This contradiction implies that f must be zero.

It is of obvious interest now to display some instances in which $E_T(F)$ is closed. Consider the following class of operators, introduced in [1].

Definition. A bounded linear operator T on the Banach space X is called *well-decomposable,* if for every open covering $\{U, V\}$ of $\boldsymbol{C}$ there is a bounded linear operator R on X, an integer n and subspaces $Y, Z \in \operatorname{Lat}(T)$ for which

$$\sigma(T \mid Y) \subset U\,, \quad \sigma(T \mid Z) \subset V$$

$$R(X) \subset Y, \quad (I - R)(X) \subset Z$$

and

$$C(T)^n (R) = 0\,,$$

(where $C(T)(R) = TR - RT$ and $C(T)^k R = C(T)^{k-1} (TR - RT)$, for $k = 2, 3, \ldots$).

If T is well-decomposable and if $E_T(\emptyset) = \{0\}$ (which means that T has no non-trivial divisible subspace) then it is shown in [1, Proposition 3.2] that $E_T(F)$ is closed whenever F is closed. We give a simpler version of the proof here.

Proposition 3.7. *Suppose T is well-decomposable and $E_T(\emptyset) = \{0\}$. If $F \subset \boldsymbol{C}$ is closed then $E_T(F)$ is closed.*

Proof. Let U be an open neighborhood of F and choose R so that for certain $Y, Z \in \operatorname{Lat}(T)$ we have $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset \boldsymbol{C} \setminus F$, $RX \subset Y$ and $(I - R)(X) \subset Z$, and let n be chosen so that $C(T)^n (R) = 0$. We then have that $R\ E_T(F) \subset E_T(F)$. This is proved in [1], so we do not reproduce the details here.

Since $\sigma(T \mid Z) \subset \boldsymbol{C} \setminus F$ it follows that $Z \subset E_T(\boldsymbol{C} \setminus F)$. Hence $(I - R)\, E_T(F) \subset (I - R) X \subset Z \subset E_T(\boldsymbol{C} \setminus F)$. But $R\, E_T(F) \subset E_T(F)$ implies that $(I - R)\, E_T(F) \subset E_T(F)$, hence

$$(I - R)\, E_T(F) \subset E_T(F) \cap E_T(\boldsymbol{C} \setminus F) = \{0\}\,.$$

It follows that $E_T(F) = R\, E_T(F)$, so that $E_T(F) \subset Y$ from which $\overline{E_T(F)} \subset Y$. Since $\sigma(T \mid Y) \subset U$ and since $(T - \lambda)\, E_T(F) = E_T(F)$ for all $\lambda \notin U$ it follows that $\sigma(T \mid \overline{E_T(F)} \subset U$. Since U is an arbitrary neighborhood we conclude that $\sigma(T \mid \overline{E_T(F)}) \subset F$, hence $\overline{E_U(F)} \subset E_T(F)$, by maximality.

In numerous cases an algebraic description of the largest T-divisible subspace $E_T(\emptyset)$ yields a similar algebraic formula for $E_T(F)$, for any closed set F. Such results have been obtained for generalized scalar operators by Vrbová [18] and by Foias and Vasilescu [8] and for normal operators on Hilbert space by Pták and Vrbová [15]. Here is a more general version for super decomposable operators. Recall that a bounded linear operator T is super-decomposable if it is well decomposable and the commutation index $n = 1$ for any covering $\{U, V\}$ of $\boldsymbol{C}$ and the corresponding operator R. In other words, $RT = TR$.

Proposition 3.8. *Suppose T is super decomposable, and suppose there is an integer q for which*

$$\bigcap_{\lambda \in \boldsymbol{C}} (T - \lambda)^q X = E_T(\emptyset)\,.$$

Then for every closed $F \subset \boldsymbol{C}$ we have

$$E_T(F) = \bigcap_{\lambda \notin F} (T - \lambda)^q X .$$

Proof. Let $F \subset \boldsymbol{C}$ be closed. Since $(T - \lambda) E_T(F) = E_T(F)$ for every $\lambda \notin F$ the inclusion $E_T(F) \subseteq W_0 := \bigcap_{\lambda \notin F} (T - \lambda)^q X$ is trivial. Let U be an open neighborhood of F and choose R so that $RT = TR$ and such that there are spaces $Z, Y \in \operatorname{Lat}(T)$ with $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset \boldsymbol{C} \setminus F$, $RX \subset Y$, $(I - R) X \subset Z$. Suppose $x \in W_0$ and note that $y := (I - R) x \in Z$ so that $y_\lambda := (T - \lambda)^{-q} y$ is well defined for every $\lambda \in F$. Moreover, since $x = (T - \lambda)^q x_\lambda$ for every $\lambda \notin F$ we may define, for $\lambda \notin F$, $y_\lambda := (I - R) x_\lambda$. Then, for $\lambda \in F$, $(T - \lambda)^q y_\lambda = y$ and for $\lambda \notin F$ $(T - \lambda)^q y_\lambda = (T - \lambda)^q (I - R) x_\lambda = (I - R)(T - \lambda)^q x_\lambda = (I - R) x = y$. It follows that $y \in \bigcap_{\lambda \notin \boldsymbol{C}} (T - \lambda)^q X = E_T(\emptyset)$. Hence if $x \in W_0$ then $x = (I - R) x + Rx \in E_T(\emptyset) + Y \subset E_T(U)$. U being arbitrary we conclude that $W_0 \subseteq E_T(F)$.

A readily accessible example to which this description applies is that of a commutative semi-simple and regular Banach algebra A. If $a \in A$ then the multiplication operator M_a is super decomposable [14, Corollary 2.4]. Moreover, if $x \in \bigcap_{\lambda \in \boldsymbol{C}} (M_a - \lambda) A$ and h is a multiplicative linear functional on A let $\lambda = h(a)$ and select a_λ so that $x = (a - \lambda) a_\lambda$. Then $h(x) = 0$, hence $x = 0$ by semi-simplicity. Consequently we obtain for F closed in $\boldsymbol{C}$

$$E_{M_a}(F) = \bigcap_{\lambda \notin F} (a - \lambda) A .$$

Remark. At this stage it is tempting to add to the large collection of definitions of families of operators whose spectra possess some kind of local decomposability one expressed in terms of the $E_T(F)$ spaces. This matter will be pursued in a future paper so suffice it here to suggest that a bounded linear operator T be called *algebraically decomposable* if every open cover $\{U, V\}$ of $\boldsymbol{C}$ yields a splitting of the Banach space

$$X = E_T(U) + E_T(V) .$$

Evidently, this class of algebraically decomposable operators is rather large, containing not only all decomposable operators but also all operators satisfying property (δ) [1].

On the other hand, if we further require that $E_T(F)$ be closed whenever F is closed, then we obtain a subclass of the decomposable operators without non-trivial divisible subspaces, possibly exactly this class.

4. AUTOMATIC CONTINUITY

There is a series of results in the literature describing classes of operators $T \in B(X)$ and $S \in B(Y)$, where X, Y are Banach spaces, for which every linear map $\theta : X \rightsquigarrow Y$ intertwining S and T (i.e. $S\theta = \theta T$, or equivalently $C(S, T)\,\theta = 0$) is continuous exactly when every eigenvalue λ of S has the property that codim $(T-\lambda)X < \infty$ and, additionally, either T is algebraic or S has no non-trivial divisible subspace [11], [14], [1].

The significant feature of many of these arguments is that $\theta X_T(F) \subset E_S(F)$ for every closed F. Moreover, it is necessary to assume closedness of $\{X_T(F)\}$ and of $\{E_S(F)\}$ to be able to "localize" the possible discontinuity of θ to some finite set. The natural set of assumptions that emerge are that T be decomposable and that $E_S(F)$ be closed whenever F is closed. As a result $E_S(\emptyset) = \{0\}$ becomes part of the assumptions of our present result. On the other hand the intertwining condition is relaxed considerably, namely to $C(S, T)^n\,\theta = 0$ for some n.

Theorem 4.1. *Suppose T is a bounded decomposable operator on the Banach space X, and that S is a bounded linear operator on the Banach space Y for which $E_S(F)$ is closed for all closed F.*

Consider the class of linear maps $\theta\colon X \rightsquigarrow Y$ for which $C(S, T)^n\,\theta = 0$ for some n (depending on θ). Then every θ in this class is continuous if and only if (S, T) has no critical eigenvalues (this means that if λ is an eigenvalue for S, then codim $(T-\lambda)X < \infty$.)

Proof. The necessity of the condition that (S, T) have no critical eigenvalues is classical and easy [11]. So let us assume that (S, T) have no critical eigenvalues and consider $\theta\colon X \rightsquigarrow Y$ for which $C(S, T)^n\,\theta = 0$ for some n.

We first establish that $\theta X_T(F) \subset E_S(F)$, for any closed $F \subseteq \boldsymbol{C}$; actually the proof of this is sufficiently similar to that given in [1, proof of Proposition 3.2] that we skip the details (as we also did in the proof of Proposition 3.7).

Now, by [2, Theorem 4.3 e)] there is a finite set $F \subset \boldsymbol{C}$ such that if

$$\tau(\theta) := \{y \in Y \mid \exists x_n \to 0 \text{ and } \theta x_n \to y\}$$

then $\tau(\theta) \subset E_S(F)$.

Arrange the points of $F = \{\lambda_1, \dots, \lambda_m\}$ in an infinite sequence $\{\mu_j\}$ (in which each λ_j appears infinitely often) and let $T_j := T - \mu_j$, $j = 1, 2, \dots$.

By [12], there is an $N \in \boldsymbol{N}$ so that

$$\tau(\theta T_1 \dots T_q) = \tau(\theta T_1 \dots T_N)$$

for all $q \geqq N$. Thus, if $p(T) := T_1 \dots T_N$ then we have found a polynomial p with all its roots in F for which

$$\tau(\theta\, p(T)) = \tau(\theta\, p(T)\,(T-\lambda_j)^q)$$

for $j = 1, \dots, m$ and for any $q \in \boldsymbol{N}$. Letting $\psi := \theta\, p(T)$ we actually have a bit more:

$$(*) \qquad \tau(\psi) = \tau(\psi(T-\lambda)^q)$$

for any $\lambda \in \boldsymbol{C}$ and any $q \in \boldsymbol{N}$.

For $\lambda \in F$ this has just been established. And if $\lambda \notin F$ choose $\{U_1, U_2\}$ open so that $\boldsymbol{C} = U_1 \cup U_2$ and so that $\lambda \notin \bar{U}_1$, while $F \subset U_1$ and $F \cap \bar{U}_2 = \emptyset$. Since T

is decomposable

$$X = X_T(\bar{U}_1) + X_T(\bar{U}_2)$$

and since

$$\psi X_T(\bar{U}_2) \subset \theta X_T(\bar{U}_2) \subset E_S(\bar{U}_2)$$

it follows that

$$\tau(\psi|_{X_T(\bar{U}_2)}) \subseteq E_S(F) \cap E_S(\bar{U}_2) = \{0\},$$

so that ψ is continuous on $X_T(\bar{U}_2)$. It follows that $\tau(\psi) = \tau(\psi|_{X_T(\bar{U}_1)})$ and since $T - \lambda$ is invertible on $X_T(\bar{U}_1)$, $(*)$ follows.

Next, with $\tau := \tau(\psi)$ let

$$Z := [\tau + S\tau + S^2\tau + \ldots + S^{n-1}\tau]^- .$$

It is an easy consequence of $C(S, T)^n\, \theta = 0$ that Z is S-invariant. We also note that $Z \subset E_S(F)$. We want to establish that

$$(SZ)^- = Z .$$

Since

$$\tau = \tau(\psi T^n) = \tau\left(\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} S^{n-k}\psi T^k\right) = \left[S\tau\left(\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} S^{n-(k+1)}\psi T^k\right)\right]^-$$

and since

$$\tau\left(\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} S^{n-(k+1)}\psi T^k\right) \subseteq \tau(S^{n-1}\psi) + \tau(S^{n-2}\psi T) + \ldots + \tau(\psi T^{n-1}) =$$
$$= (S^{n-1}\tau)^- + (S^{n-2}\tau)^- + \ldots + \tau$$

(remember that $\tau = \tau(\psi T^k)$, $k = 0, \ldots, n-1$) we conclude that

$$S^{n-1}\tau + S^{n-2}\tau + \ldots + \tau$$

is dense in $\tau(\sum_{k=0}^{n-1}(-1)^k \binom{n}{k} S^{n-(k+1)}\psi T^k)$. Thus

$$\tau \subset (SZ)^- .$$

But then

$$\tau + S\tau + \ldots + S^{n-1}\tau \subseteq (SZ + S\tau + \ldots + S^{n-1}\tau)^- =$$
$$= (SZ + S(\tau + \ldots + S^{n-2}\tau))^- = (SZ)^-$$

so that $Z = (SZ)^-$.

Precisely the same argument would have shown that $Z = ((S - \lambda_j)\, Z)^-$, $j = 1, \ldots, m$. Hence, by the Mittag-Leffler theorem [16] there is a dense linear subspace $W \subset Z$ on which

$$(S - \lambda_j)\, W = W, \quad j = 1, \ldots, m .$$

Thus

$$W \subset E_S(C \setminus F)$$

so that

$$W \subset E_S(F) \cap E_S(C \setminus F) = \{0\} .$$

This shows that $Z = \{0\}$, hence $\tau(\psi) = \{0\}$.

We have proved that $\theta\, p(T)$ is continuous.

Consider now two possibilities: if $\lambda_1, \ldots, \lambda_m$ all are eigenvalues of S then $p(T)\, X$ is of finite codimension in X. Let G be a closed neighborhood of F. Then $p(T)\, X \cap$ $\cap\, X_T(G)$ is of finite codimension in $X_T(G)$. But $X_T(G)$ is T-absorbent [17, Theorem IV.4.2], hence $p(T)\, X \cap X_T(G) = p(T)\, X_T(G)$ is of finite codimension in $X_T(G)$ (and closed). Since $\theta\, p(T)$ is continuous, the continuity of θ on $X_T(G)$ is established. As we saw before (in connection with establishing (*)), that is enough to ensure continuity of θ in this case.

If there is $\lambda_j \in F$ for which $S - \lambda_j$ is 1-1, we argue first by induction to reduce the number of such points in F to one: suppose there are $q > 1$ non-eigenvalues for S in F and suppose continuity of θ has been established whenever S is a mapping with less than q eigenvalues in the singularity set F. If $\{\lambda_1, \ldots, \lambda_q\}$ is an enumeration of the non-eigenvalues in F, write $\{\lambda_1, \ldots, \lambda_q\} = F_1 \cup F_2$ as a disjoint union of non-empty sets and let $G_j \subset F_j$ be disjoint closed neighborhoods. Since

$$\theta(X_T(G_j)) \subset E_S(G_j)$$

and

$$\tau(\theta|_{X_T(G_j)}) \subset E_S(F) \cap E_S(G_j) \subset E_S(F \setminus F_{3-j})$$

it follows by inductive hypothesis that $\theta|_{X_T(G_1)}$ and $\theta|_{X_T(G_2)}$ are continuous. But $X_T(G_1 \cup G_2) = X_T(G_1) \oplus X_T(G_2)$ so by the open mapping theorem continuity of θ on $X_T(G_1 \cup G_2)$ follows. The continuity of θ then follows as before.

We still have left to establish the result for the case $F = \{\lambda_1\} = \{0\}$, say, and S assumed 1-1. From the preceding parts of the proof we know that θT^r is continuous for some r. Since $C(S, T)^n\, \theta = 0$ implies that $C(S, T)^{n+r}\, \theta = 0$ we may assume that $C(S, T)^n\, \theta = 0$ and θT^n is continuous. From the next lemma it then follows that θ is continuous. This completes the proof.

Lemma 4.2. *Suppose $\theta T^n = \alpha_1 S\theta T^{n-1} + \ldots + \alpha_n S^n\theta$, where $\alpha_1, \ldots, \alpha_n$ are all non-zero. Suppose θT^n is continuous and S is 1-1. Then θ is continuous.*

Proof. Since

$$\alpha_1 S\theta T^{n-1} + \ldots + \alpha_n S^n\theta$$

is continuous we have

$$\{0\} = \tau(\alpha_1 S\theta^{n-1} + \ldots + \alpha_n S^n\theta) = S\tau(\alpha_1 \theta T^{n-1} + \ldots + \alpha_n S^{n-1}\theta)$$

and hence $\alpha_1 \theta T^{n-1} + \ldots + \alpha_n S^{n-1}\theta$ is continuous (S being 1-1). Multiply this equation by T^{n-1} and observe that all but the last term

$$\alpha_n S^{n-1}\theta T^{n-1}$$

contain θT^m, where $m \geqq n$. Hence $S^{n-1}\theta T^{n-1}$ and so θT^{n-1} are continuous. We then obtain that

$$\alpha_2 S^2\theta T^{n-2} + \ldots + \alpha_n S^n\theta$$

is continuous and hence that

$$\alpha_2 \theta T^{n-2} + \ldots + \alpha_n S^{n-2}\theta$$

is continuous. Repeat the argument; we end up with both

$$\alpha_{n-1}\theta T + \alpha_n S\theta$$

and θT^2 being continuous, hence also $S\theta T$, and therefore θT, continuous. It follows, finally, that $S\theta$, and consequently θ, are continuous.

5. MODULE DERIVATIONS

As an application of Theorem 4.1 and its proof consider a commutative Banach algebra A with identity to which [14, Theorem 2.3] applies; thus, A may be regular and semi-simple, or A may have a totally disconnected maximal ideal space Φ_A. In particular, A may be radical with identity adjoined. Consider also a Banach A-module $\mathfrak{M}$; thus we have a continuous algebra homomorphism $\varrho\colon A \curvearrowright B(\mathfrak{M})$ which maps the identity of A to the identity on $\mathfrak{M}$. Suppose $D\colon A \curvearrowright \mathfrak{M}$ is a derivation; this means that D is linear and

$$D(a_1 a_2) = \varrho(a_1)\, Da_2 + \varrho(a_2)\, Da_1 \quad \text{for all} \quad a_1, a_2 \in A\,.$$

Quite a few results on continuity of derivations are already in the literature [3, 4, 5, 9, 13]. More follow from Theorem 4.1. This is because if $a_0 \in A$ then $C(\varrho(a_0), a_0)^2\, D = 0$: if $a \in A$ then

$$C(\varrho(a_0), a_0)^2\, Da = C(\varrho(a_0), a_0)\,[\varrho(a_0)\, Da - D(a_0 a)] =$$
$$= \varrho(a_0^2)\, Da - \varrho(a_0)\, D(a_0 a) - \varrho(a_0)\, D(a_0 a) + D(a_0^2 a) = 0\,.$$

A special case of derivations is the point derivations, where $\mathfrak{M}$ is the complex plane (in which case the module action ϱ is given by some $\lambda \in \Phi_A$).

To phrase our result we also need the notion of a scalar module: a Banach A-module N is scalar if

$$\varrho(a)\, m = \lambda_a m \quad \text{for all} \quad a \in A\,, \quad m \in N\,,$$

where $\lambda_a \in \mathbf{C}$. This means that the module action ϱ is given by a linear functional, necessarily multiplicative (and continuous). Note that a Banach A-module $\mathfrak{M}$ contains scalar submodules if and only if it contains one-dimensional submodules. We then have

Theorem 5.1. *Suppose A is a singly generated commutative Banach algebra with identity e and generator z. Suppose $\mathfrak{M}$ is a Banach A-module on which $\varrho(z)$ is an operator for which $E_{\varrho(z)}(F)$ is closed for every closed $F \subset \mathbf{C}$. Suppose also that multiplication by z in A is a decomposable operator.*

Then there are discontinuous derivations from A to $\mathfrak{M}$ if and only if there is a multiplicative linear functional $\lambda \in \Phi_A$ with respect to which $\mathfrak{M}$ contains a scalar submodule and with respect to which A possesses discontinuous point derivations.

This result is an analogue of [10, Corollary 4.2]. It is also related to [5, Theorem 3.3] and to [3, Corollary 4.6].

Proof. It is easy to see that if $\lambda \in \Phi_A$, if $Cs \subset \mathfrak{M}$ is a one-dimensional submodule with module action λ and if $\delta: A \frown C$ is a discontinuous linear functional for which $\delta(a_1 a_2) = \lambda(a_1)\,\delta(a_2) + \lambda(a_2)\,\delta(a_1)$ for all $a_1, a_2 \in A$, then

$$Da := \delta(a)\, s$$

is a discontinuous derivation into $\mathfrak{M}$.

Suppose conversely that $D: A \frown \mathfrak{M}$ is a discontinuous derivation. Our assumptions are exactly what Theorem 4.1 requires. We conclude that the pair $(z, \varrho(z))$ has a critical eigenvalue. In particular $\varrho(z)$ has an eigenvalue, say λ_0. Let $s \neq 0$ be an eigenvector. It is straightforward, using the single generation of A, and the continuity of ϱ, to conclude that Cs is a submodule of $\mathfrak{M}$. This provides us with a multiplicative linear functional $\lambda \in \Phi_A$ (for which $\lambda(z) = \lambda_0$).

We must show that A has a discontinuous point derivation at $M_\lambda := \ker \lambda$. Suppose not. Since point derivations are characterized by their vanishing on e and on M_λ^2 this means that M_λ^2 is closed and of finite codimension.

For a derivation D we may introduce the continuity ideal

$$I(D) := \{a \in A \mid \varrho(a)\,\tau(D) = \{0\}\},$$

which is easily seen to be a closed ideal. In [13, Proposition 2.8] is shown that there is a constant $C > 0$ so that

$$\|D(a_1 a_2)\| \leqq C\|a_1\|\;\|a_2\| \quad \text{for all} \quad a_1, a_2 \in I(D);$$

moreover, if A is separable (which it is here, by single generation) then D is continuous on $I(D)^2$ [13, Corollary 2.11]. We shall change the given derivation D to a derivation D_1 for which $I(D_1) = M_\lambda$ but such that F_1 is still discontinuous. This cannot be, however, since M_λ^2 is closed and of finite codimension. The derivation D_1 may be obtained by means of the prime ideal theorem [4], but we can get it directly from the proof of Theorem 4.1. Early in that proof we obtained a polynomial p (which we may take to have minimal degree) for which $D(p(z)\,\cdot)$ is continuous. From the definition of a derivation it then follows that $p(\varrho(z))\,D$ is continuous. We also noted that the roots of p are located in some finite set F for which

$$\tau(D) \subset E_{\varrho(t)}(F).$$

We may even assume that F is exactly the set of roots of p: since $p(\varrho(z))\,\tau(D) = \{0\}$ we may cancel all factors $\varrho(z) - \lambda$ which are 1-1; moreover, if

$$p(\varrho(z)) = (\varrho(z) - \lambda_1)^{r_1} \dots (\varrho(z) - \lambda_n)^{r_n},$$

then by the absorbency of $E_{\varrho(z)}(\{\lambda_1, \dots, \lambda_n\})$ it follows that since $p(\varrho(z))\,\tau(D) = \{0\}$),

$$\tau(D) \subset E_{\varrho(z)}(\{\lambda_1, \dots, \lambda_n\}).$$

We can say more: let

$$p_z(t) := p(t)/(t - \lambda_k)^{r_k}, \quad k = 1, \dots, n.$$

Then

$$p_k(\varrho(z))\,\tau(D) \subset \ker\,(\varrho(z) - \lambda_k)^{r_k}, \quad k = 1, \dots, n,$$

and since $\{p_1, \ldots, p_n\}$ has greatest common divisor 1 we can find polynomials $q_1, \ldots, q_n$ so that $p_1 q + \ldots + p_n q_n = 1$; consequently

$$\tau(D) = \ker\left((\varrho(z) - \lambda_1)^{r_1} \mid \tau(D)\right) \oplus \ldots \oplus \ker\left((\varrho(z) - \lambda_n)^{r_n} \mid \tau(D)\right).$$

We may assume that $F = \{\lambda_1, \ldots, \lambda_n\}$ is minimal with respect to containing $\tau(D)$. Then each of the maps

$$(\varrho(z) - \lambda_j)^k, \quad k = 1, \ldots, r_j, \quad j = 1, \ldots, n,$$

acting on $\tau(D)$ cannot have dense range (if one of them did, the Mittag-Leffler theorem would imply the existence of a dense subspace on which $\varrho(z) - \lambda_j$ would be surjective, contradicting the minimality of F).

These remarks show that if $Q(t) = p(t)/(t - \lambda_j)$ for some j, then

$$Q(\varrho(z))\, \tau(D) \neq \{0\},$$

which means that the derivation $Q(\varrho(z))\, D$ is discontinuous, but also

$$(\varrho(z) - \lambda_j)\, Q(\varrho(z))\, (D) = \{0\},$$

which means that the continuity ideal $I(Q(\varrho(z))\, D)$ is $\ker \lambda$, where $\lambda \in \Phi_A$ is chosen so that $\lambda(z) = \lambda_j$. Thus $D_1 = Q(\varrho(z))\, D$ is the discontinuous derivation with continuity ideal $M_\lambda := \ker \lambda$, that we want.

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