Czechoslovak Mathematical Journal

Ján Jakubík

Closure operators on the lattice of radical classes of lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 1, 71–77

Persistent URL: `http://dml.cz/dmlcz/102201`

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

CLOSURE OPERATORS ON THE LATTICE OF RADICAL CLASSES OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received September 11, 1985)

The notion of radical class of lattice ordered groups was introduced in [7]; cf. also [8], [1], [9], [4], [10]. In this note there will be investigated a question proposed by M. Darnel [4] concerning permutability of certain closure operators on the lattice of radical classes of lattice ordered groups.

1. PRELIMINARIES

We recall the basic notions and some notation.

Let $\mathcal{G}$ be the class of all lattice ordered groups. When considering a subclass X of $\mathcal{G}$ we always assume that the zero group $\{0\}$ belongs to X and that X is closed with respect to isomorphisms.

A subclass R of $\mathcal{G}$ is said to be a *radical class* [7] if it is closed with respect to convex l-subgroups and with respect to joins of convex l-subgroups. It is known that every variety of lattice ordered groups is a radical class [6].

Let $\mathcal{R}$ be the collection of all radical classes; $\mathcal{R}$ is partially ordered by inclusion. Then $\mathcal{R}$ is a "complete lattice" in the sense that if $\mathcal{R}_1$ is a subcollection of $\mathcal{R}$, then $\sup \mathcal{R}_1$ and $\inf \mathcal{R}_1$ do exist in $\mathcal{R}$. (Cf. [7].)

1.1. Theorem. (Cf. [4], Thm. 5.1.) *For any radical class R, there exist unique minimal radical classes R^s and R^h, closed with respect to l-subgroups and l-homomorphic images, respectively, that contain R. Moreover, the collection of s-closed and h-closed radical classes form complete lattices under inclusion.*

It is clear that the mappings $R \to R^s$ and $R \to R^h$ are closure operators on the lattice $\mathcal{R}$.

In [4] it is remarked that in a surprising number of cases (though not all), R^{hs} and R^{sh} are varieties and that this indicates that the s-closure and the h-closure might be strongly linked in some way. Next, the question is raised in [4], whether or not the relation

$$R^{sh} = R^{hs} \tag{1}$$

is valid for each radical class R.

Let us denote by $\mathscr{R}_1$ the collection of all radical classes $\mathscr{R}$ for which the relation (1) fails to hold. In this paper it will be shown that the collection $\mathscr{R}_1$ is rather large. Namely, the following result will be established:

1.2. Theorem. *There exists an injective mapping of the class of all cardinals into the collection $\mathscr{R}_1$.*

2. A CONSTRUCTION

In this section a linearly ordered group G will be constructed which will be applied below in proving Theorem 1.2.

If G_1 is any linearly ordered group, then each subgroup of G_1 is linearly ordered by the induced linear order.

The additive group of all reals (all rational numbers) with the natural linear order will be denoted by R_0 (or by R'_0, respectively).

For each $i \in R'_0$ let $A_i = R_0$. Next, let A^0 be the lexicographic product

$$A^0 = \Gamma A_i \quad (i \in R'_0)$$

(cf. [5]). The elements of A^0 will be written in the form $a = \langle \ldots, a_i, \ldots \rangle$ $(i \in R'_0)$. The support $S(a)$ of the element a is defined by

$$S(a) = \{i \in R'_0 : a_i \neq 0\} .$$

Let A be the subgroup of A^0 consisting of all elements of A^0 with finite support.

Let $B = R'_0$. For $a \in A$ and $b \in B$ we denote

$$a^b = \langle \ldots, a'_i, \ldots \rangle \ (i \in R'_0) ,$$

where $a'_i = a_{i-b}$ for each $i \in R'_0$.

Let B_0 be the set of all pairs (b, a) with $b \in B$ and $a \in A$. For $(b_i, a_i) \in B_0$ $(i = 1, 2)$ we put $(b_1, a_1) \leqq (b_2, a_2)$ if either $b_1 < b_2$, or $b_1 = b_2$ and $a_1 \leqq a_2$. We define the operation $+$ on B_0 by putting

$$(b_1, a_1) + (b_2, a_2) = (b_1 + b_2,\ a_1^{b_2} + a_2) .$$

Then B_0 turns out to be a linearly ordered group. Let

$$A^{01} = \{(b, a) \in B_0 : b = 0\} .$$

The following assertion follows immediately from the definition of B_0.

2.1. Lemma. *A^{01} is an l-ideal of B_0. If K is an l-ideal in B_0 with $\{0\} \neq K \neq B_0$, then $K = A^{01}$.*

Let α be a cardinal, $\alpha > \aleph_0$. Let I_α be the first ordinal with card $I_\alpha = \alpha$ and let J_α be a linearly ordered set dual to I_α. For each $j \in J_\alpha$ let $C_j = R_0$. Put

$$C_0 = \Gamma C_j \quad (j \in J_\alpha) .$$

Let C be the subgroup of C_0 consisting of all elements of C_0 having a finite support.

Put

$$G_0 = C \circ B_0 ,$$

where $\circ$ denotes the operation of lexicographic product. The elements of G_0 can be written as triples $g = (c, b, a)$ with $c \in C$, $b \in B$ and $a \in A$. Denote

$$f(g) = \sum a_i + \sum c_j \quad (i \in R_0', \ j \in J_\alpha) .$$

Put

(2) $$G = \{g \in G_0 : f(g) \text{ is an integer}\} .$$

Then G is a subgroup of G_0; thus G is a linearly ordered group.

Let J_1 be a subset of J_α such that either $J_1 = \emptyset$ or J_1 is an ideal of the linearly ordered set J_α. Denote

$$G^1(J_1) = \{g = (c, b, a) \in G : c_j = 0 \quad \text{for each} \quad j \in J_\alpha \setminus J_1\} ,$$
$$G^2 \quad = \{g = (c, b, a) \in G : c \ = 0 \quad \text{and} \quad b = 0\} .$$

From 2.1 we obtain:

2.2. Lemma. *Both $G^1(J_1)$ and G^2 are l-ideals of G. If K is an l-ideal of G with $\{0\} \neq K \neq G$, then either $K = G^1(J_1)$ for some J_1 or $K = G^2$.*

Also, in view of the definition of G we have:

2.3. Lemma. *Let K_1 be a convex subgroup of G. Then some of the following conditions is satisfied:*

(i) *$K_1 = G^1(J_1)$ for some J_1.*

(ii) *K_1 is a convex subgroup of G^2.*

Lemma 2.3 implies:

2.4. Lemma. *Let G' be a linearly ordered group. Suppose that there exist subgroups G_i' $(i \in I)$ of G' such that*

(i) *$G = \bigcup_{i \in I} G_i'$,*

(ii) *for each G_i' there exists a convex subgroup of G which is isomorphic to G_i'.*

Then G' is isomorphic to a convex subgroup of G.

Let R be the radical class of lattice ordered groups generated by the linearly ordered group G.

From 2.4 and Theorem 3.4, [8] we infer:

2.5. Lemma. *The radical class R is the class of all lattice ordered groups which can be expressed (up to isomorphism) as direct sums of some convex subgroups of G.*

Now we shall construct a linearly ordered group H_2 belonging to R^{sh}.

Denote

$$H = \{g = (c, b, a) \in G : b = 0\} .$$

Then H is a subgroup of G, whence $H \in R^s$. Let I be an ideal of the linearly ordered set R_0' such that $I \neq R_0'$. Put

$$H_1 = \{g = (c, b, a) \in H : c = 0 \ \text{and} \ a_i = 0 \ \text{for each} \ i \in R_0' \setminus I\} .$$

H_1 is an l-ideal of the linearly ordered group H. In view of (2) we obtain:

2.6. Lemma. *The linearly ordered group* $H_2 = H/H_1$ *is isomorphic to the linearly ordered group*

$$C = \Gamma A_i \quad (i \in R'_0 \setminus I) .$$

For any subclass X of $\mathcal{G}$ we denote by

Sub X – the class of all l-subgroups of lattice ordered groups belonging to X;

Hom X – the class of all homomorphic images of lattice ordered groups belonging to X.

Let Y be the class of all linearly ordered groups K having the property that K is isomorphic to some convex subgroup of G. From the construction of G and from 2.6 we obtain

2.7. Lemma. *The linearly ordered group* H_2 *does not belong to the class* Sub Hom Y.

Clearly $H_2 \in R^{sh}$. Moreover, H_2 contains a strong unit (cf. [5]).

3. THE RADICAL CLASS R^{hs}

3.1. Lemma. *Let* K_m $(m \in M)$ *be lattice ordered groups and let* $K = \sum_{m \in M} K_m$. *Let* K_0 *be an l-ideal of* K *and for each* $m \in M$ *let* K_{0m} *be the projection of* K_0 *into* K_m. *Then the lattice ordered group* K/K_0 *is isomorphic to the direct sum* $\sum_{m \in M} K_m/K_{0m}$.

The proof is easy.

From 3.1 and 2.5 we obtain:

3.2. Lemma. *The class* Hom R *is the class of all lattice ordered groups which can be expressed (up to isomorphism) as direct sums of linearly ordered groups belonging to* Hom Y.

For any lattice ordered group L we denote by $c(L)$ the system of all convex l-subgroups of L; the system $c(L)$ is partially ordered by inclusion. In fact, $c(L)$ is a complete lattice. The lattice operations in $c(L)$ will be denoted by $\bigvee^c$ and $\bigwedge^c$. (The operation $\bigwedge^c$ coincides with the set-theoretic intersection.)

Let H_2 be as in Section 2.

3.3. Lemma. *Let* K_m $(m \in M)$ *be linearly ordered groups belonging to* $c(H_2)$ *such that* $\bigvee^c_{m \in M} K_m = H_2$. *Then there is* $m \in M$ *such that* $K_m = H_2$.

For any $X \subseteq \mathcal{G}$ we denote by X_r the radical class generated by X. From 3.2 and Proposition 5.5, [4] it follows:

3.4. Lemma. $R^h = (\text{Hom } Y)_r$.

Now since Hom Y is a class of linearly ordered groups, $(\text{Hom } Y)_r$ can be obtained by means of Thm. 3.4 in [8]; from this theorem, from 3.3 and 2.7 we infer:

3.5. Lemma. *The linearly ordered group* H_2 *does not belong to the radical class* R^h.

3.6. Lemma. *Let D_i $(i \in I)$ be lattice ordered groups. Suppose that K is an l-subgroup of the direct sum $\sum_{i \in I} D_i$ such that* (i) *K is a linearly ordered group, and* (ii) *K has a strong unit. Then there exists $i \in I$ such that the projection $k \to k_i$ is an isomorphism of K into D_i.*

Proof. Let e be a strong unit in K. Put $I_1 = \{i \in I : e_i \neq 0\}$. The set I_1 is finite. For each $k \in K$ there exists a positive integer n such that $-ne \leqq k \leqq ne$. Hence if $i \notin I_1$, then $k_i = 0$. Therefore K is an l-subgroup of $\sum D_i$ $(i \in I_1)$.

There exists a minimal subset I_2 of I_1 having the property that the mapping

$$k \to \langle \ldots, k_i, \ldots \rangle \quad (i \in I_2) \tag{3}$$

is an isomorphism. Assume that card $I_2 \geqq 2$. Choose $i_2 \in I_2$. Then the mapping

$$k \to k_{i_2}$$

is a homomorphism of K into D_{i_2}, but it fails to be an isomorphism. Thus there is $0 < k' \in K$ such that

$$k'_{i_2} = 0 . \tag{4}$$

Put $I_3 = I_2 \setminus \{i_2\}$. The mapping

$$k \to \langle \ldots, k_i, \ldots \rangle \quad (i \in I_3) \tag{5}$$

is a homomorphism of K into $\sum D_i$ $(i \in I_3)$, but (in view of the minimality of I_2) the mapping (5) fails to be an isomorphism. Hence there is $0 < k'' \in K$ such that

$$k''_i = 0 \quad \text{for each} \quad i \in I_3 . \tag{6}$$

We distinguish two cases.

a) $k'' \leqq k'$. Then from (4) we infer that $k''_{i_2} = 0$, hence $k''_i = 0$ for each $i \in I_2$; in view of (3) we arrive at a contradiction.

b) $k' < k''$. Then according to (6) we have $k'_i = 0$ for each $i \in I_3$ and hence $k'_j = 0$ for each $j \in I_2$. This contradicts the fact that the mapping (3) is an isomorphism. Therefore card $I_2 = 1$, which completes the proof.

Put $Q = R^h$. Let Q'_1 be the class of all l-subgroups of elements of Q and let $Q_1 = (Q'_1)_r$. Define $Q'_2, Q_2, Q'_3, Q_3, \ldots$ analogously. Then we have (cf. [4], Section 5)

$$Q^s = \bigvee Q_i \quad (i = 1, 2, \ldots) . \tag{7}$$

Denote $Q_0 = Q$.

Let us denote by R_d the class of all lattice ordered groups having the property that each upper bounded disjoint subset is finite.

In view of Thm. 3.4, [8] we have $R \subseteq R_d$. Then from 3.4 we obtain $R^h \subseteq R_d$. Finally, from [4], Lemma 5.4 and from (7) we get that the following lemma is valid:

3.7. Lemma. $Q^s \subseteq R_d$.

3.8. Lemma. *Let $K' \in R_d$. Let K and K_i $(i \in I)$ be elements of $c(K')$ such that $K = \bigvee^c_{i \in I} K_i$. Suppose that K has a strong unit. Then there exists a finite subset $\{K_j\}_{j \in J}$ of $c(K)$ such that* (i) *for each $j \in J$ there exists $i \in I$ with $K_j \in c(K_i)$, and* (ii) *$K = \sum_{j \in J} K_j$.*

Proof. Let e be a strong unit of K. Then there exists a finite subset I_1 of I and for each $i \in I_1$ there exists $0 < e_i \in K_i$ such that $e = \sum_{i \in I_1} e_i$. Then we have $K = \bigvee^c_{i \in I_1} K_i$. For each $i \in I_1$ let K'_i be the convex l-subgroup of K_i generated by the element e_i. The relation

(8) $$K = \bigvee^c_{i \in I_1} K'_i$$

is valid and for each $i \in I_1$, e_i is a strong unit in K'_i.

Let $i \in I_1$. Then each disjoint subset of K'_i is finite. Hence according to [2] (cf. also [5], Chap. V, Section 6) K_i can be expressed as a direct sum of a finite number of lattice ordered groups K'_{it} such that each K'_{it} is a nontrivial lexico extension, i.e., $K'_{it} = \langle K''_{it} \rangle$, $K''_{it} \neq K'_{it}$.

Let us consider two such l-groups $K'_{i_1 t_1}$ and K'_{i_2, t_2}. Both of them belong to $c(K)$. In view of [3], Propos. 2.9 we have two possibilities:

(i) $K'_{i_1 t_1}$ is comparable with $K'_{i_2 t_2}$,

(ii) $K'_{i_1 t_1} \cap K'_{i_2 t_2} = \{0\}$.

Hence we can choose a finite number of these l-subgroups K'_{it} which will be denoted as K'_j ($j \in J$; J finite) such that (cf. (8))

$$K = \bigvee^c_{j \in J} K'_j$$

and the system $\{K'_j\}_{j \in J}$ is disjoint. This implies that $K = \sum_{j \in J} K'_j$.

3.9. Lemma. *H_2 does not belong to Q^s.*

Proof. By way of contradiction, assume that H_2 belongs to Q^s. Hence in view of (7) and Lemma 3.3 there exists a positive integer i such that $H_2 \in Q_i$. Let i be the least positive integer having this property.

Suppose that $i = 1$. According to 3.3 and Lemma 5.4 in [4] we must have $H_2 \in Q'_1$. Hence there is $K \in Q_0 = Q$ such that H_2 is isomorphic to an l-subgroup H'_2 of K; without loss of generality we can suppose that $H'_2 = H_2$. Let K_1 be the convex subgroup of K generated by the element e. Clearly $K_1 \in Q$.

According to 3.2 and 3.4 the radical class Q is generated by a class of linearly ordered groups. Thus in view of Propos. 3.4, [8] K_1 is a direct sum of linearly ordered groups D_i ($i \in I$). Moreover, H_2 is an l-subgroup of K_1. Each D_i belongs to Q. From 3.6 we conclude that there is $i \in I$ such that H_2 is isomorphic to D_i. Therefore H_2 belongs to Q, which is a contradiction (cf. 3.5).

Now suppose that $i > 1$. Then according to 3.3 and Lemma 5.4 in [4] we have $H_2 \in Q'_i$. Thus there exists $K \in Q_{i-1}$ such that $H_2 \in \operatorname{Sub}\{K\}$. We may suppose that H_2 is an l-subgroup of K. Let K_1 be the convex l-subgroup of K generated by the element e. Because Q_{i-1} is a radical class, we have $K_1 \in Q_{i-1}$. At the same time, H_2 is an l-subgroup of K_1.

There exist $K_i \in Q'_{i-1}$ ($i \in I$) such that $K_1 = \bigvee_{i \in I} K_i$, $K_i \in c(K_1)$ for each $i \in I$. We apply 3.7 and 3.8; let K_j ($j \in J$) be as in 3.8. Then all K_j belong to Q'_{i-1} as well. According to 3.8 and 3.6 there exists $j \in J$ such that H_2 is isomorphic to an l-subgroup of K_j. At the same time, there exists $K'_j \in Q_{i-2}$ such that K_j is an l-subgroup of K'_j. Hence H_2 is isomorphic to an l-subgroup of K'_j and therefore $H_2 \in Q'_{i-1}$, which is a contradiction with respect to the minimality of i.

Since $Q^s = R^{hs}$, from 3.9 and from the relation $H_2 \in R^{sh}$ we obtain:

3.10. Proposition. $R^{sh} \neq R^{hs}$.

In view of the construction introduced in Section 2, the linearly ordered group G and the radical class R were defined by means of a cardinal α, $\alpha > \aleph_0$. Let us now write $G(\alpha)$ and $R(\alpha)$ instead of G and R.

By using Lemma 5.4 in [4] we can easily verify that if α and β are cardinals with $\aleph_0 < \alpha < \beta$, then $G(\beta)$ does not belong to $R(\alpha)$. As a corollary we obtain:

3.11. Proposition. *Let α and β be cardinals, $\aleph_0 < \alpha < \beta$. Then* $R(\alpha) \neq R(\beta)$.

Let C be the class of all cardinals greater than $\aleph_0$. In view of 3.11 there exists an injective mapping of the class C into the class $\mathscr{R}_1$ (cf. Section 1 for the notation); from this we infer that Theorem 1.1 is valid.

References

[1] *P. Conrad:* *K*-radical classes of lattice ordered groups. Algebra Carbondale 1980, Lecture Notes in Mathematics 848, Springer Verlag 1981, 186—207.

[2] *P. Conrad:* The structure of lattice ordered groups with a finite number of disjoint elements. Michigan Math. Journ. *7*, 1960, 171—180.

[3] *P. Conrad:* Lattice ordered groups, Tulane Lecture Notes, Tulane University, 1970.

[4] *M. Darnel:* Closure operators on radicals of lattice ordered groups. Czechoslov. Math. J. *37*, 1987, 51—64.

[5] *L. Fuchs:* Partially ordered algebraic systems, Pergamon Press, Oxford 1963.

[6] *W. C. Holland:* Varieties of l-groups are torsion classes. Czechoslov. Math. J. *29*, 1979, 11—12.

[7] *J. Jakubík:* Radical mappings and radical classes of lattice ordered groups. Symposia Mathematica 21, Academic Press 1977, 451—477.

[8] *J. Jakubík:* Products of radical classes of lattice ordered groups. Acta Math. Univ. Comenianae *39*, 1980, 31—41.

[9] *J. Jakubík:* On *K*-radical classes of lattice ordered groups. Czechoslov. Math. J. *33*, 1983, 149—163.

[10] *N. Ja. Medvedev:* On the lattice of radicals of a finitely generated l-group. (In Russian.) Math. Slovaca *33*, 1983, 185—188.

Author's address: 040 01 Košice, Ždanovova 6, Czechoslovakia (Matematický ústav SAV).