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A NEW AND MORE POWERFUL CONCEPT OF THE PU-INTEGRAL

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0. INTRODUCTION

Let $f: \mathbb{R}^n \to \mathbb{R}$ have compact support. The PU-integral $(PU) \int f(x)\, dx$ was introduced [1] as a limit (in a specific sense) of integral sums $\sum_{j=1}^{k} f(t^j) \int \vartheta_j(x)\, dx$, $\{\vartheta_1, \vartheta_2, \ldots, \vartheta_k\}$ being a partition of unity (hence the PU-integral). The limiting process involved in the definition of the PU-integral resulted in the following properties of the PU-integral:

(0.1) $(PU) \int f(x)\, dx \in \mathbb{R}$ for every PU-integrable f.

(0.2) The map $f \mapsto (PU) \int f(x)\, dx$ is linear (on the set of PU-integrable functions).

(0.3) If $f: \mathbb{R}^n \to \mathbb{R}$ has compact support and is Lebesgue integrable, then it is PU-integrable and the two integrals coincide.

(0.4) The PU-integral is a true extension of the Lebesgue integral, since f is PU-integrable and $(PU) \int f(x)\, dx = 0$ if there exists such a $g: \mathbb{R}^n \to \mathbb{R}$ that g has compact support, is differentiable at every $x \in \mathbb{R}^n$ and $f = \partial g/\partial x_1$. It is not difficult to find such a g that $\int |f(x)|\, dx = \infty$ so that, in general, the PU-integral is a nonabsolutely convergent integral.

(0.5) The usual transformation formula holds for diffeomorphisms and the PU-integral. This property makes it possible to extend the PU-integration to differentiable manifolds.

(0.6) Stokes' theorem can be proved on differentiable manifolds for $(n-1)$-forms which are differentiable at every point (or in $\mathbb{R}^n$ for vector fields which are differentiable at every point).

However, the assumption in (0.4) that g is to be differentiable at every point is essential; if it is dropped for a single point and replaced by the assumption of continuity of g at this particular point then $(PU) \int f(x)\, dx$ need not exist, and a similar situation takes place with Stokes' theorem in (0.6).

The aim of this paper is to relax the limiting process in the definition of the PU-integral in such a way that weaker conditions on g in (0.4) be sufficient for the existence of $(PU) \int f(x)\, dx$: It is sufficient to assume that g is differentiable at every $x \in \mathbb{R}^n \setminus W$ provided one of the following conditions holds:

(0.7) W is a hyperplane and g is continuous at every point of W (in fact, W may be an $(n - 1)$-dimensional manifold);
(0.8) W is a small set (in the sense of (5.4)) and g is bounded;
(0.9) W is a one-point set, $W = \{w\}$, and $|g(x)| = o(\|x - w\|^{1-n})$ in a neighbourhood of w.

Moreover, we prove that the product $f\chi$ is PU-integrable provided f is PU-integrable and χ is of class C^1.

Section 1 contains some auxiliary concepts and results, in Section 2 the definition of the PU-integral is introduced, and in the subsequent sections transformation of the PU-integral, multiplication of PU-integrable functions and Stokes' theorem are treated.

First version of this treatment was published as a preprint [2]. However, since then the manuscript has undergone substantial changes concerning the fundamental definitions as well as the organization of the proofs.

1. PU-PARTITIONS

If $M \subset \mathbb{R}^n$, we denote by ∂M, Int M and Cl M (or $\overline{M}$) the boundary, interior and closure of M, respectively. The Euclidean space $\mathbb{R}^n$ is viewed as a Hilbert space, that is, we set

$$\|x\| = (x_1^2 + \ldots + x_n^2)^{1/2} ,$$
$$B(y, \alpha) = \{x \in \mathbb{R}^n;\ \|x - y\| < \alpha\} ,$$

and represent linear functionals as vectors: for example, $\varphi(x) = \sum_{i=1}^{n} \varphi_i x_i$ with $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{R}^n$. If $f\colon \mathbb{R}^n \to \mathbb{R}$ then supp f stands for the support of f, Df is its differential,

$$\|Df\| = \left(\sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2\right)^{1/2} .$$

1.1. Definition (cf. [1]). Let $M \subset \mathbb{R}^n$ be compact. A family

(1.1) $$\Delta = \{(t^j, \vartheta_j);\ j = 1, \ldots, k\}$$

where k is a positive integer, $t^j \in M$, $\vartheta_j\colon \mathbb{R}^n \to [0, 1]$ are C^1-functions with compact supports satisfying

(1.2) $$0 \leqq \vartheta(t) = \sum_{j=1}^{k} \vartheta_j(t) \leqq 1 \quad \text{for all} \quad t \in \mathbb{R}^n ,$$

(1.3) $$\operatorname{Int}\{t \in \mathbb{R}^n;\ \vartheta(t) = 1\} \supset M ,$$

is called *a PU-partition of* M. (The letters PU stand for "partition of unity". For technical reasons, a finite set different from $\{1, 2, \ldots, k\}$ is sometimes used as the index set for a PU-partition.)

Any function $\delta\colon M \to (0, +\infty)$ will be called *a gauge on* M.

If δ is a gauge on M, then the PU-partition (1.1) is said to be *δ-fine* if

$$\text{supp}\, \vartheta_j \subset B(t^j, \delta(t^j)), \quad j = 1, \ldots, k. \tag{1.4}$$

For Δ defined by (1.1) denote

$$\varrho_j = \sup \{\|x - t^j\|;\ x \in \text{supp}\, \vartheta_j\}.$$

Then Δ is δ-fine iff $\varrho_j < \delta(t^j)$, $j = 1, \ldots, k$.

Let $\alpha > 0$, $K > 1$ be constants. We introduce the following conditions concerning the PU-partition (1.1):

$$\vartheta_j(x) < (1 + \alpha)\, \vartheta_j(t^j) \quad \text{for} \quad x \in \mathbb{R}^n; \tag{1.5}$$

$$\vartheta_j(x) = \vartheta_j(t^j) \quad \text{for} \quad x \in \bar{B}(t^j, \varrho_j/K); \tag{1.6}$$

$$\int \|D\, \vartheta_j(x)\|\, dx < K/\varrho_j \int \vartheta_j(x)\, dx. \tag{1.7}$$

Notice that (1.5), (1.7) immediately imply

$$\int \|D\, \vartheta_j(x)\|\, dx < \varkappa_1 K(1 + \alpha)\, \varrho_j^{n-1}\, \vartheta_j(t^j), \tag{1.7*}$$

where $\varkappa_1 = \int_{B(0,1)} dx$.

1.2. Remark. Notice that the integration in (1.7), (1.7*) is in fact over a compact set. Throughout the paper, we will always omit the specification of the integration domain provided it is the whole $\mathbb{R}^n$.

1.3. Proposition. *For every positive integer n there is a constant $\varkappa = \varkappa(n) \geqq 2$ such that for every compact set $M \subset \mathbb{R}^n$ and every gauge δ on M there is a δ-fine PU-partition Δ defined by* (1.1) *satisfying*

$$\vartheta_j(x) \leqq 1 \quad \textit{for} \quad x \in \mathbb{R}^n, \tag{1.5'}$$

$$\vartheta_j(x) = 1 \quad \textit{for} \quad x \in \bar{B}(t^j, \varrho_j/\varkappa(n)), \tag{1.6'}$$

$$\int \|D\, \vartheta_j(x)\|\, dx < \varkappa(n)/\varrho_j \int \vartheta_j(x)\, dx. \tag{1.7'}$$

Proof. Since in the proof we deal mostly with intervals and their unions, it is more convenient to make use of the maximum norm (parallelly with the Euclidean one). We denote

$$|x| = \max \{|x_i|;\ i = 1, 2, \ldots, n\},$$

$$U(t, \delta) = \mathop{\times}_{i=1}^{n} [t_i - \delta, t_i + \delta] \quad \text{(a closed cube with center } t \text{ and edge } 2\delta).$$

Following the idea of proof of Proposition 1.1 [1], we shall first find a system

$$\tilde{\Delta} = \{(t^j, D^j);\ j = 1, \ldots, k\} \tag{1.8}$$

where $D^j \subset \mathbb{R}^n$ and $t^j \in D^j$ satisfy the conditions

$$M \subset \text{Int} \bigcup_{j=1}^{k} D^j; \tag{1.9}$$

(1.10) Int $D^i \cap$ Int $D^j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, k$;

(1.11) each D^j, $j = 1, \ldots, k$, is the union of a finite number of compact intervals;

if $\tilde{\varrho}_j = \sup \{\|x - t^j\|;\ x \in D^j\}$, then for $j = 1, \ldots, k$ we have

(1.12) $$\tilde{\varrho}_j \leqq \tfrac{1}{2}\delta(t^j)\,;$$

(1.13) $$U(t^j, \tfrac{1}{4}\tilde{\varrho}_j) \subset D^j \subset U(t^j, \tilde{\varrho}_j)\,;$$

(1.14) $$m_{n-1}(\partial D^j) \leqq \tilde{\varkappa}(n)\, \tilde{\varrho}_j^{-1}\, m_n(D^j)$$

where m_ν stands for the ν-dimensional Lebesgue measure and $\tilde{\varkappa}(n)$ is a constant depending only on the dimension n.

We shall describe an algorithm which results in such a partition. Choose a decreasing sequence $\frac{1}{2} > \eta_1 > \eta_2 > \ldots > \eta_l > \ldots > \frac{1}{4}$.

Step 1: Find $t^1 \in M$ such that

$$\delta(t^1) > \frac{\eta_2}{\eta_1} \sup \{\delta(t);\ t \in M\}$$

and denote

$$W_1 = U(t^1, \tfrac{1}{2}\eta_1\, \delta(t^1))\,, \quad U_1 = U(t^1, \eta_1\, \delta(t^1))\,,$$
$$V_{11} = U_1\,.$$

Let us assume that after l steps we have points t^j, $j = 1, \ldots, l$ and sets W_j, U_j, V_{jm}, $j = 1, \ldots, l$, $m = j, j+1, \ldots, l$. If $M \setminus \operatorname{Int} \bigcup_{j=1}^{l} U_j = \emptyset$, the algorithm stops. Otherwise, the algorithm is continued by

Step $(l + 1)$: Find

(1.15) $$t^{l+1} \in M \setminus \operatorname{Int} \bigcup_{j=1}^{l} U_j$$

such that

(1.16) $$\delta(t^{l+1}) > \frac{\eta_{l+2}}{\eta_{l+1}} \sup \{\delta(t);\ t \in M \setminus \operatorname{Int} \bigcup_{j=1}^{l} U_j\}$$

and set

(1.17) $$W_{l+1} = U(t^{l+1}, \tfrac{1}{2}\eta_{l+1}\, \delta(t^{l+1}))\,,$$
$$U_{l+1} = U(t^{l+1}, \eta_{l+1}\, \delta(t^{l+1}))\,,$$
$$V_{j,l+1} = V_{j,l} \setminus \operatorname{Int} W_{l+1}\,, \quad V_{l+1,l+1} = U_{l+1} \setminus \operatorname{Int} \bigcup_{j=1}^{l} V_{j,l+1}\,.$$

It is clear from the construction that each $V_{j,m}$, $m = j, j+1, \ldots, l+1$ is the union of a finite number of intervals and that the sets $V_{1m}, V_{2m}, \ldots, V_{mm}$ are nonoverlapping. Moreover, it is seen from (1.15), (1.16) that

(1.18) $$\eta_1\, \delta(t^1) > \eta_2\, \delta(t^2) > \ldots > \eta_{l+1}\, \delta(t^{l+1})\,.$$

By (1.15) we have $t^r \in M \setminus \operatorname{Int} U_j$ for $j < r \leqq m$, so that

(1.19) $$\|t^r - t^j\| \geqq \eta_j\, \delta(t^j)\,.$$

Further, it can be proved that the system of sets resulting by the algorithm has the following properties:

(1.20) $$W_j \cap W_m = \emptyset \quad \text{for} \quad j < m \leqq l + 1 ;$$

(1.21) $$W_j \subset V_{jm} \subset U_j \quad \text{for} \quad j \leqq m \leqq l + 1 ;$$

(1.22) $$\bigcup_{j=1}^{m} V_{jm} = \bigcup_{j=1}^{m} U_j \quad \text{for} \quad m = 1, \ldots, l .$$

(The proof is rather technical but not difficult.)

Now we will prove that the algorithm comes to an end after a finite number k of steps because of

$$M \subset \operatorname{Int} \bigcup_{j=1}^{k} U_j .$$

Suppose the contrary. Since M is compact and the sets W_m are pairwise disjoint (cf. (1.20)), the sum $\sum_{m=1}^{\infty} \eta_m^n \, \delta^n(t^m)$ converges and since $\eta_m > \frac{1}{4} > 0$ we have

(1.23) $$\lim_{m \to \infty} \delta(t^m) = 0 .$$

There is $s \in \bigcap_{m=1}^{\infty} (M \setminus \operatorname{Int} \bigcup_{j=1}^{m} U_j)$ since the sets on the right hand side are nonempty and compact. However, (1.16) implies that $\delta(t^m) > \eta_{m+1} \, \delta(s)/\eta_m > \frac{1}{2} \, \delta(s) > 0$ since $\frac{1}{2} > \eta_m > \eta_{m+1} > \frac{1}{4}$. This contradicts (1.23) and consequently, the algorithm stops after a finite number k of steps.

Set $D^j = V_{jk}, j = 1, \ldots, k$.

Then the system (1.8) satisfies (1.9)–(1.14). Indeed, (1.9)–(1.11) follow from the construction, (1.12) and (1.13) follow from (1.21), (1.17) and the inequality $\frac{1}{4} < \eta_j < \frac{1}{2}$. The only point requiring a detailed discussion of its proof is (1.14).

Let us first introduce two lemmas.

Lemma 1. *Let $a \in \mathbb{R}$, $p \in \{1, \ldots, n\}$, and denote*

$$Y_{pa} = \{x \in \mathbb{R}^n ; \; x_p \geqq a\} .$$

Let $j < m \leqq k$, $\max \{t_p^j, t_p^m\} < a$, $U_j \cap Y_{pa} \neq \emptyset \neq U_m \cap Y_{pa}$. Then

$$\| t^j - t^m \| = \max \{ |t_i^j - t_i^m| ; \; i \neq p, \; i = 1, \ldots, n\} .$$

Proof. Since $a - \eta_j \, \delta(t^j) \leqq t_p^j < a$, $a - \eta_m \, \delta(t^m) \leqq t_p^m < a$, $\eta_j \, \delta(t^j) > \eta_m \, \delta(t^m) > 0$, we have

$$|t_p^m - t_p^j| < \eta_j \, \delta(t^j) ;$$

but (1.19) implies that

$$|t_i^m - t_i^j| \geqq \eta_j \, \delta(t^j) \quad \text{for at least one} \quad i \in \{1, \ldots, n\} .$$

Lemma 2. *Define* $\omega(1) = 4$, $\omega(r + 1) = 2(r + 1) \, \omega(r) + 2^{r+1}$ *for* $r = 1, 2, \ldots$. *Let* $1 < m \leqq k$, $q \in \mathbb{R}^n$, $Q = U(q, \eta_m \, \delta(t^m))$,

$$L = \{l ; \; l < m, \; U_l \cap Q \neq \emptyset\} .$$

Then, denoting by $|L|$ the number of elements of L, we have

$$|L| \leqq \omega(n) . \tag{1.24}$$

Proof proceeds by induction on n.

Let $n = 1$. Denote

$$\begin{aligned} L_1 &= \{l \in L;\ q - \eta_m \delta(t^m) \leqq t^l \leqq q\} , \\ L_2 &= \{l \in L;\ q \leqq t^l \leqq q + \eta_m \delta(t^m)\} , \\ L_3 &= \{l \in L;\ t^l < q - \eta_m \delta(t^m)\} , \\ L_4 &= \{l \in L;\ q + \eta_m \delta(t^m) < t^l\} . \end{aligned}$$

For $i = 1, 2$, (1.18) and (1.19) imply that L_i contains at most one element (recall that $l < m$). The same holds for $i = 3, 4$. Indeed, suppose e.g. that $j, r \in L_3$, $j < r$. As in the proof of Lemma 1 we have

$$q - \eta_m \delta(t^m) - \eta_j \delta(t^j) \leqq t^j < q - \eta_m \delta(t^m)$$

and analogously with j replaced by r; hence

$$|t^j - t^r| < \max \{\eta_j \delta(t^j),\ \eta_r \delta(t^r)\} = \eta_j \delta(t^j)$$

but this inequality contradicts (1.19). The proof for L_4 is analogous, hence $|L| \leqq 4 = \omega(1)$.

Now suppose that (1.24) holds for $n \leqq \nu$. Let $n = \nu + 1$ and put

$$\begin{aligned} L_0 &= \{l \in L;\ \|t^l - q\| \leqq \eta_m \delta(t^m)\} = \{l \in L,\ t^l \in Q\} , \\ L_{-i} &= \{l \in L;\ t^l_i < q_i - \eta_m \delta(t^m)\} , \\ L_i &= \{l \in L;\ q_i + \eta_m \delta(t^m) < t^l_i\} , \end{aligned}$$

$i = 1, \ldots, \nu + 1$. Then

$$L = \bigcup_{i=-(\nu+1)}^{\nu+1} L_i .$$

First we estimate $|L_0|$. By halving all edges of Q we obtain $2^{\nu+1}$ cubes with edges of length $\eta_m \delta(t^m)$; since $l < m$ for $l \in L$, (1.18) and (1.19) imply that each of these cubes contains at most one t^l with $l \in L$, hence $|L_0| \leqq 2^{\nu+1}$.

For $i \in \{1, \ldots, \nu + 1\}$ let P_i denote the i-th projection, i.e.

$$\begin{aligned} P_i x &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\nu+1}) , \\ P_i M &= \{P_i x;\ x \in M\} \quad \text{for} \quad M \subset \mathbb{R}^{\nu+1} . \end{aligned}$$

If $j, r \in L_i$, $j < r$, then applying Lemma 1 we obtain (using again (1.19))

$$\|P_i t^j - P_i t^r\| = \|t^j - t^r\| \geqq \eta_j \delta(t^j) .$$

At the same time, the definition of L obviously yields the inclusion

$$L_i \subset \{l;\ l < m,\ P_i U_l \cap P_i Q \neq \emptyset\} .$$

The dimension of $P_i U_l$, $P_i Q$ being ν, we can apply Lemma 2 concluding that $|L_i| \leqq$

$\leqq \omega(v)$ and, quite analogously, $|L_{-i}| \leqq \omega(v)$. Hence

$$|L| \leqq \sum_{i=-(v+1)}^{v+1} |L_i| \leqq 2(v+1)\,\omega(v) + 2^{v+1} = \omega(v+1)\,.$$

Let us now proceed to the proof proper of (1.14). First we shall prove the inclusion

$$\partial V_{jk} \subset \bigcup_{i=1}^{j} \partial U_i \cup \bigcup_{p=1}^{k} \partial W_p\,, \quad j = 1, \ldots, k\,. \tag{1.25}$$

Using induction on k, we notice that for $k = 1$,

$$\partial V_{11} = \partial U_1 \subset \partial U_1 \cup \partial W_1\,.$$

If (1.25) holds for some k, then for $j \leqq k$ we have $\partial V_{j,k+1} = \partial(V_{jk} \setminus \operatorname{Int} W_{k+1})$. Using the elementary inclusion $\partial(A \setminus B) \subset \partial A \cup \partial B$ and the induction hypothesis we obtain

$$\partial V_{j,k+1} \subset \partial V_{jk} \cup \partial W_{k+1} \subset \bigcup_{i=1}^{j} \partial U_i \cup \bigcup_{p=1}^{k} \partial W_p \cup \partial W_{k+1} = \bigcup_{i=1}^{j} \partial U_i \cup \bigcup_{p=1}^{k+1} \partial W_p\,.$$

Finally, applying this inclusion with $j \leqq k$ we conclude

$$\partial V_{k+1,k+1} = \partial(U_{k+1} \setminus \operatorname{Int} \bigcup_{j=1}^{k} V_{j,k+1}) \subset \partial U_{k+1} \cup \bigcup_{j=1}^{k} \partial V_{j,k+1} \subset$$

$$\subset \partial U_{k+1} \cup \bigcup_{j=1}^{k} \partial U_j \cup \bigcup_{p=1}^{k+1} \partial W_p = \bigcup_{j=1}^{k+1} \partial U_j \cup \bigcup_{p=1}^{k+1} \partial W_p\,.$$

The proof of (1.25) is complete.

Denote $Z(j) = \{i;\, i < j,\ U_i \cap U_j \neq \emptyset\}$; by Lemma 2 we have $|Z(j)| \leqq \omega(n)$. Since $V_{jk} \subset U_j$ for $j = 1, \ldots, k$ and the sets U_i, W_p are compact intervals, we can rewrite (1.25) as

$$\partial V_{jk} \subset \bigcup_{i \in Z(j)} (U_j \cap \partial U_i) \cup \bigcup_{i \in Z(j)} (U_j \cap \partial W_i) \cup \partial U_j \cup \bigcup_{p=j}^{k} (U_j \cap \partial W_p)\,. \tag{1.26}$$

Taking into account the elementary inequality

$$\max\{m_{n-1}(U_j \cap \partial U_i),\ m_{n-1}(U_j \cap \partial W_i)\} \leqq m_{n-1}(\partial U_j)\,, \quad i < j$$

(recall that U_i, U_j, W_i are intervals) and the inequality

$$m_{n-1}(U_j \cap \partial W_p) \leqq 2n\, m_{n-1}(\partial U_j \cap W_p)\,, \quad p > j \tag{1.27}$$

(its proof is sketched in Remark 1.4 at the end of this section) we conclude from (1.26), (1.20) and the inequality $|Z(j)| \leqq \omega(n)$ that

$$m_{n-1}(\partial V_{jk}) \leqq 2\,\omega(n)\, m_{n-1}(\partial U_j) + 2n\, m_{n-1}(\partial U_j) + m_{n-1}(\partial U_j) \leqq$$

$$\leqq [2\,\omega(n) + 2n + 1]\, m_{n-1}(\partial U_j)\,.$$

By virtue of (1.17) we have

$$m_{n-1}(\partial U_j) = 2n[2\eta_j\, \delta(t^j)]^{n-1} = \frac{2^n n}{\eta_j\, \delta(t^j)}\, m_n(W^j)$$

and, since $D^j = V_{jk}$, it follows from (1.21) and the definition of $\tilde{\varrho}_j$ that

$$m_{n-1}(\partial D^j) \leqq \frac{1}{\tilde{\varrho}_j}\, 2^n n[2\,\omega(n) + 2n + 1]\, m_n(D^j)\,.$$

This completes the proof of (1.14) with the constant $\tilde{\varkappa}(n) = 2^n n[2\,\omega(n) + 2n + 1]$.

Using the just constructed system (1.8), we can find the desired PU-partition (1.1) analogously to [1], using smooth approximations of the characteristic functions of the sets D^j as the functions ϑ_j. The properties of (1.8), in particular (1.12)–(1.14), imply that (1.1) obtained in the suggested manner satisfies (1.2)–(1.4) and (1.5')–(1.7') with a constant $\varkappa(n) > \tilde{\varkappa}(n)$, say $\varkappa(n) = \tilde{\varkappa}(n) + 1$.

The proof of Proposition is complete, which justifies the definition which we will introduce in the next section.

1.4. Remark. Let us sketch the proof of the inequality (1.27). Since (1.27) evidently holds if $\operatorname{Int} U_j \cap \operatorname{Int} W_p = \emptyset$, we may assume without loss of generality that

$$\operatorname{Int} U_j \cap \operatorname{Int} W_p \neq \emptyset\,. \tag{1.28}$$

Obviously $U_j \cap \partial W_p \subset \partial(U_j \cap W_p)$, so that $m_{n-1}(U_j \cap \partial W_p) \leqq m_{n-1}(\partial(U_j \cap W_p))$ and it is sufficient to prove

$$m_{n-1}(\partial(U_j \cap W_p)) \leqq 2n\, m_{n-1}(\partial U_j \cap W_p)\,. \tag{1.29}$$

We have (by the definitions of U_j, W_p, $p > j$, (1.18) and (1.28))

$$U_j \cap W_p = \mathop{\times}_{i=1}^{n} [\alpha_i, \beta_i]\,,$$

where for every i one of the following cases occurs:

(i) $[\alpha_i, \beta_i] = [t_i^p - \frac{1}{2}\eta_p\,\delta(t^p),\ t_i^p + \frac{1}{2}\eta_p\,\delta(t^p)]$,
(ii) $[\alpha_i, \beta_i] = [t_i^p - \frac{1}{2}\eta_p\,\delta(t^p),\ t_i^j + \eta_j\,\delta(t^j)]$, $t_i^j < t_i^p - \frac{1}{2}\eta_p\,\delta(t^p)$,
(iii) $[\alpha_i, \beta_i] = [t_i^j - \eta_j\,\delta(t^j),\ t_i^p + \frac{1}{2}\eta_p\,\delta(t^p)]$, $t_i^p + \frac{1}{2}\eta_p\,\delta(t^p) < t_i^j$.

Moreover, $\beta_i - \alpha_i > 0$ for every i, $\beta_i - \alpha_i < \eta_p\,\delta(t^p)$ in cases (ii) and (iii), and there exists at least one i such that either case (ii) or (iii) occurs (cf. (1.19)). Put

$$F_i^+ = \{x;\ x_i = \beta_i,\ x_l \in [\alpha_l, \beta_l] \ \text{ for } \ l \neq i\}\,,$$
$$F_i^- = \{x;\ x_i = \alpha_i,\ x_l \in [\alpha_l, \beta_l] \ \text{ for } \ l \neq i\}\,.$$

F_i^+ and F_i^- are all faces of $U_j \cap W_p$. Find such an s that

$$\beta_s - \alpha_s = \min_i\,(\beta_i - \alpha_i)\,.$$

Then one of cases (ii), (iii) occurs for $i = s$; for instance, let it be case (ii).

Then $F_s^+ \subset \partial U_j$, $m_{n-1}(F_s^+) \geqq m_{n-1}(F_i^+)$, $m_{n-1}(F_i^+) = m_{n-1}(F_i^-)$, $i = 1, 2, \ldots, n$ and (1.29) follows since $F_s^+ \subset W_p$ evidently holds.

Proposition proved in the previous section justifies the following definition.

2.1. Definition. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function with compact support. For a PU-partition (1.1) of supp f, set

$$S(f, \Delta) = \sum_{j=1}^{k} f(t^j) \int \vartheta_j(x)\, dx\,. \tag{2.1}$$

Let $q \in \mathbb{R}$ satisfy the following condition:

for every $\varepsilon > 0$ there is $\alpha > 0$ such that for every $K > 1$ there is a gauge δ on supp f such that

$$|q - S(f, \Delta)| \leqq \varepsilon$$

for every δ-fine PU-partition (1.1) of supp f which satisfies (1.5)–(1.7).

Then f is said to be *PU-integrable*, q is its *PU-integral* and we write

$$q = (\mathrm{PU}) \int f\, dx\,.$$

2.2. Remarks. 1. Definition 2.1 has good sense since Proposition 1.3 guarantees – for any gauge δ and every $\alpha > 0$, $K \geqq \varkappa(n)$ – existence of δ-fine PU-partitions satisfying (1.5)–(1.7).

2. It is the small values of ε and α, and large values of K which are important, as is immediately seen from Definition 2.1. Consequently, in our considerations we may restrict ourselves, without affecting the definition, to values $\varepsilon < \varepsilon_0$, $\alpha < \alpha_0$, $K > K_0$, where $\varepsilon_0 > 0$, $\alpha_0 > 0$, $K_0 \geqq 1$ are arbitrary but fixed constants. In particular, it is of no consequence that for $K < \varkappa(n)$ there need not exist PU-partitions with the desired properties.

The notion of PU-integral was introduced in [1] by an analogous definition in which the conditions (1.5)–(1.7) were replaced by

$$\sum_{j=1}^{k} \int \|x - t^j\| \, \|D\, \vartheta_j(x)\| \, dx \leqq K\,. \tag{2.2}$$

It is easy to verify that (1.7) implies (2.2) (with K enlarged if necessary), hence every function PU-integrable in the sense of [1] is PU-integrable in the sense of the above definition (and the two integrals coincide). Since the PU-integral from [1] is a true extension of the Lebesgue integral, so is the PU-integral from Definition 2.1. From now on, we shall stick to our Definition 2.1 when dealing with PU-integrability.

The PU-integral evidently has the following properties:

(i) the PU-integral of a nonnegative PU-integrable function is nonnegative;

(ii) if f is PU-integrable, $c \in \mathbb{R}$, then cf is PU-integrable and $(\mathrm{PU}) \int cf\, dx = c(\mathrm{PU}) \int f\, dx$.

However, to prove additivity we have to proceed analogously as in [1], introducing a modified notion of the PUI-integral.

2.3. Definition. Let $\mathscr{I}$ be a compact interval in $\mathbb{R}^n$, let $f\colon \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} f \subset \operatorname{Int} \mathscr{I}$. Let $q \in \mathbb{R}$ satisfy the condition from Definition 2.1 with the only change that (1.1) is a PU-partition of $\mathscr{I}$ (instead of $\operatorname{supp} f$). Then f is said to be *PUI-integrable*, q is its *PUI-integral* and we write $q = (\mathrm{PUI}) \int f \, \mathrm{d}x$.

A proof that $f_1 + f_2$ is PUI-integrable and $(\mathrm{PUI}) \int (f_1 + f_2) \, \mathrm{d}x = (\mathrm{PUI}) \int f_1 \, \mathrm{d}x + (\mathrm{PUI}) \int f_2 \, \mathrm{d}x$ provided f_i are PUI-integrable, $\operatorname{supp} f_i \subset \operatorname{Int} \mathscr{I}$ for $i = 1, 2$, is straightforward. In the next theorem we assert the equivalence of Definitions 2.1 and 2.3 (hence also the independence of the PUI-integral of the choice of the interval $\mathscr{I}$). Thus, this theorem yields additivity in the above sense also for the PU-integral.

2.4. Theorem. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ have compact support* $\operatorname{supp} f \subset \operatorname{Int} \mathscr{I} \subset \mathbb{R}^n$, *$\mathscr{I}$ a compact interval. Then f is PU-integrable if and only if it is PUI-integrable and*

$$(\mathrm{PU}) \int f \, \mathrm{d}x = (\mathrm{PUI}) \int f \, \mathrm{d}x$$

holds provided one of the integrals exists.

Proof. The "only if" part is easy; we refer the reader to [1] for details. The main step of the proof is the restriction of a δ-fine PU-partition of the interval $\mathscr{I}$ to a δ-fine PU-partition of $\operatorname{supp} f$. Such a restriction is trivial if we assume (which we may) that $\bar{B}(x, \delta(x)) \cap \operatorname{supp} f = \emptyset$ for $x \in \mathscr{I} \setminus \operatorname{supp} f$.

However, the "if" part consists primarily in the converse process, that is, in extending a δ-fine PU-partition of $\operatorname{supp} f$ to that of $\mathscr{I}$ without violating the requirements imposed on the "admissible" PU-partitions, which is a much more complicated matter. After preparatory Lemmas 2.5 and 2.6, the existence of such an extension is established in Lemma 2.8.

First we introduce three auxiliary function ψ, μ, ν. Let ψ satisfy the following conditions:

(i) $\psi\colon \mathbb{R} \to [0, 1]$ is of class C^1,

(ii) $\operatorname{supp} \psi = [-1, 1]$, $\psi(x) > 0$ for $x \in (-1, 1)$,

(iii) $\psi(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$,

(iv) $\psi(x) < 1$ for $\frac{1}{2} < |x| < 1$.

Further, let β be a real number, $0 < \beta < \frac{1}{2}$, and let μ satisfy the following conditions.

(v) $\mu\colon \mathbb{R} \to [0, 1 + 2\beta]$ is of class C^1,

(vi) $\operatorname{supp} \mu = [-1, 1]$, $\mu(x) > 0$ for $x \in (-1, 1)$,

(vii) $\mu(x) = 1 + x$ for $x \in [-\beta, \beta]$,

(viii) $\mu(x) < \min \{1 + x, 1 + 2\beta\}$ for $\beta < |x| < 1$.

Finally, let $\nu\colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\nu(u_1, u_2, \ldots, u_n) = \mu(u_1) \, \psi((u_2^2 + u_3^2 + \ldots + u_n^2)^{1/2}) \, .$$

We introduce the following constants:

$$\varkappa_1 = m_n(B(0, 1))$$

(the measure of the unit ball in $\mathbb{R}^n$),

$$\varkappa_2 = \max\{\|H\| \det H^{-1};\ H \in M_n,\ \|H - I\| \leqq \tfrac{1}{2}\},$$
$$\varkappa_3 = \max\{\det H;\ H \in M_n,\ \|H - I\| \leqq \tfrac{1}{2}\},$$
$$\varkappa_4 = \max\{\det H^{-1};\ H \in M_n,\ \|H - I\| \leqq \tfrac{1}{2}\},$$

where M_n is the set of all $(n \times n)$-matrices, I is the unit matrix;

$$\varkappa_5 = \int v(x)\,dx,$$
$$\varkappa_6 = \varkappa_5^{-1} \int \|D\, v(x)\|\,dx,$$
$$\varkappa_7 = \varkappa_2\varkappa_3\varkappa_6,$$
$$\varkappa_8 = 4\varkappa_1^{-1}\varkappa_4\varkappa_5.$$

2.5. Lemma. *Let $w \in \mathbb{R}^n$, $\sigma > 0$. Let $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and satisfy the conditions*

$$\Phi(w) = 0, \quad D\,\Phi(w) = I,$$
$$\|D\,\Phi(x) - I\| \leqq \tfrac{1}{2} \quad \text{for} \quad \|x - w\| \leqq \sigma.$$

Then

$$\Phi(\bar{B}(w, \tfrac{1}{3}\sigma)) \subset \bar{B}(0, \tfrac{1}{2}\sigma) \subset \Phi(\bar{B}(w, \sigma)) \subset \bar{B}(0, \tfrac{3}{2}\sigma) \tag{2.3}$$

(*the inclusions hold also with open balls instead of the closed ones*).

Proof. By assumption we have $\frac{1}{2} \leqq \|D\,\Phi(x)\| \leqq \frac{3}{2}$ provided $\|x - w\| \leqq \sigma$. Using the identity

$$\Phi(x) = \int_0^1 D\Phi(w + \lambda(x - w))\,d\lambda(x - w)$$

(recall that $\Phi(w) = 0$), we immediately obtain

$$\|\Phi(x)\| \leqq \tfrac{3}{2}\|x - w\|,$$

which yields the first and last inclusion in (2.3). To prove the middle inclusion, let $z \in \bar{B}(0, \frac{1}{2}\sigma)$, that is, $\|z\| \leqq \frac{1}{2}\sigma$. Set

$$x_0 = w, \quad x_{i+1} = x_i - \Phi(x_i) + z, \quad i = 1, 2, \ldots;$$

then

$$x_{i+2} - x_{i+1} = x_{i+1} - x_i - (\Phi(x_{i+1}) - \Phi(x_i)).$$

Substituting for $\Phi(x_{i+1}) - \Phi(x_i)$ from the integral identity analogous to that introduced above and proceeding in a standard manner we prove $\|x_{i+2} - x_{i+1}\| \leqq \leqq \frac{1}{2}\|x_{i+1} - x_i\|$ and, by induction, $\|x_i - w\| < \sigma$, $i = 0, 1, 2, \ldots$. Hence there is x, $x = \lim_{i\to\infty} x_i$, $\|x - w\| \leqq \sigma$. Since evidently $\Phi(x) = z$, the inclusion is proved.

2.6. Lemma. *Let $K_1 > 1$, $0 < \theta < \frac{1}{6}$, $w \in \mathbb{R}^n$. Let $\varphi_0: \mathbb{R}^n \to [0, 1]$ be of class C^1 with a compact support satisfying*

$$\varphi_0(w) \neq 0 \neq D\,\varphi_0(w),$$

and denote

$$\omega_0 = \sup\{\|x - w\|;\ x \in \operatorname{supp} \varphi_0\}.$$

Further, let

$$(2.4)\qquad \bar{B}\left(w, \frac{\omega_0}{K_1}\right) \subset \operatorname{supp} \varphi_0 ;$$

$$(2.5)\qquad \|D\,\varphi_0(x) - D\,\varphi_0(w)\| \leqq \tfrac{1}{2}\gamma \quad \textit{for} \quad x \in \bar{B}\left(w, \frac{\omega_0}{K_1}\right),$$

$$\textit{where} \quad \gamma = \|D\,\varphi_0(w)\| ;$$

$$(2.6)\qquad \varphi_0(x) \leqq (1 + \theta)\,\varphi_0(w)\,, \quad x \in \mathbb{R}^n ;$$

$$(2.7)\qquad \int \|D\,\varphi_0(x)\|\,\mathrm{d}x \leqq \frac{K_1}{\omega_0} \int \varphi_0(x)\,\mathrm{d}x\,.$$

Then for every constants β, K_2 *with* $0 < \beta < \frac{1}{2}$, $K_2 > \max\{9, (\varkappa_8\theta)^{1/(n+1)}\}$ *there are functions* $E\varphi_0, F\varphi_0\colon \mathbb{R}^n \to [0, 1]$ *of class* C^1 *such that*

$$(2.8)\qquad E\varphi_0 + F\varphi_0 = \varphi_0 ;$$

$$(2.9)\qquad E\,\varphi_0(x) > 0 \quad \textit{provided} \quad \varphi_0(x) > 0\,, \quad E\,\varphi_0(w) > \tfrac{1}{2}\,\varphi_0(w) ;$$

$$(2.10)\qquad E\,\varphi_0(x) = E\,\varphi_0(w) \quad \textit{for} \quad x \in \bar{B}\left(w, \frac{2\beta\omega_0}{3K_1K_2}\right);$$

$$(2.11)\qquad E\,\varphi_0(x) \leqq (1 + \theta)\left(1 - \frac{4}{K_2}\right)^{-1} E\,\varphi_0(w) ;$$

$$(2.12)\qquad \int \|DE\,\varphi_0(x)\|\,\mathrm{d}x \leqq \left(1 - \theta\varkappa_8 K_2^{-(n+1)}\right)^{-1} \left(K_1 + \theta\varkappa_7\varkappa_8 K_1 K_2^{-n}\right) .$$

$$.\ \omega_0^{-1} \int E\,\varphi_0(x)\,\mathrm{d}x\,.$$

Denote

$$\omega_1 = \sup\{\|x - w\|;\ x \in \operatorname{supp} F\varphi_0\}\,.$$

Then, moreover,

$$(2.13)\qquad \frac{2^{3/2}\,\omega_0}{3K_1K_2} \leqq \omega_1 \leqq \frac{2^{3/2}\omega_0}{K_1K_2} ;$$

$$(2.14)\qquad \bar{B}\left(w, \frac{\beta\omega_1}{3}\right) \subset \operatorname{supp} F\varphi_0 ;$$

$$(2.15)\qquad \|DF\,\varphi_0(x) - DF\,\varphi_0(w)\| \leqq \tfrac{1}{2}\gamma \quad \textit{for} \quad x \in \bar{B}\left(w, \frac{2\beta\omega_0}{3K_1K_2}\right);$$

$$(2.16)\qquad F\,\varphi_0(x) \leqq (1 + 2\beta)\,F\,\varphi_0(w) ;$$

$$(2.17)\qquad \int \|DF\,\varphi_0(x)\|\,\mathrm{d}x \leqq \frac{4\varkappa_4}{\omega_1} \int F\,\varphi_0(x)\,\mathrm{d}x\,.$$

Remarks. 1. Let us mention some simple consequences of (2.8)–(2.17). Since both $E\varphi_0$, $F\varphi_0$ are nonnegative, (2.8) together with (2.9) implies

$$(2.18)\qquad \operatorname{supp} E\varphi_0 = \operatorname{supp} \varphi_0\,.$$

By (2.13) we have $\frac{1}{6}\beta\omega_1 \leqq 2^{1/2}\beta\omega_0/(3K_1K_2)$ and thus (2.10) together with (2.8) yields $E\,\varphi_0(x) = E\,\varphi_0(w)$ and $DF\,\varphi_0(x) = D\,\varphi_0(x)$ for $x \in \bar{B}(w, \frac{1}{6}\beta\omega_1)$; hence $\gamma = \|D\,\varphi_0(w)\| = \|DF\,\varphi_0(w)\|$ and (2.15) may be modified to

$$\|DF\,\varphi_0(x) - DF\,\varphi_0(w)\| \leqq \tfrac{1}{2}\gamma \quad \text{for} \quad x \in \bar{B}(w, \tfrac{1}{6}\beta\omega_1)\,. \tag{2.19}$$

2. Notice that E, F are not uniquely determined by the conditions (2.8)–(2.17). However, in the course of proof of Lemma 2.6 formulas for $E\varphi_0$, $F\varphi_0$ will be given. This will enable us to view E, F as operators.

Proof of Lemma 2.6. Recall that we assume $\gamma = \|D\,\varphi_0(w)\| > 0$. Choose an orthonormal system in $\mathbb{R}^n$

$$e^1, e^2, \ldots, e^n \quad \text{with} \quad e^1 = \gamma^{-1} D\,\varphi_0(w)\,. \tag{2.20}$$

Introducing in $\mathbb{R}^n$ new coordinates corresponding to this orthonormal system we have

$$x = (x_1, x_2, \ldots, x_n) \Leftrightarrow x = x_1e^1 + x_2e^2 + \ldots + x_ne^n\,.$$

Define a mapping $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi\colon x \mapsto (\gamma^{-1}(\varphi_0(x) - \varphi_0(w)), (x - w, e^2), \ldots, (x - w, e^n))\,. \tag{2.21}$$

Assume $x \in \bar{B}(w, \omega_0/K_1)$, $y \in \mathbb{R}^n$. Then

$$D\,\Phi(x)\,y = \gamma^{-1}y_1 D\,\varphi_0(x) + (0, y_2, \ldots, y_n)\,,$$
$$[D\,\Phi(x) - D\,\Phi(w)]\,y = \gamma^{-1}y_1(D\,\varphi_0(x) - D\,\varphi_0(w))$$

and consequently,

$$\|D\,\Phi(x) - D\,\Phi(w)\| \leqq \gamma^{-1}\|D\,\varphi_0(x) - D\,\varphi_0(w)\| \leqq \tfrac{1}{2}$$

by (2.5). Hence Φ satisfies the assumptions of Lemma 2.5 with $\sigma = \omega_0/K_1$.

Lemma 2.5 implies that $\Phi\colon \bar{B}(w, \omega_0/K_1) \to \mathbb{R}^n$ is an injection, and thus (2.3) yields

$$\bar{B}\left(0, \frac{\sigma}{2}\right) \subset \Phi(\bar{B}(w, \sigma)) \subset \bar{B}\left(0, \frac{3\sigma}{2}\right), \tag{2.22}$$

$$\bar{B}\left(w, \frac{\sigma}{3}\right) \subset \Phi^{-1}\left(\bar{B}\left(0, \frac{\sigma}{2}\right)\right) \subset \bar{B}(w, \sigma)$$

for any σ, $0 < \sigma \leqq \omega_0/K_1$. Further, if $x \in \bar{B}(w, \omega_0/K_1)$, $u = \Phi(x)$, then we may write $u_1 = \gamma^{-1}(\varphi_0(x) - \varphi_0(w))$, hence

$$\varphi_0(x) = \varphi_0(w) + \gamma u_1 \tag{2.23}$$

and, since $u \in \Phi(\bar{B}(w, \omega_0/K_1))$ we may also write

$$\varphi_0(\Phi^{-1}(u)) = \left[\varphi_0(\Phi^{-1}(u)) - \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,u\right)\right] + \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,u\right) \tag{2.24}$$

or

$$\varphi_0(x) = \left[\varphi_0(x) - \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right)\right] + \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right).$$

Recalling the definition of v, we notice that

$$\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} \geqq 2^{1/2} \quad \text{implies} \quad v(v) = 0\,, \tag{2.25}$$

$$v\left(\frac{K_1K_2}{\omega_0}\,u\right) > 0 \quad \text{implies} \quad \|u\| < \frac{2^{1/2}\omega_0}{K_1K_2}\,.$$

Setting $\sigma = 2^{3/2}\omega_0/(K_1K_2)$ we have $0 < \sigma < \omega_0/K_1$ (since $K_2 > 9$). Hence the last inclusion in (2.22) reads

$$\Phi^{-1}\left(\bar{B}\left(0, \frac{2^{1/2}\omega_0}{K_1K_2}\right)\right) \subset \bar{B}\left(w, \frac{2^{3/2}\omega_0}{K_1K_2}\right)$$

and from the second implication in (2.25) we conclude

$$v\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right) = 0 \quad \text{provided} \quad \frac{2^{3/2}\omega_0}{K_1K_2} \leqq \|x - w\| \leqq \frac{\omega_0}{K_1}\,. \tag{2.26}$$

Let us define

$$F\,\varphi_0(x) = \begin{cases} \dfrac{\gamma\omega_0}{K_1K_2}\, v\left(\dfrac{K_1K_2}{\omega_0}\,\Phi(x)\right) & \text{for} \quad \|x - w\| \leqq \dfrac{\omega_0}{K_1}\,, \\ 0 & \text{for} \quad \|x - w\| > \dfrac{\omega_0}{K_1}\,; \end{cases} \tag{2.27}$$

$$E\,\varphi_0(x) = \varphi_0(x) - F\,\varphi_0(x)\,.$$

It is easily seen that $F\varphi_0, E\varphi_0\colon \mathbb{R}^n \to \mathbb{R}$ are of class C^1. Using (2.20) and (2.5) we obtain

$$D\,\varphi_0(x)\,e^1 = D\,\varphi_0(w)\,e^1 + [D\,\varphi_0(x) - D\,\varphi_0(w)]\,e^1 \geqq$$
$$\geqq \gamma - \|D\,\varphi_0(x) - D\,\varphi_0(w)\| \geqq \tfrac{1}{2}\gamma\,.$$

Consequently, the identity

$$\varphi_0(w) - \varphi_0\left(w - \frac{\omega_0}{K_1}\,e^1\right) = \int_0^{\omega_0/K_1} D\varphi_0\left(w + \left(\lambda - \frac{\omega_0}{K_1}\right)e^1\right)e^1\,d\lambda$$

implies

$$\varphi_0(w) \geqq \frac{\gamma\omega_0}{2K_1}\,; \tag{2.28}$$

hence

$$\frac{\gamma\omega_0}{2K_1} \leqq 1\,.$$

Taking into account points (i), (v) in the definition of the functions ψ, μ we find that (2.16) holds, that is,

$$F\,\varphi_0(x) \leqq (1 + 2\beta)\,\frac{\gamma\omega_0}{K_1K_2}\,.$$

This togehter with (2.28) and the conditions imposed on β, K_2 in Lemma 2.6 yields (recall that $K_2 > 9$)

$$(2.29) \qquad F\varphi_0(x) \leqq \frac{2\gamma\omega_0}{K_1K_2} \leqq \frac{4}{K_2}\,\varphi_0(w) < \tfrac{1}{2}\varphi_0(w) \leqq \tfrac{1}{2},$$

hence $F\varphi_0 \colon R^n \to [0, 1]$ as required. Moreover, the above inequality implies in particular $F\,\varphi_0(w) < \frac{1}{2}\,\varphi_0(w)$, and since (2.8) holds by definition (cf. (2.27)), the second inequality in (2.9), that is, $E\,\varphi_0(w) > \frac{1}{2}\,\varphi_0(w)$, holds.

To prove the first inequality in (2.9), notice that (2.27) and (2.26) imply $F\,\varphi_0(x) = 0$, and thus $E\,\varphi_0(x) = \varphi_0(x)$, for x satisfying $\|x - w\| \geqq 2^{3/2}\omega_0/(K_1K_2)$. If $\|x - w\| < 2^{3/2}\omega_0/(K_1K_2)$ holds and $u = \Phi(x)$, then (cf. (2.23), (2.27))

$$(2.30) \quad E\,\varphi_0(x) = \varphi_0(x) - F\,\varphi_0(x) = \varphi_0(w) + \gamma u_1 - \frac{\gamma\omega_0}{K_1K_2}\,\nu\left(\frac{K_1K_2}{\omega_0}\,u\right) \geqq$$

$$\geqq \varphi_0(w) + \gamma u_1 - \frac{\gamma\omega_0}{K_1K_2}\,\mu\left(\frac{K_1K_2}{\omega_0}\,u_1\right),$$

and at the same time $|u_1| \leqq 3\,.\,2^{1/2}\omega_0/(K_1K_2)$ (cf. (2.22)). If $\omega_0/(K_1K_2) \leqq |u_1| \leqq 3\,.\,2^{1/2}\omega_0/(K_1K_2)$ then $\mu((K_1K_2/\omega_0)\,u_1) = 0$ and, since $K_2 > 9$, we have by (2.28)

$$E\,\varphi_0(x) \geqq \varphi_0(w) - \gamma\,\frac{3\,.\,2^{1/2}\omega_0}{K_1K_2} \geqq \frac{\gamma\omega_0}{K_1}\left(\frac{1}{2} - \frac{3\,.\,2^{1/2}}{K_2}\right) > 0\,.$$

If $|u_1| \leqq \omega_0/(K_1K_2)$, then $\mu((K_1K_2/\omega_0)\,u_1) \leqq 1 + (K_1K_2/\omega_0)\,u_1$ (see (vii) in the definition of μ) and consequently,

$$E\,\varphi_0(x) \geqq \varphi_0(w) + \gamma u_1 - \frac{\gamma\omega_0}{K_1K_2}\left(1 + \frac{K_1K_2}{\omega_0}\,u_1\right) =$$

$$= \varphi_0(w) - \frac{\gamma\omega_0}{K_1K_2} \geqq \frac{\gamma\omega_0}{K_1}\left(\frac{1}{2} - \frac{1}{K_2}\right) > 0\,.$$

The proof of (2.9) is complete. Moreover, since $\varphi_0(x) = 0$ evidently implies $E\,\varphi_0(x) = 0$, we have proved that $E\varphi_0 \colon R^n \to [0, 1]$ as required.

To prove (2.10), assume $\|u\| \leqq \beta\omega_0/(K_1K_2)$, $x = \Phi^{-1}(u)$. Then $\nu((K_1K_2/\omega_0)\,u) = 1 + (K_1K_2/\omega_0)\,u_1$ and similarly as in (2.30) we obtain

$$(2.31) \qquad E\,\varphi_0(x) = \varphi_0(x) - \frac{\gamma\omega_0}{K_1K_2}\,\nu\left(\frac{K_1K_2}{\omega_0}\,u\right) =$$

$$= \varphi_0(w) + \gamma u_1 - \frac{\gamma\omega_0}{K_1K_2}\left(1 + \frac{K_1K_2}{\omega_0}\,u_1\right) = \varphi_0(w) - \frac{\gamma\omega_0}{K_1K_2}\,.$$

If $\|x - w\| \leqq 2\beta\omega_0/(3K_1K_2)$, then by (2.22) $\|\Phi(x)\| \leqq \beta\omega_0/(K_1K_2)$ and (2.31) implies (2.10).

The inequality (2.11) follows by (2.29). Indeed, we have (cf. (2.6))

$$E\,\varphi_0(x) \leqq \varphi_0(x) \leqq (1 + \theta)\,\varphi_0(w)$$

and, by (2.29),

$$E\,\varphi_0(w) = \varphi_0(w) - F\,\varphi_0(w) \geqq \left(1 - \frac{4}{K_2}\right)\varphi_0(w)\,,$$

which combined gives (2.11).

The first implication in (2.25) yields

$$E\varphi_0(\Phi^{-1}(u)) = 0 \quad \text{for} \quad \|u\| \geqq \frac{2^{1/2}\omega_0}{K_1K_2}\,.$$

On the other hand, it follows from (2.22) that if $\|x - w\| \geqq 2^{3/2}\omega_0/(K_1K_2)$ then $\|\Phi(x)\| \geqq 2^{1/2}\omega_0/(K_1K_2)$ and consequently, $F\,\varphi_0(x) = 0$. Recalling the definition of ω_1, we conclude that $\omega_1 \leqq 2^{3/2}\omega_0/(K_1K_2)$.

Put $v = (\omega_0/(K_1K_2))\,(1, 1, 0, \ldots, 0)$. Then by points (ii), (vi) in the definitions of ψ, μ we have

$$\nu\left(\frac{K_1K_2}{\omega_0}\lambda v\right) > 0 \quad \text{for} \quad |\lambda| < 1\,,$$

that is,

$$F\varphi_0(\Phi^{-1}(\lambda v)) > 0 \quad \text{for} \quad |\lambda| < 1\,.$$

Using the inclusions (2.22) (with open balls instead of closed ones – see the note in Lemma 2.5) we find that $\|\Phi^{-1}(\lambda v) - w\| \geqq |\lambda|\,2^{3/2}\omega_0/(3K_1K_2)$; hence $\omega_1 \geqq$ $\geqq 2^{3/2}\omega_0/(3K_1K_2)$ and (2.13) is proved.

To prove (2.14), notice that for $\|u\| < \omega_0/K_1K_2$ we have $\nu((K_1K_2/\omega_0)\,u) > 0$ and hence $F\varphi_0(\Phi^{-1}(u)) > 0$. The inclusions (2.22) yield

$$\bar{B}\left(w, \frac{2\omega_0}{3K_1K_2}\right) \subset \Phi^{-1}\left(\bar{B}\left(0, \frac{\omega_0}{K_1K_2}\right)\right),$$

which implies $\bar{B}(w, 2\omega_0/(3K_1K_2)) \subset \operatorname{supp} F\varphi_0$. Combining this result with (2.13) and the inequality $0 < \beta < \frac{1}{2}$ we obtain (2.14).

The inequality (2.15) is a direct consequence of (2.8), (2.10) and (2.5).

It remains to prove (2.17) and (2.12). Recall that

$$F\varphi_0(\Phi^{-1}(u)) = \frac{\gamma\omega_0}{K_1K_2}\,\nu\left(\frac{K_1K_2}{\omega_0}\,u\right) \quad \text{for} \quad u \in \Phi\left(\bar{B}\left(w, \frac{\omega_0}{K_1}\right)\right),$$

$$\bar{B}\left(0, \frac{\omega_0}{2K_1}\right) \subset \Phi\left(\bar{B}\left(w, \frac{\omega_0}{K_1}\right)\right), \quad \nu\left(\frac{K_1K_2}{\omega_0}\,u\right) = 0 \quad \text{provided} \quad \|u\| \geqq \frac{2^{1/2}\omega_0}{K_1K_2}$$

(cf. (2.27), (2.22), (2.25), respectively). Differentiation of the first formula leads to

$$(DF\varphi_0)\,(\Phi^{-1}(u))\,D\,\Phi^{-1}(u) = \gamma(D\nu)\left(\frac{K_1K_2}{\omega_0}\,u\right),$$

$$DF\,\varphi_0(x) = \gamma(D\nu)\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right)D\,\Phi(x)\,.$$

Consequently,

$$\int \|DF\,\varphi_0(x)\|\,dx \leqq \gamma \int \left\|Dv\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right)\right\| \|D\,\Phi(x)\|\,dx =$$

$$= \gamma \int \left\|Dv\left(\frac{K_1K_2}{\omega_0}\,u\right)\right\| \|D\Phi(\Phi^{-1}(u))\|\,|\det D\Phi^{-1}(u)|\,du \leqq$$

$$\leqq \gamma\varkappa_2 \int \left\|Dv\left(\frac{K_1K_2}{\omega_0}\,u\right)\right\| du = \frac{\gamma\varkappa_2\omega_0^n}{(K_1K_2)^n} \int \|Dv(v)\|\,dv \leqq$$

$$\leqq \frac{\gamma\varkappa_2\varkappa_6\omega_0^n}{(K_1K_2)^n} \int v(v)\,dv = \gamma\varkappa_2\varkappa_6 \int v\left(\frac{K_1K_2}{\omega_0}\,u\right) du =$$

$$= \frac{\varkappa_2\varkappa_6K_1K_2}{\omega_0} \int \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,u\right) du =$$

$$= \frac{\varkappa_2\varkappa_6K_1K_2}{\omega_0} \int F\,\varphi_0(x)\,|\det D\,\Phi(x)|\,dx \leqq$$

$$\leqq \frac{\varkappa_2\varkappa_3\varkappa_6K_1K_2}{\omega_0} \int F\,\varphi_0(x)\,dx \leqq \frac{4\varkappa_7}{\omega_1} \int F\,\varphi_0(x)\,dx\,,$$

hence (2.17) holds. (The last inequality follows from (2.13).)

To prove (2.12) we estimate the integral on the left-hand side of the inequality using (2.8), (2.7) and the result just obtained when proving (2.17):

$$\int \|DE\,\varphi_0(x)\|\,dx \leqq \int \|D\,\varphi_0(x)\|\,dx + \int \|DF\,\varphi_0(x)\|\,dx \leqq \tag{2.32}$$

$$\leqq \frac{K_1}{\omega_0} \int \varphi_0(x)\,dx + \frac{\varkappa_7K_1K_2}{\omega_0} \int F\,\varphi_0(x)\,dx\,.$$

Now we will treat the two terms on the right-hand side separately. From (2.5) we easily obtain that $\|D\,\varphi_0(x)\| \geqq \frac{1}{2}\gamma$ for $x \in \bar{B}(w, \omega_0/K_1)$; hence

$$\varphi_0\left(w + \frac{\omega_0}{K_1}\,\frac{D\,\varphi_0(w)}{\|D\,\varphi_0(w)\|}\right) - \varphi_0(w) \geqq \tfrac{1}{2}\gamma\,\frac{\omega_0}{K_1}\,.$$

Combining this inequality with (2.6) we have

$$(1 + \theta)\,\varphi_0(w) - \varphi_0(w) \geqq \tfrac{1}{2}\gamma\,\frac{\omega_0}{K_1}\,,$$

$$\varphi_0(w) \geqq \frac{\gamma\omega_0}{2\theta K_1}\,.$$

On the other hand, (2.5) also yields $\|D\,\varphi_0(x)\| \leqq \frac{3}{2}\gamma$ for $x \in \bar{B}(w, \omega_0/K_1)$ and thus

$$\varphi_0(x) \geqq \varphi_0(w) - \tfrac{3}{2}\gamma\,\frac{\omega_0}{K_1} \geqq \frac{\gamma\omega_0}{2\theta K_1}\,(1 - 3\theta) \geqq \frac{\gamma\omega_0}{4\theta K_1}$$

(recall that $0 < \theta < \frac{1}{6}$) holds for $x \in \bar{B}(w, \omega_0/K_1)$, which implies

$$\int \varphi_0(x)\,dx \geqq \frac{\gamma\omega_0}{4\theta K_1}\,\varkappa_1\left(\frac{\omega_0}{K_1}\right)^n = \frac{\varkappa_1\gamma}{4\theta}\left(\frac{\omega_0}{K_1}\right)^{n+1}.$$

Further,

$$\int F\,\varphi_0(x)\,dx = \int \frac{\gamma\omega_0}{K_1K_2}\,v\left(\frac{K_1K_2}{\omega_0}\,\Phi(x)\right)dx =$$

$$= \frac{\gamma\omega_0}{K_1K_2}\int v\left(\frac{K_1K_2}{\omega_0}\,u\right)\left|\det D\,\Phi^{-1}(u)\right|\,du \leqq$$

$$\leqq \varkappa_4\,\frac{\gamma\omega_0}{K_1K_2}\int v\left(\frac{K_1K_2}{\omega_0}\,u\right)du = \varkappa_4\varkappa_5\gamma\left(\frac{\omega_0}{K_1K_2}\right)^{n+1}.$$

Combined with the previous inequality, this yields

$$\int F\,\varphi_0(x)\,dx \leqq \frac{4\varkappa_4\varkappa_5\theta}{\varkappa_1}\,K_2^{-(n+1)}\int \varphi_0(x)\,dx\,, \tag{2.33}$$

from which we conclude

$$\int E\,\varphi_0(x)\,dx = \int \varphi_0(x)\,dx - \int F\,\varphi_0(x)\,dx \geqq \int \varphi_0(x)\,dx\left[1 - \varkappa_8\theta K_2^{-(n+1)}\right]. \tag{2.34}$$

Returning to (2.32) and making use of (2.33), (2.34) we conclude

$$\int \left\|DE\,\varphi_0(x)\right\|\,dx \leqq \left(K_1 + \varkappa_7\varkappa_8\theta K_1K_2^{-n}\right)\omega_0^{-1}\int \varphi_0(x)\,dx \leqq$$
$$\leqq \left(1 - \varkappa_8\theta K_2^{-(n+1)}\right)^{-1}\left(K_1 + \varkappa_7\varkappa_8\theta K_1K_2^{-n}\right)\omega_0^{-1}\int E\,\varphi_0(x)\,dx\;;$$

thus, (2.12) is established, the proof of Lemma 2.6 being now complete.

Put $\varphi_1 = F\varphi_0$. Then ω_1 plays the same role with respect to φ_1 as ω_0 did with respect to φ_0. Let us find conditions under which we can repeat the process from Lemma 2.6, that is, under which we can start with the pair φ_1, ω_1 instead of φ_0, ω_0, and construct $E\varphi_1, F\varphi_1$. To this end we have to guarantee that conditions (2.4)–(2.7) are satisfied with φ_1, ω_1 instead of φ_0, ω_0. That this is the case follows from (2.13)–(2.17) provided the constants K_1, θ, β satisfy some additional conditions ensuring that after passing to φ_1, ω_1 we have the same constants in (2.4)–(2.7) as before. Let us now find these conditions.

The inclusion

$$B\left(w, \frac{\omega_1}{K_1}\right) \subset \operatorname{supp}\varphi_1 \tag{2.4_1}$$

will be satisfied, in virtue of (2.14), if

$$\frac{\beta}{3} \geqq \frac{1}{K_1}. \tag{2.35}$$

By (2.15)

$$\left\|D\,\varphi_1(x) - D\,\varphi_1(w)\right\| \leqq \tfrac{1}{2}\gamma \tag{2.5_1}$$

will hold for $x \in \bar{B}(w, \omega_1/K_1)$ provided

$$\omega_1 \leqq \frac{2\beta\omega_0}{3K_2}. \tag{2.36}$$

(Notice that $\|D\varphi_1(w)\| = \|DF\varphi_0(w)\| = \|D\varphi_0(w)\| = \gamma$.) Further,

$$\varphi_1(x) \leqq (1+\theta)\,\varphi_1(w) \quad \text{for} \quad x \in \mathbb{R}^n \tag{2.6$_1$}$$

follows from (2.16) provided

$$2\beta \leqq \theta, \tag{2.37}$$

and finally, (2.17) implies that

$$\int \|D\varphi_1(x)\|\,dx \leqq \frac{K_1}{\omega_1}\int \varphi_1(x)\,dx \tag{2.7$_1$}$$

holds provided

$$K_1 \geqq 4\varkappa_4. \tag{2.38}$$

Taking into account (2.13), we see that (2.36) holds if $2^{3/2}/K_1 \leqq 2\beta/3$; so both (2.35), (2.36) will certainly hold if we assume

$$\beta K_1 \geqq 6. \tag{2.39}$$

In what follows, let us assume that (2.37)–(2.39) hold. Let N be a positive integer, and put

$$\varphi_{i+1} = F\varphi_i, \quad \omega_{i+1} = \sup\{\|x - w\|;\ x \in \operatorname{supp}\varphi_{i+1}\},$$
$$i = 0, 1, 2, \ldots, N-1.$$

It follows from (2.8), (2.9), (2.13) and (2.16) that

$$\varphi_0(x) = E\varphi_0(x) + E\varphi_1(x) + \ldots + E\varphi_{N-1}(x) + \varphi_N(x), \tag{2.40}$$

$$\varphi_i(x) \leqq (1+2\beta)^i\, 2^{-i}, \tag{2.41}$$

$$\omega_i \leqq \left(\frac{2^{3/2}}{K_1K_2}\right)^i \omega_0 \tag{2.42}$$

for $x \in \mathbb{R}^n$.

Rewriting (2.8)–(2.12) for the functions $E\varphi_i$, $i = 0, 1, \ldots, N-1$, we obtain

$$E\varphi_i + F\varphi_i = \varphi_i; \tag{2.43}$$

$$E\varphi_i(x) > 0 \quad \text{provided} \quad \varphi_i(x) > 0, \quad E\varphi_i(w) > \tfrac{1}{2}\varphi_i(w); \tag{2.44}$$

$$E\varphi_i(x) = E\varphi_i(w) \quad \text{for} \quad x \in \bar{B}\left(w, \frac{2\beta\omega_i}{3K_1K_2}\right); \tag{2.45}$$

$$E\varphi_i(x) \leqq (1+\theta)\left(1 - \frac{4}{K_2}\right)^{-1} E\varphi_i(w); \tag{2.46}$$

$$\int \|DE\varphi_i(x)\|\,dx \leqq [1 - \theta\varkappa_8 K_2^{-(n+1)}]^{-1} \cdot (K_1 + \theta\varkappa_7\varkappa_8 K_1 K_2^{-n})\,\omega_i^{-1} \int E\varphi_i(x)\,dx. \tag{2.47}$$

Let us now introduce a system

$$\Pi = \{(z^m, \zeta_m);\ m = 1, 2, \ldots, p\},$$

where $z_m \in \mathbb{R}^n$, $\zeta_m : \mathbb{R}^n \to [0, 1]$ are of class C^1 with nonempty compact support; denote $\zeta(x) = \sum_{m=1}^{p} \zeta_m(x)$, $\sigma_m = \sup\{\|x - z^m\|;\ x \in \operatorname{supp} \zeta_m\}$. Assume that there are constants $\theta_1 > 0$, $K_3 > 1$ such that

(2.48) $$\zeta(x) \leqq 1, \quad x \in \mathbb{R}^n;$$

(2.49) $$\zeta_m(x) < (1 + \theta_1)\, \zeta_m(z^m), \quad x \in \mathbb{R}^n;$$

(2.50) $$\zeta_m(x) = \zeta_m(z^m), \quad x \in \bar{B}\left(z^m, \frac{\sigma_m}{K_3}\right);$$

(2.51) $$\int \|D\, \zeta_m(x)\|\, dx < \frac{K_3}{\sigma_m} \int \zeta_m(x)\, dx$$

for $m = 1, 2, \ldots, p$. (Note that (2.49) implies $\zeta_m(z^m) > 0$, hence $z^m \in \operatorname{Int} \operatorname{supp} \zeta_m$.)

We are now ready to introduce a definition which will be needed in the sequel.

2.7. Definition. Let $\varepsilon_1 > 0$. A system

$$\Pi' = \{(z^m, \zeta'_m);\ m = 1, 2, \ldots, p\},$$

where $\zeta'_m : \mathbb{R}^n \to [0, 1]$ are of class C^1, is called *an ε_1-modification of the system Π* if, denoting

$$\zeta'(x) = \sum_{m=1}^{p} \zeta'_m(x),$$

$$\sigma'_m = \sup\{\|x - z^m\|;\ x \in \operatorname{supp} \zeta'_m\},$$

we have

(2.52) $$\sigma'_m \leqq \tfrac{4}{3}\sigma_m, \quad \zeta'_m(x) \geqq \zeta_m(x), \quad x \in \mathbb{R}^n;$$

(2.53) $$\zeta'(x) \leqq 1, \quad x \in \mathbb{R}^n;$$

(2.54) $$\zeta'_m(x) \leqq (1 + \theta_1)\, \zeta'_m(z^m), \quad x \in \mathbb{R}^n;$$

(2.55) $$\zeta'_m(x) = \zeta_m(z^m), \quad x \in \bar{B}\left(z^m, \frac{\sigma'_m}{2K_3}\right);$$

(2.56) $$\int \|D\, \zeta'_m(x)\|\, dx \leqq \frac{2K_3}{\sigma'_m} \int \zeta'_m(x)\, dx;$$

(2.57) $$\int [\zeta'_m(x) - \zeta_m(x)]\, dx < \varepsilon_1.$$

2.8. Lemma. *Let $0 < \varepsilon_1 < 1$, $0 < \theta_1 < 1$, $K_3 > 1$, and let $M \subset \mathbb{R}^n$ be compact. Let Π be the system introduced above (and satisfying (2.48)–(2.51)). Further, let $z^m \in M$, $m = 1, 2, \ldots, p$, and let δ_1 be a gauge on M.*

Let β, θ, K_1, K_2 be constants satisfying the assumptions of Lemma 2.6, the con-

ditions (2.37)–(2.39),

(2.58) $$(1 + \theta) \Big/ \left(1 - \frac{4}{K_2}\right) \leqq 1 + \theta_1$$

and

(2.59) $$K_1 > \varkappa(n)^n \,(3\varkappa(n) + 1)$$

with $\varkappa(n)$ *from Proposition* 1.3.

Then, for any K_4 *satisfying*

(2.60) $$K_4 > \max\{\varkappa(n), \tfrac{3}{2}K_1K_2/\beta, (K_1 + \theta\varkappa_7\varkappa_8K_1K_2^{-n})/(1 - \theta\varkappa_8K_2^{-(n+1)})\},$$

there exists an ε_1*-modification* Π' *of the system* Π *and such a system*

$$\Lambda = \{(s^l, \lambda_l);\ l = 1, 2, \ldots, L\}$$

that the following conditions are fulfilled:

(2.61) $$\Lambda \cup \Pi' \quad \text{is a PU-partition of } M\,;$$

(2.62) $$\operatorname{supp} \lambda_l \subset \bar{B}(s^l, \delta_1(s^l))\,;$$

(2.63) $$\lambda_l(x) \leqq (1 + \theta_1)\,\lambda_l(s^l)\,, \quad x \in \mathbb{R}^n\,;$$

(2.64) $$\lambda_l(x) = \lambda_l(s^l)\,, \quad x \in \bar{B}(s^l, \tau_l/K_4)$$

where $\tau_l = \sup\{\|x - s^l\|;\ x \in \operatorname{supp}\lambda_l\}$;

(2.65) $$\int \|D\,\lambda_l(x)\|\,dx \leqq \frac{K_4}{\tau_l}\int \lambda_l(x)\,dx\,.$$

Proof. Let us choose a bounded open set G; $M \subset G \subset \mathbb{R}^n$, and denote $\mu = \max\{m_n(G), 1\}$,

$$Z = \{x \in \mathbb{R}^n;\quad \zeta(x) = 1\}\,,$$
$$T = \{x \in \mathbb{R}^n;\ D\,\zeta(x) = 0\}\,.$$

Then

$$M = (M \cap \operatorname{Int} T) \cup (M \setminus T) \cup (M \cap Z \cap \partial T) \cup (M \cap \partial T \setminus Z)\,,$$

the union on the right-hand side being disjoint. For every $u \in (M \setminus T) \cup (M \cap \partial T)$ there exists an integer $q(u)$, $1 \leqq q(u) \leqq p$, such that

$$D\,\zeta_{q(u)}(u) \neq 0 \quad \text{if} \quad u \in M \setminus T\,,$$
$$u \in \operatorname{Cl}\{x \in \mathbb{R}^n;\ D\,\zeta_{q(u)}(x) \neq 0\} \quad \text{if} \quad u \in M \cap \partial T\,.$$

Let δ_2 be a gauge on M satisfying the following conditions:

(2.66) $$\delta_2(x) \leqq \min\{1, \delta_1(x)\}\,, \quad x \in M\,,$$

(2.67) $$\bar{B}(u, \delta_2(u)) \subset G\,, \quad u \in M\,,$$

(2.68) $$\bar{B}(u, \delta_2(u)) \subset \operatorname{Int} T\,, \quad u \in M \cap \operatorname{Int} T\,;$$

if $u \in M \setminus T$, $m = q(u)$, $x \in \bar{B}(u, \delta_2(u))$, then

(2.69) $$\bar{B}(u, \delta_2(u)) \subset \mathbb{R}^n \setminus T\,,$$

(2.70) $$\|D\,\zeta(x) - D\,\zeta(u)\| \leqq \tfrac{1}{2}\|D\,\zeta(u)\|\,,$$

(2.71) $$\zeta(x) < (1 + \tfrac{1}{2}\varepsilon_1)\,\zeta(u)\,,$$

(2.72) $$B(u, \delta_2(u)) \subset B(z^m, \sigma_m) \setminus \bar{B}(z^m, \sigma_m/K_3)\,,$$

(2.73) $$\|D\,\zeta(u)\|\,\delta_2(u) < \tfrac{2}{3}\theta(1 - \zeta(u))\,;$$

if $u \in Z \cap \partial T$, $m = q(u)$, $x \in \bar{B}(u, \delta_2(u))$, then

(2.74) $$\delta_2(u) < \frac{\sigma_m}{3K_3}\,,$$

(2.75) $$\|D\,\zeta(x)\| < \frac{\varepsilon_m^*}{2\,\varkappa(n)^{n+1}\,\mu}\,;$$

if $u \in \partial T \setminus Z$, $m = q(u)$, $x \in \bar{B}(u, \delta_2(u))$, then

(2.76) $$\delta_2(u) < \frac{\sigma_m}{3K_3}\,,$$

(2.77) $$\|D\,\zeta(x)\| < (1 - \zeta(u))\,\frac{\varepsilon_m^*}{2\,\varkappa(n)^{n+1}\,\mu}\,;$$

in (2.75) and (2.77), ε_m^* is a constant,

$$0 < \varepsilon_m^* \leqq \min\Big\{(1 + \theta_1)\,\zeta_m(z^m) - \sup\{\zeta_m(x);\ x \in \mathbb{R}^n\}\,,$$
$$\frac{K_3}{\sigma_m}\int \zeta_m(x)\,\mathrm{d}x - \int \|D\,\zeta_m(x)\|\,\mathrm{d}x\,,\ \ \frac{\varepsilon_1}{\mu}\,,\ \ 2\,\varkappa(n)^{n+1}\,\mu\Big\}\,.$$

It is easy to verify that it is indeed possible to satisfy the conditions (2.66)–(2.77). Notice that $\zeta(u) \neq 0$ in (2.71) since otherwise we should have $D\,\zeta(u) = 0$ but $u \notin T$. Further, the set on the right-hand side of the inclusion (2.72) is open and contains the point u ($u \notin T$ and $m = q(u)$, hence $D\,\zeta_m(u) \neq 0$ and (2.50) yields the result). Finally, in (2.73) and (2.77) we have $\zeta(u) < 1$ since $u \notin Z$, while (2.49), (2.51) make it possible to choose ε_m^* a positive constant.

Let
$$\Delta = \{(t^j, \vartheta_j);\ j = 1, 2, \ldots, k\}$$
be a δ_2-fine PU-partition of M from Proposition 1.3, that is, Δ fulfils (1.5')–(1.7').

The modification Π' and the system Λ are constructed as follows:

(i) If $t^j \in M \cap \operatorname{Int} T$, we include the pair $(t^j, (1 - \zeta)\,\vartheta_j)$ into the system Λ (if, at the same time, $t^j \in Z$, then (2.70) implies $\zeta(x) = 1$ for $x \in B(t^j, \delta_2(t^j))$ and the corresponding pair will be omitted).

(ii) If $t^j \in M \setminus T$ then we put $\varphi_0 = (1 - \zeta)\,\vartheta_j$, $w = t^j$ in Lemma 2.6 and use it repeatedly N_j-times (N_j an integer to be fixed later). The pairs $(t^j, E\varphi_0)$, $(t^j, E\varphi_1)$, … …, $(t^j, E\varphi_{N_j - 1})$ are put in the system Λ while the function $\varphi_{N_j} = F^{N_j}((1 - \zeta)\,\vartheta_j)$ is added to $\zeta_{q(t^j)}$.

(iii) If $t^j \in M \cap Z \cap \partial T$ then we add the function $(1 - \zeta)\,\vartheta_j$ to $\zeta_{q(t^j)}$.

(iv) If $t^j \in (M \cap \partial T) \setminus Z$ then the pair $(t^j, (1 - (\varepsilon^*_{q(t^j)}\varrho_j/(2\,\varkappa(n)^{n+1}\,\mu))(1 - \zeta(t^j))\,\vartheta_j)$ is included into the system Λ, while the function $[(\varepsilon^*_{q(t^j)}\varrho_j/(2\,\varkappa(n)^{n+1}\,\mu))(1 - \zeta(t^j)) + \zeta(t^j) - \zeta]\,\vartheta_j$ is added to $\zeta_{q(t^j)}$.

Following the notation introduced in Lemma 2.6, let us denote the pairs that form the system Λ by (s^l, λ_l), $l = 1, 2, \ldots, L$.

On the other hand, taking into account the above described procedure, we may write

$$\zeta'_m = \zeta_m + \sum_{\substack{t^j \in M \setminus T \\ q(t^j) = m}} F^{N_j}((1 - \zeta)\,\vartheta_j) + \tag{2.78}$$

$$+ \sum_{\substack{t^j \in M \cap Z \cap \partial T \\ q(t^j) = m}} (1 - \zeta)\,\vartheta_j + \sum_{\substack{t^j \in (M \cap \partial T) \setminus Z \\ q(t^j) = m}} \left[\frac{\varepsilon^*_m \varrho_j}{2\,\varkappa(n)^{n+1}\,\mu}(1 - \zeta(t^j)) + \zeta(t^j) - \zeta\right]\vartheta_j\,.$$

Our task now is to prove (2.52)–(2.57) and (2.61)–(2.65). Let us start with the latter set of conditions.

Taking into account (2.40) we find from (i)–(iv) that

$$\sum_{l=1}^{L} \lambda_l(x) + \sum_{m=1}^{p} \zeta'_m(x) = \sum_{j=1}^{k} \vartheta_j(x)\,(1 - \zeta(x)) + \zeta(x)\,, \quad x \in \mathbb{R}^n\,; \tag{2.79}$$

the right-hand side of the identity is always less than or equal to one, the equality holding if and only if $\sum_{j=1}^{k} \vartheta_j(x) = 1$ or $\zeta(x) = 1$. Thus, (2.61) is proved.

Since Δ is δ_2-fine, (2.62) follows from (2.66) and (i), (ii), (iv).

Now, let $(s^l, \lambda_l) \in \Lambda$, $s^l \in M \cap \operatorname{Int} T$. Then there exists such j that

$$(s^l, \lambda_l) = (t^j, (1 - \zeta)\,\vartheta_j)\,, \quad t^j \in M \cap \operatorname{Int} T$$

(cf. (i)). By (2.68) we have $\zeta(x) = \zeta(t^j)$ for $x \in \bar{B}(t^j, \delta_2(t^j))$ (recall the definition of T), hence $(1 - \zeta(x))\,\vartheta_j(x) = (1 - \zeta(t^j))\,\vartheta_j(x)$ and (2.63)–(2.65) evidently hold as a consequence of (1.5')–(1.7'), since we have $K_4 > \varkappa(n)$ by (2.60).

If $(s^l, \lambda_l) \in \Lambda$ and $s^l \in (M \cap \partial T) \setminus Z$, then we proceed quite similarly (notice that by (iv), $\lambda_l = \text{const.}\ \vartheta_j$ in this case).

If $(s^l, \lambda_l) \in \Lambda$ and $s^l \in M \setminus T$, then there are such j, i that

$$(s^l, \lambda_l) = (t^j, EF^i[(1 - \zeta)\,\vartheta_j])\,, \quad t^j \in M \setminus T\,.$$

As mentioned in (ii), in this case we apply Lemma 2.6. To justify its application we have to verify (2.4)–(2.7), where $w = t^j$, $\varphi_0 = (1 - \zeta)\,\vartheta_j$.

If $D\,\zeta(x) \neq 0$ then obviously $0 < \zeta(x) < 1$ and since $t^j \in M \setminus T$, (2.69) implies $1 - \zeta(x) > 0$ for $x \in \bar{B}(t^j, \delta_2(t^j))$. Consequently, $\varphi_0(x) \neq 0$ if and only if $\vartheta_j(x) \neq 0$, and hence $\omega_0 \mathrel{=\!\!\!\!/} \varrho_j$. Therefore, by (1.6') from Proposition 1.3, (2.4) holds since (2.59) guarantees $K_1 > \varkappa(n)$. Further, $\varphi_0(x) = 1 - \zeta(x)$ for $x \in \bar{B}(w, \omega_0/K_1)$, hence $\gamma = \|D\,\zeta(w)\|$ and (2.5) follows from (2.70).

By virtue of (2.70) and (2.73), for $x \in \bar{B}(w, \omega_0)$ we have

$$(2.80) \qquad 1 - \zeta(x) \leqq 1 - \zeta(w) + \left|\int_0^1 D\zeta(w + \eta(x - w))\, d\eta(x - w)\right| \leqq$$
$$\leqq 1 - \zeta(w) + \tfrac{3}{2}\|D\,\zeta(w)\|\,\omega_0 \leqq (1 + \theta)(1 - \zeta(w))$$

(notice that (2.70) implies $\|D\,\zeta(x)\| \leqq \frac{3}{2}\|D\,\zeta(w)\|$), and since $\vartheta_j(x) \leqq \vartheta_j(w)$, (2.6) immediately follows.

Finally, (2.80) together with (1.7'), (2.70) and (2.73) yields

$$(2.81) \qquad \int \|D\,\varphi_0(x)\|\,dx \leqq \max\{1 - \zeta(x);\ x \in \bar{B}(w, \omega_0)\} \int \|D\,\vartheta_j(x)\|\,dx +$$
$$+ \int_{B(w,\omega_0)} \|D\,\zeta(x)\|\,dx \leqq$$
$$\leqq (1 + \theta)(1 - \zeta(w))(\varkappa(n)/\varrho_j) \int \vartheta_j(x)\,dx + \tfrac{3}{2}\|D\,\zeta(w)\|\,\varkappa_1\omega_0^n \leqq$$
$$\leqq (1 + \theta)(1 - \zeta(w))\,\varkappa(n)\,\varkappa_1\varrho_j^{n-1} + \tfrac{3}{2}\varkappa_1\,\tfrac{2}{3}\theta(1 - \zeta(w))\,\omega_0^{n-1} =$$
$$= [(1 + \theta)\,\varkappa(n) + \theta]\,\varkappa_1(1 - \zeta(w))\,\omega_0^{n-1}$$

(by (2.66), $\omega_0 \leqq 1$). For $x \in \bar{B}(w, \omega_0)$, analogously as in (2.80), we obtain

$$1 - \zeta(x) \geqq (1 - \theta)(1 - \zeta(w))\,.$$

For $x \in \bar{B}(w, \omega_0/\varkappa(n))$ we have $\vartheta_j(x) = 1$, hence

$$(1 - \theta)(1 - \zeta(w))\,\varkappa_1\left(\frac{\omega_0}{\varkappa(n)}\right)^n \leqq \int_{B\left(w,\frac{\omega_0}{\varkappa(n)}\right)} (1 - \zeta(x))\,\vartheta_j(x)\,dx \leqq \int \varphi_0(x)\,dx\,,$$

and combining this inequality with (2.81) we conclude that

$$\int \|D\,\varphi_0(x)\|\,dx \leqq [(1 + \theta)\,\varkappa(n) + \theta]\,\frac{\varkappa(n)^n}{\omega_0(1 - \theta)} \int \varphi_0(x)\,dx$$

and (2.7) holds by (2.59) (recall that $\theta \leqq \frac{1}{6}$).

Thus, we have shown that the assumptions of Lemma 2.8 guarantee that we may use Lemma 2.6 repeatedly as described above. We choose N_j so large that

$$(2.82) \qquad \int \varphi_{N_j}(x)\,dx \leqq \frac{\varepsilon_m^*}{\mu} \int \vartheta_j(x)\,dx\,, \quad \int \|D\,\varphi_{N_j}(x)\|\,dx \leqq \frac{\varepsilon_m^*}{\mu} \int \vartheta_j(x)\,dx\,,$$

$$(2.83) \qquad \varphi_{N_j}(x) \leqq \frac{\varepsilon_m^*}{\mu} \quad \text{for} \quad x \in \mathbb{R}^n\,, \quad \omega_{N_j} \leqq \frac{\varrho_j}{\varkappa(n)}$$

(again $m = q(t^j)$). Such a choice is possible in virtue of (2.41), (2.42) and (2.17) since the pair $(\varphi_{N_j}, \omega_{N_j})$ satisfies (2.4)–(2.7) with φ_0, ω_0 replaced by $\varphi_{N_j}, \omega_{N_j}$.

Recall that we are now dealing with the case $(s^l, \lambda_l) \in \Lambda$, $s^l \in M \setminus T$ which corresponds to point (ii) of the construction of Λ, so that λ_l is actually some $E\varphi_i$. Consequently, (2.44)–(2.47) hold. In particular, (2.44) implies

$$\omega_i = \sup\{\|x - t^j\|;\ x \in \operatorname{supp} F^i[(1 - \zeta)\,\vartheta_j]\} =$$
$$= \sup\{\|x - t^j\|;\ x \in \operatorname{supp} EF^i[(1 - \zeta)\,\vartheta_j]\} = \tau_l\,,$$

and (2.45)–(2.47) yield (2.63)–(2.65) by virtue of the assumptions (2.58), (2.60).

Thus (2.63)–(2.65) hold for all $(s^l, \lambda_l) \in \Lambda$. It remains to prove (2.52)–(2.57), i.e., that Π' is an ε_1-modification of Π.

The second inequality in (2.52) is trivial, the inequality (2.53) follows from (2.79). The rest of the proof will be based on the formula (2.78) for ζ'_m.

If $t^j \in M \setminus T$ then $\operatorname{supp} \vartheta_j \subset B(t^j, \delta_2(t^j)) \subset B(z^{q(t^j)}, \sigma_{q(t^j)})$ by (2.72); if $t^j \in \partial T$, then $\sigma'_j \leqq \operatorname{dist}(z^{q(t^j)}, t^j) + \frac{1}{3}\sigma_{q(t^j)}$ (cf. (2.74) or (2.76)), and it follows from the properties of $q(u)$ that the first summand on the right-hand side is not greater than $\sigma_{q(t^j)}$. Hence the first inequality in (2.52) holds in both cases.

Further, to prove (2.55) we notice that for $t^j \in M \setminus T$, (2.72) implies $\operatorname{supp} \vartheta_j \cap B(z^{q(t^j)}, \sigma_{q(t^j)}/K_3) = \emptyset$, while for $t^j \in \partial T$, either (2.74) or (2.76) yields $\operatorname{supp} \vartheta_j \cap B(z^{q(t^j)}, \frac{2}{3}\sigma_{q(t^j)}/K_3) = \emptyset$ since $\operatorname{dist}(z^{q(t^j)}, t^j) \geqq \sigma_{q(t^j)}/K_3$ (see (2.50)). Hence (2.55) always holds.

Now we proceed to prove (2.54) and (2.57). Again we distinguish two cases. If $t^j \in M \setminus T$ then $\operatorname{supp} \varphi_{N_j} \subset \bar{B}(t^j, \omega_{N_j})$ and by (2.83) we have $\operatorname{supp} \varphi_{N_j} \subset \bar{B}(t^j, \varrho_j/\varkappa(n))$. Hence for $x \in \operatorname{supp} \varphi_{N_j}$ we have $\vartheta_j(x) = 1$ by (1.5') and (2.83) yields (we denote $q(t^j) = m$ again)

$$\varphi_{N_j}(x) \leqq \frac{\varepsilon^*_m}{\mu} \leqq \varepsilon^*_m \, \vartheta_j(x) \,, \quad x \in \mathbb{R}^n \tag{2.84}$$

(recall that $\mu \geqq 1$).

If $t^j \in \partial T \cap Z$, then taking into account the definition of Z we have $\zeta(t^j) = 1$, hence (2.75) yields

$$1 - \zeta(x) = \zeta(t^j) - \zeta(x) \leqq \tfrac{1}{2}\varrho_j \varepsilon^*_m/\mu$$

for $x \in B(t^j, \varrho_j)$ and, since we have assumed $\varrho_j \leqq 1$, $\mu \geqq 1$, we have

$$(1 - \zeta(x))\, \vartheta_j(x) \leqq \varepsilon^*_m \, \vartheta_j(x) \,. \tag{2.85}$$

Finally, let $t^j \in \partial T \setminus Z$ (and $q(t^j) = m$ again). Then by (2.77) we have

$$|\zeta(t^j) - \zeta(x)| \leqq \tfrac{1}{2}\varrho_j(1 - \zeta(t^j)) \frac{\varepsilon^*_m}{\varkappa(n)^{n+1}\, \mu}$$

for $x \in \bar{B}(t^j, \varrho_j)$, hence

$$0 \leqq \left[\frac{\varepsilon^*_m \varrho_j}{2\, \varkappa(n)^{n+1}\, \mu}(1 - \zeta(t^j)) + \zeta(t^j) - \zeta(x)\right] \vartheta_j(x) \leqq \tag{2.86}$$
$$\leqq \varepsilon^*_m \varrho_j (1 - \zeta(t^j))\, \vartheta_j(x) \leqq \varepsilon^*_m \, \vartheta_j(x) \,.$$

Inserting (2.84)–(2.86) into (2.78) and taking into account the definition of ε^*_m (just after the formula (2.77)) as well as (2.49) and the evident inequality $\sum \vartheta_j(x) \leqq 1$ we conclude that (2.54) is valid.

Further, we obtain from (2.78) that

$$\int [\zeta'_m(x) - \zeta_m(x)] \, dx \leqq \varepsilon^*_m \int \sum_{j=1}^{k} \vartheta_j(x) \, dx \leqq \frac{\varepsilon_1}{\mu} \int_G dx = \varepsilon_1 \,,$$

which proves (2.57).

It remains to prove (2.56). Let $t^j \in M \setminus T$, $q(t^j) = m$. Since $\varphi_{N_j} = F^{N_j}((1-\zeta)\,\vartheta_j)$, we obtain from (2.82)

$$(2.87)\qquad \int \|DF^{N_j}((1-\zeta(x))\,\vartheta_j(x))\|\,\mathrm{d}x \leqq \frac{\varepsilon_m^*}{\mu}\int \vartheta_j(x)\,\mathrm{d}x\,.$$

Let $t^j \in Z \cap \partial T$. By (2.75) we have

$$\max\{1-\zeta(x);\ x \in \bar{B}(t^j, \varrho_j)\} \leqq \frac{\varepsilon_m^* \varrho_j}{2\,\varkappa(n)\,\mu}$$

(notice that $\zeta(t^j) = 1$ since $t^j \in Z$). Further, again by (2.75) we have (cf. (1.7'))

$$(2.88)\qquad \int_{B(t^j,\varrho_j)} \|D\,\zeta(x)\|\,\mathrm{d}x \leqq \varkappa_1 \varrho_j^n \frac{\varepsilon_m^*}{2\,\varkappa(n)^n\,\mu} \leqq \frac{\varepsilon_m^*}{2\mu}\int \vartheta_j(x)\,\mathrm{d}x$$

(since $\vartheta_j(x) = 1$ for $x \in B(t^j, \varrho_j/\varkappa(n))$), which yields

$$(2.89)\qquad \int \|D[(1-\zeta(x))\,\vartheta_j(x)]\|\,\mathrm{d}x \leqq$$

$$\leqq \int \|D\,\vartheta_j(x)\|\,\mathrm{d}x\,\frac{\varepsilon_m^* \varrho_j}{2\,\varkappa(n)\,\mu} + \int_{B(t^j,\varrho_j)} \|D\,\zeta(x)\|\,\mathrm{d}x \leqq$$

$$\leqq \frac{\varkappa(n)}{\varrho_j}\int \vartheta_j(x)\,\mathrm{d}x\,\frac{\varepsilon_m^* \varrho_j}{2\,\varkappa(n)\,\mu} + \frac{\varepsilon_m^*}{2\mu}\int \vartheta_j(x)\,\mathrm{d}x \leqq \frac{\varepsilon_m^*}{\mu}\int \vartheta_j(x)\,\mathrm{d}x\,.$$

Finally, let $t^j \in \partial T \setminus Z$. By (2.77) we have

$$\max\left\{\frac{\varepsilon_m^* \varrho_j}{2\,\varkappa(n)^{n+1}\,\mu}(1-\zeta(t^j)) + \zeta(t^j) - \zeta(x)\,;\ x \in \bar{B}(t^j, \varrho_j)\right\} \leqq$$

$$\leqq \frac{\varepsilon_m^* \varrho_j}{\varkappa(n)^{n+1}\,\mu}(1-\zeta(t^j))\,;$$

using (2.77) instead of (2.75) we find that (2.88) again holds. Consequently (cf. (1.7')),

$$(2.90)\qquad \int \left\|D\left[\left(\frac{\varepsilon_m^* \varrho_j}{2\,\varkappa(n)^{n+1}\,\mu}(1-\zeta(t^j)) + \zeta(t^j) - \zeta(x)\right)\vartheta_j(x)\right]\right\|\,\mathrm{d}x \leqq$$

$$\leqq \int \|D\,\vartheta_j(x)\|\,\mathrm{d}x\,\frac{\varepsilon_m^* \varrho_j}{\varkappa(n)^{n+1}\,\mu} + \int_{B(t^j,\varrho_j)} \|D\,\zeta(x)\|\,\mathrm{d}x \leqq$$

$$\leqq \frac{\varkappa(n)}{\varrho_j}\int \vartheta_j(x)\,\mathrm{d}x\,\frac{\varepsilon_m^* \varrho_j}{\varkappa(n)^{n+1}\,\mu} + \frac{\varepsilon_m^*}{2\mu}\int \vartheta_j(x)\,\mathrm{d}x \leqq \frac{\varepsilon_m^*}{\mu}\int \vartheta_j(x)\,\mathrm{d}x\,.$$

Again inserting (2.87), (2.89), (2.90) into (2.78) and taking into account (2.21) and the definition of ε_m^*, we conclude that (2.56) holds. The proof of Lemma 2.8 is complete.

Our next step is to prove the "if" part of Theorem 2.4, that is:

Let $f: \mathbb{R}^n \to \mathbb{R}$ have a compact support $\operatorname{supp} f \subset \operatorname{Int} \mathscr{I}$, where $\mathscr{I}$ is a compact interval. Let (PUI) $\int f(x)\,\mathrm{d}x$ exist. Then (PU) $\int f(x)\,\mathrm{d}x$ exists and the two integrals are equal.

Let $\varepsilon > 0$. Set $\varepsilon_1 = \frac{1}{2}\varepsilon$ and find $\alpha > 0$ corresponding to ε_1 according to Definition

2.3 (of the PUI-integral). Choose $\theta = \min\left(\frac{1}{6}, \frac{1}{2}\alpha\right)$; β and K_1 satisfying $0 < \beta \leqq \frac{1}{2}$, (2.37)–(2.39) and (2.59); $K_2 > \max\{9, (\varkappa_8\theta)^{1/(n+1)}\}$ (see Lemma 2.6) such that $(1 + \theta)/(1 - 4/K_2) \leqq 1 + \alpha$; θ_1 satisfying (2.58); $K_3 > 1$, and K_4 satisfying (2.60).

Given $K > 1$, set $K^* = \max\{K, 2K_3, K_4\}$ and find a gauge δ_1 on $\mathscr{I}$ such that for every δ-fine ($\delta = \frac{4}{3}\delta_1$) PU-partition Ξ of $\mathscr{I}$ satisfying (1.5)–(1.7) with K replaced by K^* we have

$$\left|(\mathrm{PUI}) \int f(x)\,\mathrm{d}x - S(f, \Xi)\right| < \varepsilon_1 .$$

Now, let Π be a δ_1-fine PU-partition of $\sup f$ satisfying (2.49)–(2.51) (which is the same as (1.5)–(1.7) with $\alpha, K, t^j, \vartheta_j$ replaced by $\theta_1, K_3, z^m, \zeta_m$, respectively). Construct $\Lambda \cup \Pi' = \Xi$ according to Lemma 2.8 with $M = \mathscr{I}$. Then Ξ is a PU-partition of $\mathscr{I}$ (cf. (2.61)); it is δ-fine by (2.52), (2.62). Further, Ξ satisfies (1.5) by (2.54), (2.63) and the choice of θ_1 which guarantees $\theta_1 \leqq \alpha$; it satisfies (1.6) by (2.55), (2.64) and the choice of K^*; finally, it satisfies (1.7) by (2.56), (2.65) and the choice of K^*. Hence

$$\left|(\mathrm{PUI}) \int f(x)\,\mathrm{d}x - \sum_{l=1}^{L} f(s^l) \int \lambda_l(x)\,\mathrm{d}x - \sum_{m=1}^{p} f(z^m) \int \zeta'_m(x)\,\mathrm{d}x\right| \leqq \varepsilon_1 .$$

Since Π is a PU-partition of $\operatorname{supp} f$, we have $\operatorname{supp} f \subset \operatorname{Int} Z$ and, since obviously $s^l \notin \operatorname{Int} Z$, we have $f(s^l) = 0$, $l = 1, 2, \ldots, L$. By (2.57) we conclude

$$\left|(\mathrm{PUI}) \int f(x)\,\mathrm{d}x - \sum_{m=1}^{p} f(z^m) \int \zeta_m(x)\,\mathrm{d}x\right| < 2\varepsilon_1 = \varepsilon ,$$

which proves that (PU) $\int f(x)\,\mathrm{d}x$ exists and is equal to (PUI) $\int f(x)\,\mathrm{d}x$. This completes the proof of Theorem 2.4.

3. TRANSFORMATION THEOREM

3.1. Theorem. *Let $f: \mathbb{R}^n \to \mathbb{R}$ with compact support be PU-integrable, let $G \subset \mathbb{R}^n$ be open and bounded. Let $\varphi: G \to \varphi(G)$ be a C^1-diffeomorphism,* $\operatorname{supp} f \subset \varphi(G)$. *Then $(f \circ \varphi)\,|\det D\varphi|$ is PU-integrable and*

$$(3.1) \qquad (\mathrm{PU}) \int f(x)\,\mathrm{d}x = (\mathrm{PU}) \int f(\varphi(y))\,|\det D\,\varphi(y)|\,\mathrm{d}y .$$

(We put formally $(f \circ \varphi)\,|\det D\varphi| = 0$ on $\mathbb{R}^n \setminus G$.)

Proof. Without loss of generality we may assume that there exist such $c \geqq 1$ and $\varrho > 0$ that

$$(3.2) \qquad \begin{aligned} &\|D\,\varphi(\eta)\| \leqq c\,, \quad |\det D\,\varphi(\eta)| \leqq c \quad \text{for} \quad \eta \in G\,,\\ &\|D\,\varphi^{-1}(\xi)\| \leqq c\,, \quad |\det D\,\varphi^{-1}(\xi)| \leqq c \quad \text{for} \quad \xi \in \varphi(G)\,,\\ &\bar{B}(y, \varrho) \subset G \quad \text{for} \quad y \in \varphi^{-1}(\operatorname{supp} f)\,,\\ &\bar{B}(x, \varrho) \subset \varphi(G) \quad \text{for} \quad x \in \operatorname{supp} f\,. \end{aligned}$$

It follows from (3.2) that

$$(3.3) \qquad \begin{aligned} &\|\varphi(y) - \varphi(\eta)\| \leqq c\|y - \eta\| \quad \text{for} \quad y \in \varphi^{-1}(\operatorname{supp} f)\,, \quad \eta \in \bar{B}(y, \varrho)\,,\\ &\|\varphi^{-1}(x) - \varphi^{-1}(\xi)\| \leqq c\|x - \xi\| \quad \text{for} \quad x \in \operatorname{supp} f\,, \quad \xi \in \bar{B}(x, \varrho)\,. \end{aligned}$$

Let $\alpha > 0$, $K' > 1$, and let $\delta_1 \colon \varphi^{-1}(\operatorname{supp} f) \to (0, \infty)$ be a gauge.
Assume that

$$\Delta' = \{(s^j, \zeta_j);\ j = 1, \ldots, k\}$$

is a δ_1-fine PU-partition of $\varphi^{-1}(\operatorname{supp} f)$. Put

$$\sigma_j = \sup\{\|y - s^j\|;\ y \in \operatorname{supp} \zeta_j\}$$

and assume that (1.5)–(1.7) is fulfilled for Δ' (with $t^j, \vartheta_j, \varrho_j, K$ replaced by $s^j, \zeta_j, \sigma_j, K'$, respectively).

Put $t^j = \varphi(s^j)$, $\vartheta_j = \zeta_j \circ \varphi^{-1}$. Then it is not difficult to see that

$$\Delta = \{(t^j, \vartheta_j);\ j = 1, \ldots, k\} \tag{3.4}$$

is a PU-partition of $\operatorname{supp} f$. It follows from (3.3) that Δ is δ_2-fine with

$$\delta_2(x) = c\delta_1(\varphi^{-1}(x)), \quad x \in \operatorname{supp} f. \tag{3.5}$$

We shall prove that (1.5)–(1.7) hold for Δ provided

$$K = c^4 K' \tag{3.6}$$

(ϱ_j has been defined after (1.4)). Observe that (3.3) implies

$$\sigma_j \leqq c\varrho_j, \quad \varrho_j \leqq c\sigma_j.$$

Since $t^j = \varphi(s^j)$, we have (for $x \in \mathbb{R}^n$)

$$\vartheta_j(x) = \zeta_j(\varphi^{-1}(x)) < (1 + \alpha)\,\zeta_j(s^j) = (1 + \alpha)\,\vartheta_j(\varphi(s^j)) = (1 + \alpha)\,\vartheta_j(t^j)$$

and (1.5) is fulfilled.

Let $x \in B(t^j, \varrho_j/K)$. Then $\varphi^{-1}(x) = y \in B(s^j, c\sigma_j/K) \subset B(s^j, \sigma_j/K')$, hence $\vartheta_j(x) = \zeta_j(\varphi^{-1}(x)) = \zeta_j(y) = \zeta_j(s^j) = \vartheta_j(t^j)$ and (1.6) holds.

Now, let us prove (1.7). We have

$$\int \|D\,\vartheta_j(x)\|\,dx \leqq \int \|D\,\zeta_j(y)\|\,\|D\,\varphi^{-1}(\varphi(y))\|\,|\det D\,\varphi(y)|\,dy \leqq$$
$$\leqq c^2 \int \|D\,\zeta_j(y)\|\,dy \leqq \frac{c^2 K'}{\sigma_j} \int \zeta_j(y)\,dy \leqq \frac{c^4 K'}{\varrho_j} \int \vartheta_j(x)\,dx$$

and (1.7) holds for Δ.

After the preliminary considerations let us proceed to the proof proper. Let $\varepsilon > 0$. Find $\alpha > 0$ from Definition 2.1. Given $K' > 1$, find a gauge δ on $\operatorname{supp} f$ such that

$$\left|(\mathrm{PU})\int f(x)\,dx - S(f, \Delta)\right| \leqq \tfrac{1}{2}\varepsilon \tag{3.7}$$

holds for every δ-fine PU-partition Δ of $\operatorname{supp} f$ satisfying (1.5)–(1.7) with $K = c^4 K'$. Assume in addition that

$$|\det D\,\varphi(y) - \det D\,\varphi(\eta)| \leqq \frac{\varepsilon}{2\,m_n(G)\,[1 + |f(\varphi(y))|]} \tag{3.8}$$
$$\text{for} \quad y \in \varphi^{-1}(\operatorname{supp} f), \quad \eta \in B(y, \delta(y))$$

(which can be achieved by decreasing δ if necessary).

Let Δ' be a $c^{-1}\delta$-fine PU-partition of $\varphi^{-1}(\operatorname{supp} f)$ satisfying (1.5)–(1.7) with $s^j, \zeta_j, \sigma_j, K'$ instead of $t^j, \vartheta_j, \varrho_j, K$, respectively. Define Δ by (3.4). Then Δ is a δ-fine PU-partition of $\operatorname{supp} f$ (cf. (3.5)) satisfying (1.5)–(1.7) with $K = c^4 K'$, so that (3.7) holds. By easy calculation we have

$$S(f, \Delta) = \sum_{j=1}^{k} f(\varphi(s^j)) \int \zeta_j(y) \left|\det D\,\varphi(y)\right| \mathrm{d}y$$

and by virtue of (3.8) we find that

$$\left|S(f, \Delta) - S((f \circ \varphi) \left|\det D\varphi\right|, \Delta')\right| \leqq \tfrac{1}{2}\varepsilon\,.$$

This together with (3.7) yields

$$\left|(\mathrm{PU}) \int f(x)\,\mathrm{d}x - S((f \circ \varphi)\left|\det D\varphi\right|, \Delta')\right| \leqq \varepsilon$$

and the proof of Theorem 3.1 is complete.

4. MULTIPLICATION OF PU-INTEGRABLE FUNCTIONS

4.1. Theorem. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ with a compact support be PU-integrable. Let $G \supset \operatorname{supp} f$ be an open bounded set, let $\chi\colon G \to \mathbb{R}$ be of class C^1. Then the function $f\chi$ is PU-integrable.*

We will first prove a less general result.

4.2. Theorem. *Let the assumptions of Theorem 4.1 be fulfilled, let $D\chi(x) \neq 0$ for $x \in G$. Then the function $f\chi$ is PU-integrable.*

Proof. Without loss of generality we will assume that

$$\chi\colon G \to \left[\tfrac{1}{4}, \tfrac{3}{4}\right]. \tag{4.1}$$

To prove our theorem we will use the analogue of the Bolzano-Cauchy condition, that is, we will estimate the difference of two integral sums corresponding to sufficiently fine PU-partitions.

Let $\varepsilon > 0$; find $\alpha > 0$ corresponding to ε according to Definition 2.1 of the PU-integral. Given $K > 1$, find a gauge δ on $\operatorname{supp} f$ corresponding to ε, α, K according to the same definition. Then there is a constant $b > 0$ and a gauge δ_1 on $\operatorname{supp} f$, $\delta_1(x) \leqq \delta(x)$ for $x \in \operatorname{supp} f$, such that the following proposition is true.

Proposition. *Let*

$$\Theta_i = \{(t^j, \vartheta_j);\ j \in J_i\}\,, \quad i = 1, 2\,, \quad J_1 \cap J_2 = \emptyset$$

be δ_1-fine PU-partitions of $\operatorname{supp} f$ *satisfying*

$$\vartheta_j(x) < \left(1 + \frac{\alpha}{5}\right) \vartheta_j(t^j)\,, \quad x \in \mathbb{R}^n\,; \tag{4.2}$$

$$\vartheta_j(x) = \vartheta_j(t^j)\,, \quad x \in B\left(t^j, \left(\frac{b\varrho_j}{K}\right)\right); \tag{4.3}$$

(4.4) $$\int \|D\,\vartheta_j(x)\|\,dx \leqq \frac{K}{b\varrho_j}\int \vartheta_j(x)\,dx\,,$$

where $\varrho_j = \sup\{\|x - t^j\|;\ x \in \operatorname{supp}\vartheta_j\}$. *Set*

(4.5) $$\chi_j = \begin{cases}\chi & \text{for } j \in J_1\\ 1-\chi & \text{for } j \in J_2\,.\end{cases}$$

Then there exists a δ_1*-fine PU-partition of* $\operatorname{supp} f$,

$$\Lambda = \{(t^j, \lambda_{jl});\ j \in J_1 \cup J_2,\ l = 0, 1, \ldots, L_j\}\,,\quad L_j \geqq 0\,,$$

such that Λ *satisfies* (1.5)–(1.7) *with* ϑ_j *replaced by* λ_{jl}, *and*

(4.6) $$\int |\vartheta_j(x)\,\chi_j(x) - \sum_{l=0}^{L_j}\lambda_{jl}(x)|\,dx \leqq \varepsilon(1 + \sum_{j\in J_1\cup J_2}|f(t^j)|)^{-1}\,,$$
$$j \in J_1 \cup J_2\,.$$

Let us first show that Theorem 4.1 is an easy consequence of this proposition.

Let Θ_i, $i = 1, 2$, be PU-partitions of $\operatorname{supp} f$ satisfying the assumptions of Proposition. Evidently, Θ_i, $i = 1, 2$, as well as Λ from Proposition are δ-fine and satisfy (1.5)–(1.7). Following the definition of the PU-integral we have

$$|(\mathrm{PU})\int f(x)\,dx - \sum_{j,l} f(t^j)\int \lambda_{jl}(x)\,dx| \leqq \varepsilon\,;$$

hence (4.6) yields

$$|(\mathrm{PU})\int f(x)\,dx - \sum_{j\in J_1\cup J_2} f(t^j)\int \vartheta_j(x)\,\chi_j(x)\,dx)| \leqq$$
$$\leqq \varepsilon + |\sum_{j\in J_1\cup J_2} f(t^j)\int[\vartheta_j(x)\,\chi_j(x) - \sum_{l=0}^{L_j}\lambda_{jl}(x)]\,dx| \leqq 2\varepsilon\,.$$

Taking into account the definition of χ_j (cf. (4.5)) and the above mentioned fact that Θ_2 is δ-fine and satisfies (1.5)–(1.7) we obtain

$$|\sum_{j\in J_2} f(t^j)\int \vartheta_j(x)\,\chi(x)\,dx - \sum_{j\in J_1} f(t^j)\int \vartheta_j(x)\,\chi(x)\,dx| =$$
$$= |\sum_{j\in J_2} f(t^j)\int \vartheta_j(x)\,(1 - \chi_j(x))\,dx - \sum_{j\in J_1} f(t^j)\int \vartheta_j(x)\,\chi_j(x)\,dx| \leqq$$
$$\leqq |\sum_{j\in J_2} f(t^j)\int \vartheta_j(x)\,dx - (\mathrm{PU})\int f(x)\,dx| +$$
$$+ |(\mathrm{PU})\int f(x)\,dx - \sum_{j\in J_1\cup J_2} f(t^j)\int \vartheta_j(x)\,\chi_j(x)\,dx| \leqq 3\varepsilon\,.$$

We may assume that the gauge δ_1 satisfies the condition

(4.7) $$\text{if } u \in \operatorname{supp} f,\ x \in B(u, \delta_1(u)), \text{ then } B(u, \delta_1(u)) \subset G \text{ and}$$
$$|\chi(x) - \chi(u)| < \frac{\varepsilon}{(1 + |f(u)|)\,\mathscr{m}_n(G)}\,.$$

Then evidently

(4.8) $$|\sum_{j\in J_2} f(t^j)\,\chi(t^j)\int \vartheta_j(x)\,dx - \sum_{j\in J_1} f(t^j)\,\chi(t^j)\int \vartheta_j(x)\,dx| \leqq 5\varepsilon\,,$$

which is the desired analogue of the Bolzano-Cauchy condition. The existence of the integral (PU) $\int f(x)\,\chi(x)\,dx$ follows by the standard argument.

Thus we have to prove Proposition, that is, to construct a partition Λ with the required properties. To this end we will use Lemma 2.6.

For $j \in J_1 \cup J_2$ put $\varphi_0 = \vartheta_j \chi_j$, $w = t^j$, $\omega_0 = \varrho_j$. In order to justify the application of Lemma 2.6 we have first to find conditions which b, θ, K_1, δ_1 have to satisfy in order that (2.4)–(2.7) might hold. Without loss of generality we will assume $\alpha < \frac{1}{3}$ (cf. Remark 2.2). Set $\theta = \frac{1}{2}\alpha < \frac{1}{6}$.

Comparing (2.4) with (4.3), we see that (2.4) holds if

$$K_1 \geqq K/b\,. \tag{4.9}$$

Further, $\gamma = \|D(\vartheta_j \chi_j)(t^j)\| = \vartheta_j(t^j)\,\|D\,\chi(t^j)\| > 0$ in view of (4.3) and the condition $D\,\chi(x) \neq 0$, and (2.5) holds if the gauge δ_1 satisfies the inequality

$$\|D\,\chi(x) - D\,\chi(u)\| \leqq \tfrac{1}{2}\|D\,\chi(u)\|\,, \quad x \in B(u, \delta_1(u))\,. \tag{4.10}$$

Indeed, $B(t^j, \omega_0/K_1) \subset B(t^j, b\varrho_j/K)$, and for x from the bigger ball we have (cf. (4.3)) $D\,\varphi_0(x) - D\,\varphi_0(t^j) = \vartheta_j(t^j)\,[D\,\chi_j(x) - D\,\chi_j(t^j)]$.

The inequality (2.6) reads

$$\vartheta_j(x)\,\chi_j(x) \leqq (1 + \alpha/2)\,\vartheta_j(t^j)\,\chi_j(t^j)\,.$$

It is fulfilled, by virtue of (4.2), if δ_1 satisfies

$$\chi_j(x) \leqq (1 + \alpha/5)\,\chi_j(u)\,, \quad x \in B(u, \delta_1(u))\,, \tag{4.11}$$

since

$$(1 + \alpha/5)^2 \leqq 1 + \alpha/2\,.$$

Finally, to satisfy (2.7) it suffices to subject the gauge δ_1 to the condition

$$6\left(1 + \frac{\alpha}{5}\right)\|D\,\chi_j(u)\|\,\frac{K^{n-1}}{b^{n-1}}\,\delta_1(u) \leqq 1\,, \quad u \in \operatorname{supp} f\,, \tag{4.12}$$

and the constant K_1 to the condition

$$K_1 \geqq 4K/b\,. \tag{4.13}$$

Indeed, we have (by virtue of (4.1), (4.2), (4.10), (4.4), (4.12) and (4.3))

$$\int \|D(\vartheta_j \chi_j)(x)\|\,dx \leqq \max \vartheta_j(x) \int_{B(t^j,\varrho_j)} \|D\,\chi_j(x)\|\,dx + \tfrac{3}{4}\int \|D\,\vartheta_j(x)\|\,dx \leqq$$

$$\leqq \left(1 + \frac{\alpha}{5}\right)\vartheta_j(t^j)\,\tfrac{3}{2}\|D\,\chi_j(t^j)\|\,\varkappa_1 \varrho_j^n + \frac{3}{4}\frac{K}{b\varrho_j}\int \vartheta_j(x)\,dx \leqq$$

$$\leqq \left(1 + \frac{\alpha}{5}\right)\tfrac{3}{2}\|D\,\chi_j(t^j)\|\,4\,\vartheta_j(t^j)\min \chi_j(x)\,\varkappa_1\left(\frac{K}{b}\right)^n\left(\frac{b\varrho_j}{K}\right)^n +$$

$$+ \frac{3K}{b\varrho_j}\int \vartheta_j(x)\,\chi_j(x)\,dx \leqq$$

$$\leqq 6\left(1+\frac{\alpha}{5}\right)\|D\,\chi_j(t^j)\|\left(\frac{K}{b}\right)^{n-1}\frac{K}{b}\int_{B(t^j,b\varrho_j/K)}\vartheta_j(x)\min\chi_j(x)\,dx\,+$$

$$+\frac{3K}{b\varrho_j}\int\vartheta_j(x)\,\chi_j(x)\,dx\leqq\frac{4K}{b\varrho_j}\int\vartheta_j(x)\,\chi_j(x)\,dx$$

and (2.7) holds by (4.13).

Consequently, if δ_1 is a gauge on supp f satisfying $\delta_1(x) \leqq \delta(x)$ and (4.7), (4.10), (4.11), (4.12), if $\theta = \frac{1}{2}\alpha$ and if K_1 satisfies (4.13) (and, a fortiori, (4.9)), then we can apply Lemma 2.6 to $\varphi_0 = \vartheta_j\chi_j$ as desired.

Let us further assume $\beta = \frac{1}{4}\alpha$ and

(4.14) $$K_1 \geqq 4\varkappa_7\,,\quad K_1 \geqq 24/\alpha\,.$$

Then conditions (2.37)–(2.39) are satisfied, and thus we can use Lemma 2.6 repeatedly as in the proof of Lemma 2.8, obtaining for each $j \in J_1 \cup J_2$ a set of pairs

(4.15) $$(t^j, E\vartheta_j\chi_j), (t^j, EF\vartheta_j\chi_j), \ldots, (t^j, EF^{N_j-1}\vartheta_j\chi_j), (t^j, F^{N_j}\vartheta_j\chi_j)$$

(N_j are positive integers to be fixed later).

Let us list some properties that the functions appearing in (4.15) possess, denoting

$$\omega_i^j = \sup\left\{\|x - t^j\|;\ x \in \operatorname{supp} F^i\vartheta_j\chi_j\right\},\quad j \in J_1 \cup J_2\,,\quad i = 0, 1, \ldots, N_j :$$

(4.16) $$\omega_0^j = \varrho_j\,,\quad \omega_i^j = \sup\left\{\|x - t^j\|;\ x \in \operatorname{supp} EF^i\vartheta_j\chi_j\right\}$$

(cf. (2.44), (2.43)) ;

(4.17) $$EF^i\vartheta_j\,\chi_j(x) \leqq \left(1 + \frac{\alpha}{2}\right)\left(1 - \frac{4}{K_2}\right)^{-1} EF^i\vartheta_j\,\chi_j(t^j)$$

(cf. (2.46); recall that $\frac{1}{2}\alpha = \theta$) ;

(4.18) $$EF^i\vartheta_j\,\chi_j(x) = EF^i\vartheta_j\,\chi_j(t^j)\,,\quad x \in \bar{B}\left(t^j, \frac{\alpha\omega_i^j}{6K_1K_2}\right)$$

(cf. (2.45)) ;

(4.19) $$\int\|D(EF^i\vartheta_j\,\chi_j(x))\|\,dx \leqq \left(K_1 + \frac{\alpha}{2}\varkappa_7\varkappa_8K_1K_2^{-n}\right)\cdot$$

$$\cdot\left(1 - \frac{\alpha}{2}\varkappa_8K_2^{-(n+1)}\right)^{-1}\frac{1}{\omega_i^j}\int EF^i\vartheta_j\,\chi_j(x)\,dx$$

(cf. (2.47)) ;

(4.20) $$\sum_{i=0}^{N_j-1} EF^i\vartheta_j\chi_j + F^{N_j}\vartheta_j\chi_j \approx \vartheta_j\chi_j$$

(cf. (2.40)) ;

(4.21) $$F^i\vartheta_j\,\chi_j(x) \leqq \left(1 + \frac{\alpha}{2}\right)F^i\vartheta_j\,\chi_j(t^j)$$

(cf. (2.16)) ;

$$(4.22) \qquad \int \|D(F^i\vartheta_j\chi_j)(x)\| \,\mathrm{d}x \leqq \frac{4\varkappa_7}{\omega_i^j} \int F^i\vartheta_j\,\chi_j(x)\,\mathrm{d}x$$

(cf. (2.17));

$$(4.23) \qquad \frac{2^{3/2}}{3K_1K_2} \leqq \frac{\omega_{i+1}^j}{\omega_i^j} \leqq \frac{2^{3/2}}{K_1K_2}$$

(cf. (2.13)).

Comparing the inequalities (4.17), (4.18), (4.19) with (1.5)–(1.7), we see that the pairs from (4.15) except the last ones will satisfy (1.5)–(1.7) if

$$(4.24) \qquad \left(1 + \frac{\alpha}{2}\right)\left(1 - \frac{4}{K_2}\right)^{-1} < 1 + \alpha\,,$$

$$(4.25) \qquad \frac{9K_1K_2}{\alpha} < K\,,$$

$$(4.26) \qquad K_1\left(1 + \frac{\alpha}{2}\,\varkappa_7\varkappa_8 K_2^{-n}\right)\left(1 - \frac{\alpha}{2}\,\varkappa_8 K_2^{-(n+1)}\right)^{-1} < K\,.$$

Let us summarize our considerations. First we have to find α, $0 < \alpha < \frac{1}{3}$, corresponding to the given $\varepsilon > 0$. Let K be given. Without loss of generality we will assume (cf. Remark 2.2)

$$(4.27) \qquad K > \frac{160c_1}{\alpha^3}\,, \quad c_1 = \max\{4\varkappa_7, 24\}\,.$$

Choose $b = 4K\alpha/c_1$. In Lemma 2.6 choose $K_1 = c_1/\alpha$ so that (4.13) and (4.14) are satisfied. Set $\theta = \frac{1}{2}\alpha$, $\beta = \frac{1}{4}\alpha$. Choose K_2 so that (4.24), (4.25) and also the inequality $K_2 > \max\{9, (\varkappa_8\,\frac{1}{2}\alpha)^{1/(n+1)}\}$ from Lemma 2.6 are fulfilled. (Notice that (4.24) is certainly fulfilled if $K_2 > 16/\alpha$, while in view of (4.27), (4.25) is fulfilled if $K_2 < 160/9\alpha$; hence both the inequalities can be satisfied simultaneously.) The inequality (4.26) is then fulfilled as well, at least for α small enough.

To the given K find the gauge δ and choose a gauge δ_1 so that (4.7), (4.10) and (4.12) hold.

Thus, we have fixed all the constants involved in such a way that, on the basis of Lemma 2.6, we can construct the functions in (4.15) and that, moreover, the functions $EF^i\vartheta_j\chi_j$ satisfy (1.5)–(1.7).

Let us now continue in the proof proper. To construct the partition A we will use the pairs from (4.15) except the last ones of the form $(t^j, F^{N_j}\vartheta_j\chi_j)$; the functions $F^{N_j}\vartheta_j\chi_j$ from these pairs will be either added to some of the functions $EF^i\vartheta_p\chi_p$ with $i \in \{0, 1, \ldots, N_p - 1\}$ suitably chosen, or otherwise arranged in such a way that the resulting functions will still have the required properties, in particular, will satisfy (1.5)–(1.7).

For $w \in \mathbb{R}^n$ we denote

$$J_i(w) = \{j \in J_i;\ t^j = w\}\,, \quad i = 1, 2\,;$$

$$U = \{w;\ \sum_{j \in J_1(w)} \vartheta_j(w) = 1 = \sum_{j \in J_2(w)} \vartheta_j(w)\}\,;$$

$$V = \{w;\ 0 < \sum_{j \in J_1(w) \cup J_2(w)} \vartheta_j(w) < 2\}\,.$$

Evidently,

(4.28) $$U \cap V = \emptyset\,, \quad U \cup V = \{t^j;\ j \in J_1 \cup J_2\}\,.$$

It follows from the definition of the set V that

(4.29) for every $w \in V$ there is such $p(w) \in J_1 \cup J_2$ that

$$t^{p(w)} \neq w\,, \quad \vartheta_{p(w)}(w) > 0\,.$$

Further, (4.23) implies that for every $w \in V$ there is an index $i = i(w)$ such that

(4.30) $$\omega_{i(w)+1}^{p(w)} \leqq \|w - t^{p(w)}\| < \omega_{i(w)}^{p(w)}\,.$$

Since V is finite, there exists a positive integer Q such that

(4.31) $$\|w - t^{p(w)}\| > \omega_Q^{p(w)} \quad \text{for all} \quad w \in V$$

(evidently $Q > i(w)$ for all $w \in V$).

In the sequel we will assume, for $j \in J_1 \cup J_2$, $t^j \in V$:

(4.32) $$N_j \geqq Q\,,$$

$$\omega_{N_j}^j < \frac{2^{3/2} - 1}{3K_1K_2}\, \omega_{i(t^j)}^{p(t^j)}$$

$$\omega_{N_j}^j < \omega_{i(t^j)}^{p(t^j)} - \|t^{p(t^j)} - t^j\|\,.$$

(Notice that the right hand side of the last inequality is positive by (4.30), and $\lim_{i \to \infty} \omega_i^j = 0$ by (4.23).)

Let $t^j \in V$. Then we add the function $F^{N_j}\vartheta_j\chi_j$ to the function $EF^{i(t^j)}\vartheta_{p(t^j)}\chi_{p(t^j)}$. (It may happen that several functions $F^{N_j}\vartheta_j\chi_j$ – with different indices j – are added to the same function $EF^i\vartheta_m\chi_m$. In that case, however, $p(t^j) = m$ for all such j's.)

Now, let $w \in U$. Then evidently $J_1(w) \neq \emptyset \neq J_2(w)$; let us denote by $q(w)$ the number of elements in the union $J_1(w) \cup J_2(w)$. For $j \in J_1(w) \cup J_2(w)$ we replace the last pair in (4.15) by the pair

(4.33) $$\left(w, \frac{1}{q(w)}\, \Psi_w\right), \quad \text{where} \quad \Psi_w = \sum_{j \in J_1(w) \cup J_2(w)} F^{N_j}\vartheta_j\chi_j$$

(that is, given $w \in U$, we put together all pairs with $t^j = w$, thus forming a single pair (4.33)).

All pairs resulting from (4.15) by the above described modifications form the desired PU-partition Λ; in the sequel, we denote them

$$(t^j, \lambda_{j0}), (t^j, \lambda_{j1}), \ldots, (t^j, \lambda_{jL_j}) ;$$

evidently we have $L_j = N_j$ for $t^j \in U$, $L_j = N_j - 1$ for $t^j \in V$.

It follows from the construction and from (4.20) that

$$\sum_{j,l} \lambda_{jl} = \sum_j (E\vartheta_j\chi_j + \ldots + EF^{N_j-1}\vartheta_j\chi_j + F^{N_j}\vartheta_j\chi_j) =$$

$$= \sum_j \vartheta_j\chi_j = \sum_{j\in J_1} \vartheta_j\chi + \sum_{j\in J_2} \vartheta_j(1-\chi) ;$$

hence if x is such that $\sum_{j\in J_1} \vartheta_j(x) = \sum_{j\in J_2} \vartheta_j(x) = 1$ then

$$\sum_{j,l} \lambda_{jl} = 1 ,$$

which implies that Λ is a PU-partition of supp f. Moreover, it is evident that it is δ_1-fine (this follows from the fact that Θ_i, $i = 1, 2$, are δ_1-fine, and from (4.32)).

Now we have to show that Λ satisfies (4.6) and (1.5)–(1.7). This will be proved provided N_j satisfy some further conditions.

First of all, let us assume that N_j is so large that

$$(4.34) \qquad \sum_{j\in J_1\cup J_2} \int F^{N_j}\vartheta_j\chi_j \,dx < \varepsilon(1 + \sum_{j\in J_1\cup J_2} |f(t^j)|)^{-1}$$

(cf. (2.41)). Then (4.6) immediately follows from (4.20) and from the construction of Λ.

In order to fulfil (1.5), we further require that for $t^j \in V$, $k \in \{0, 1, \ldots, N_j - 1\}$, N_j is so large that

$$(4.35) \qquad \sum_{\substack{m\in J_1\cup J_2 \\ p(t^m)=j,\, i(t^m)=k}} F^{N_m}\vartheta_m\,\chi_m(x) \leqq \left[1 + \alpha - \left(1 + \frac{\alpha}{2}\right)\left(1 - \frac{4}{K_2}\right)^{-1}\right] EF^k\vartheta_j\,\chi_j(t^j)$$

(cf. (4.24) and (2.41)). This together with (4.17) yields that (1.5) is fulfilled for λ_{ml} provided $l < N_m$. If $t^m \in U$, $l = L_m = N_m$ then

$$\lambda_{ml} = \frac{1}{q(t^m)}\,\Psi_{t^m}$$

and (1.5) again holds by virtue of (4.21).

Now we will prove (1.7). By virtue of (4.23) we may and will assume that

$$(4.36) \qquad \text{if} \quad w \in U\,, \quad j, r \in J_1(w) \cup J_2(w)\,, \quad \text{then}$$

$$\frac{\omega^j_{N_j}}{\omega^r_{N_r}} \leqq 3\,.\,2^{-3/2}K_1K_2\,.$$

Indeed, this can be achieved by starting with such $s \in J_1(w) \cup J_2(w)$ that

$$(4.37) \qquad \omega^s_{N_s} = \min\{\omega^j_{N_j} : j \in J_1(w) \cup J_2(w)\}$$

and then successively increasing the other N_j's in order to fulfil (4.36) with $r = s$

(we need not worry about the inverse ratio since in view of (4.37) it never exceeds one).

This procedure has to be repeated (finitely many times), in each step omitting the minimal $\omega^s_{N_s}$.

Moreover, for $t^j \in V$, $k \in \{0, 1, \ldots, N_{j-1}\}$ let us assume

$$\sum_{\substack{m \in J_1 \cup J_2 \\ p(t^m) = j, i(t^m) = k}} \int \|F^{N_m} \vartheta_m \chi_m(x)\| \,\mathrm{d}x \leqq \tag{4.38}$$

$$\leqq [K - (1 - \theta \varkappa_8 K_2^{-(n+1)})^{-1} (K_1 + \theta \varkappa_7 \varkappa_8 K_1 K_2^{-n})] \frac{1}{\omega^j_k} \int E F^k \vartheta_j \chi_j(x) \,\mathrm{d}x$$

(notice that the expression in the square brackets is positive in view of (4.26), and again recall (2.41)).

Let us first consider a pair (t^j, λ_{jl}) with $j \in J_1 \cup J_2$, $l < N_j$. Then

$$\omega^j_l = \sup \{\|x - t^j\|;\ x \in \operatorname{supp} \lambda_{jl}\} .$$

Indeed, if $j = p(t^m)$ for some $m \in J_1 \cup J_2$ and $l = i(t^m)$, then this identity follows from the third inequality in (4.32) since this inequality implies $B(t^m, \omega^m_{N_m}) \subset$ $\subset B(t^j, \omega^j_{i(t^m)})$. In the other cases, $\lambda_{jl} \equiv EF^l \vartheta_j \chi_j$ and our identity is trivial (cf.(4.16)). Consequently, (1.7) holds (with λ_{jl}, ω^j_l instead of ϑ_j, ϱ_j) for $j \in J_1 \cup J_2$, $l < N_j$ in view of (4.19), (4.26), (4.38) and the above identity.

The last case for which we have to prove (1.7) is that of λ_{rl} with $t^r \in U$, $l = L_r$; then

$$\lambda_{rl} = \frac{1}{q(t^r)} \Psi_{t^r} .$$

Let s satisfy (4.37) with $w = t^r$, and denote

$$\tau_r = \sup \{\|x - t^r\|;\ x \in \operatorname{supp} \Psi_{t^r}\} .$$

By (4.36) we have $\omega^j_{N_j} \leqq 3 \,.\, 2^{-3/2} K_1 K_2 \omega^s_{N_s}$ for $j \in J_1(t^r) \cup J_2(t^r)$, and since $\tau_r =$ $= \max \{\omega^j_{N_j};\ j \in J_1(t^r) \cup J_2(t^r)\}$ (cf. (4.33), we conclude

$$\tau_r \leqq 3 \,.\, 2^{-3/2} K_1 K_2 \omega^s_{N_s} . \tag{4.39}$$

The inequalities (4.33), (4.22), (4.37) and (4.39) yield the estimate

$$\int \|D \Psi_{t^r}(x)\| \,\mathrm{d}x \leqq \sum_{j \in J_1(t^r) \cup J_2(t^r)} \int \|D(F^{N_j} \vartheta_j \chi_j)(x)\| \,\mathrm{d}x \leqq$$

$$\leqq \sum_{j \in J_1(t^r) \cup J_2(t^r)} \frac{4\varkappa_7}{\omega^j_{N_j}} \int F^{N_j} \vartheta_j \chi_j(x) \,\mathrm{d}x \leqq \frac{4\varkappa_7}{\omega^s_{N_s}} \int \Psi_{t^r}(x) \,\mathrm{d}x \leqq \frac{3 \,.\, 2^{1/2} K_2 K_3}{\tau_r} \int \Psi_{t^r}(x) \,\mathrm{d}x$$

and (1.7) holds provided

$$3 \,.\, 2^{1/2} K_1 K_2 < K , \tag{4.40}$$

which evidently is a weaker condition than (4.25) (recall that $\alpha < \frac{1}{3}$).

The last step is to prove (1.6) (for λ_{jl}, ω^j_l, of course). Again let us first consider the case $j \in J_1 \cup J_2$, $l < N_j$. If $j \neq p(t^m)$ for all $m \in J_1 \cup J_2$, (1.6) is obviously fulfilled. If $j = p(t^m)$ for some $m \in J_1 \cup J_2$ and $l = i(t^m)$, then the second inequality in (4.32) and the first inequality in (4.30) combined with (4.23) yield

$$B(t^m, \omega^m_{N_m}) \cap B\left(t^j, \frac{\alpha}{3K_1K_2}\, \omega^j_l\right) = \emptyset$$

(recall that $\alpha < \frac{1}{3}$). Consequently,

$$\lambda_{jl}(x) = EF^l\vartheta_j\, \chi_j(x) \quad \text{for} \quad x \in B\left(t^j, \frac{\alpha}{3K_1K_2}\, \omega^j_l\right)$$

and (1.6) follows from (4.25).

We still have to prove (1.6) for λ_{rl} with $t^r \in U$, $l = L_r$. By the definition of U and by (4.3) we have

$$\sum_{j \in J_1(t^r)} \vartheta_j(x) = 1 = \sum_{j \in J_2(t^r)} \vartheta_j(x)$$

for $x \in B(t^r, b\varrho/K)$, where $\varrho = \min\{\varrho_j;\, j \in J_1(t^r) \cup J_2(t^r)\}$. Hence (4.5) yields

$$\sum_{j \in J_1(t^r) \cup J_2(t^r)} \vartheta_j(x)\, \chi_j(x) = 1$$

for $x \in B(t^r, b\varrho/K)$, and in view of (4.20) we can write this identity in the form

(4.41)

$$\sum_{j \in J_1(t^r) \cup J_2(t^r)} [E\vartheta_j\, \chi_j(x) + EF\vartheta_j\, \chi_j(x) + \ldots + EF^{N_j-1}\vartheta_j\, \chi_j(x) + F^{N_j}\vartheta_j\, \chi_j(x)] = 1\,.$$

In this identity we can put $x = t^r$; using (4.18) in which we set $t^j = t^r$ (recall the definition of $J_1(t^r)$, $J_2(t^r)$) we obtain

$$\sum_{j \in J_1(t^r) \cup J_2(t^r)} F^{N_j}\vartheta_j\, \chi_j(x) = \sum_{j \in J_1(t^r) \cup J_2(t^r)} F^{N_j}\vartheta_j\, \chi_j(t^r) \tag{4.42}$$

for $x \in B(t^r, \sigma_1)$, where

$$\sigma_1 = \min\left\{\frac{\alpha}{3K_1K_2}\, \omega^j_{N_j-1};\, j \in J_1(t^r) \cup J_2(t^r)\right\}.$$

By (4.23) we have $2^{-3/2}K_1K_2\omega^j_{N_j} \leqq \omega^j_{N_j-1}$, hence (4.42) is valid for $x \in B(t^r, \sigma_2)$, where

$$\sigma_2 = \frac{\alpha}{3\,.\,2^{3/2}} \min\{\omega^j_{N_j};\, j \in J_1(t^r) \cup J_2(t^r)\}\,.$$

If s is the index for which (4.37) holds with $w = t^r$, then evidently

$$\sigma_2 = \frac{\alpha}{3\,.\,2^{3/2}}\, \omega^s_{N_s} \geqq \frac{\alpha}{9K_1K_2}\, \tau^r$$

(cf. (4.39)) and (1.6) holds provided (4.25) is valid.

We have already shown that the conditions concerning K_1, K_2 can be satisfied by a suitable choice of the constants. Further, the conditions imposed on N_j, i.e. (4.32), (4.34)–(4.36), (4.38) are easily satisfied by gradually increasing N_j.

Thus the proof of Proposition is complete, and Theorem 4.2 is proved as well.

4.3. Proof of Theorem 4.1. Choose an open set G_1 such that $\operatorname{supp} f \subset G_1 \subset \bar{G}_1 \subset$

$\subset G$, and a number λ so large that $D(\chi + \chi_1)(x) \neq 0$ for $x \in G_1$, where $\chi_1(x) = \lambda x_1$ (we write $x = (x_1 x_2, \ldots, x_n)$). Such a λ obviously exists. Hence $f(\chi + \chi_1)$ is PU-integrable, and the same evidently holds for $f\chi_1$.

Consequently, $f\chi = f(\chi + \chi_1) - f\chi_1$ is PU-integrable as well.

5. STOKES' THEOREM

5.1. Theorem. *Let $g: \mathbb{R}^n \to \mathbb{R}$ with compact support be differentiable at all points $x \in \mathbb{R}^n \setminus W$. For $p = 1, \ldots, n$ set*

$$f_p = \begin{cases} (\partial g/\partial x_p)(x) & \text{for } x \in \mathbb{R}^n \setminus W, \\ 0 & \text{for } x \in W. \end{cases} \tag{5.1}$$

Then f_p is PU-integrable and

$$(\mathrm{PU}) \int f_p(x)\,\mathrm{d}x = 0 \tag{5.2}$$

provided one of the following conditions is fulfilled:

(5.3) *g is continuous, $W = \{x \in \mathbb{R}^n,\ x_1 = 0\}$;*

(5.4) *W is closed, g is bounded; for every $\varepsilon > 0$ there is $\alpha' > 0$ such that for every $K > 1$ there is a gauge δ' on $\operatorname{supp} g$ such that for every δ'-fine PU-partition (1.1) of $\operatorname{supp} g$ satisfying (1.5)–(1.7) with α' instead of α the inequality*

$$\sum_{t^j \in W} \varrho_j^{-1} \int \vartheta_j(x)\,\mathrm{d}x \leqq \varepsilon$$

holds;

$$g(x) = o\{\|x\|^{1-n}\}, \quad W = \{0\}. \tag{5.5}$$

Proof. Following the idea of proof of the analogous theorem in [1], denote

$$q_t(x) = \begin{cases} g(t) + Dg(t)(x - t) & \text{for } t \notin W, \\ 0 & \text{for } t \in W. \end{cases}$$

Let $\varepsilon > 0$. Find $\alpha > 0$ according to Definition 2.1. Let $K > 1$. Find a gauge δ on $\operatorname{supp} g$ such that

$$m_n\Big(\bigcup_{x \in \operatorname{supp} g} \bar{B}(x, \delta(x))\Big) \leqq m_n(\operatorname{supp} g) + 1 = c \tag{5.6}$$

(m_n again stands for the Lebesgue measure in $\mathbb{R}^n$),

$$\bar{B}(x, 2\,\delta(x)) \cap W = \emptyset \quad \text{for} \quad x \notin W, \tag{5.7}$$

$$|g(x) - q_t(x)| \leqq (\varepsilon/K)\, c^{-1} \|x - t\| \quad \text{for} \quad t \notin W, \quad x \in B(t, \delta(t)). \tag{5.8}$$

Let Δ given by (1.1) be a δ-fine PU-partition of $\operatorname{supp} g$ satisfying (1.5)–(1.7). We have to estimate the integral sum $S(f_p, \Delta)$.

Integration by parts (with respect to x_p) together with the obvious identity

$$f_p(t^j) = \frac{\partial q_{t^j}}{\partial x_p}(x)$$

yields similarly as in [1]

$$f_p(t^j) \int \vartheta_j(x)\,\mathrm{d}x = \int \frac{\partial q_{tj}}{\partial x_p}(x)\,\vartheta_j(x)\,\mathrm{d}x = -\int q_{tj}(x)\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\,.$$

Further,

$$\sum_{j=1}^{k} \int g(x)\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x = 0$$

since $\sum_{j=1}^{k} \vartheta_j(x) = 1$ for $x \in \operatorname{supp} g$. Hence to establish the desired estimate for $S(f_p, \Delta)$ we have to establish the inequality

$$\left|\int \sum_{j=1}^{k} [g(x) - q_{tj}(x)]\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\right| \leqq \varepsilon\,. \tag{5.9}$$

Let us first estimate the terms with $t^j \notin W$:

$$\left|\int \sum_{t^j \notin W} [g(x) - q_{tj}(x)]\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\right| \leqq$$

$$\leqq \frac{\varepsilon}{K}\,c^{-1} \sum_{t^j \notin W} \int \|x - t^j\| \left|\frac{\partial \vartheta_j}{\partial x_p}(x)\right| \mathrm{d}x \leqq \varepsilon c^{-1} \int \sum_{j=1}^{k} \vartheta_j(x)\,\mathrm{d}x \leqq \varepsilon$$

by virtue of (5.6)–(5.8) and (1.7).

To estimate the terms with $t^j \in W$ we have to treat the three cases corresponding to the conditions (5.3)–(5.5) separately.

Let (5.3) be fulfilled. Without loss of generality we may assume that

$$|g(x) - g(t)| \leqq \varepsilon \quad \text{for} \quad x \in B(t, \delta(t))\,. \tag{5.10}$$

For $t^j \in W$ we have $q_{tj}(x) \equiv 0$; moreover, since ϑ_j and thus also $\partial\vartheta_j/\partial x_p$ have compact supports, we have

$$\int \frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x = 0\,.$$

Consequently, we can write

$$\left|\int \sum_{t^j \in W} [g(x) - q_{tj}(x)]\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\right| =$$

$$= \left|\int \sum_{t^j \in W} [g(x) - g(t^j)]\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\right|$$

and, using successively (5.10), (1.7*), (1.5) we obtain

$$\left|\int \sum_{t^j \in W} [g(x) - q_{tj}(x)]\,\frac{\partial \vartheta_j}{\partial x_p}(x)\,\mathrm{d}x\right| \leqq \varepsilon \sum_{t^j \in W} \int \left|\frac{\partial \vartheta_j}{\partial x_p}(x)\right| \mathrm{d}x \leqq$$

$$\leqq \varepsilon \varkappa_1 K(1 + \alpha) \sum_{t^j \in W} \varrho_j^{n-1}\,\vartheta_j(t^j) =$$

$$= \varepsilon \frac{\varkappa_1}{\varkappa_0} K^n(1 + \alpha) \sum_{t^j \in W} \int_{B(t^j, \varrho_j/K) \cap W} \vartheta_j(x)\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\,,$$

where $\varkappa_0$ is the measure of the $(n-1)$-dimensional unit ball. Since $\sum_{j=1}^{k} \vartheta_j(x) \leqq 1$ for all $x \in R^n$ and since we can assume $m_{n-1}\left(\bigcup_{t^j \in W} B(t^j, \varrho_j) \cap W\right) \leqq m_{n-1}(\operatorname{supp} g \cap W) + 1$, we eventually obtain

$$\varepsilon \frac{\varkappa_1}{\varkappa_0} K^n(1 + \alpha) \sum_{t^j \in W} \int_{B(t^j, \varrho_j/K) \cap W} \vartheta_j(x) \, dx_2 \ldots dx_n \leqq$$

$$\leqq \varepsilon \frac{\varkappa_1}{\varkappa_0} K^n(1 + \alpha) \int_{\bigcup_{t^j \in W} B(t^j, \varrho_j) \cap W} \sum_{j=1}^{k} \vartheta_j(x) \, dx_2 \ldots dx_n \leqq$$

$$\leqq \varepsilon \frac{\varkappa_1}{\varkappa_0} K^n(1 + \alpha) \left[m_{n-1}(\operatorname{supp} g \cap W) + 1\right],$$

which evidently completes the proof.

Now, let (5.4) be fulfilled, let $|g(x)| \leqq M$ for $x \in R^n$. For the given $\varepsilon > 0$, $K > 1$ find $\alpha' > 0$, δ' so that the inequality from condition (5.4) holds. Without loss of generality we may and will assume that $\alpha < \alpha'$, $\delta(x) < \delta'(x)$ for $x \in R^n$. Using the identity $q_{t^j}(x) = 0$ for $t^j \in W$, the boundedness of g, the condition (1.7) and the inequality from (5.4), we obtain

$$\left| \sum_{t^j \in W} \int [g(x) - q_{t^j}(x)] \frac{\partial \vartheta_j}{\partial x_p}(x) \, dx \right| \leqq M \sum_{t^j \in W} \int \left| \frac{\partial \vartheta_j}{\partial x_p}(x) \right| dx \leqq$$

$$\leqq MK \sum_{t^j \in W} \varrho_j^{-1} \vartheta_j(x) \, dx \leqq MK\varepsilon.$$

Finally, let (5.5) be fulfilled. Then the only terms to be estimated are those with $t^j = 0$ and their contribution reduces to

$$\sum_{t^j = 0} \left| \int_{B(0, \varrho_j)} g(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \, dx \right|$$

since $q_0(x) = 0$ again.

We divide the integration domain into two parts, $B_j' = B(0, \varrho_j) \setminus B(0, \varrho_j/K)$ and $B_j = B(0, \varrho_j/K)$, and write $g(x) = v(x) \|x\|^{1-n}$ with $\lim_{x \to 0} v(x) = 0$, $v = \sup\{|v(x)|; x \in B(0, \varrho_j)\}$. Then

$$\sum_{t^j = 0} \left| \int_{B_j'} g(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \, dx \right| \leqq vK^{n-1} \sum_{t^j = 0} \varrho_j^{1-n} \int \left| \frac{\partial \vartheta_j}{\partial x_p} \right| dx \leqq$$

$$\leqq vK^n \varkappa_1(1 + \alpha) \sum_{t^j = 0} \vartheta_j(0) = vK^n \varkappa_1(1 + \alpha)$$

by virtue of (1.7*), while (1.6) yields

$$\sum_{t^j = 0} \left| \int_{B_j} g(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \, dx \right| = 0.$$

Since $v \to 0$ with $\delta(0) \to 0$, a suitable choice of $\delta(0)$ completes the proof.

Let N be an n-manifold of class C^1 without boundary or with a boundary ∂N.

The concept of the PU-integral can be extended to differential n-forms on N in the same way as in [1]. From Theorem 5.1 and from [1], Theorem 4.2, Stokes' theorem can be proved in an analogous way as in [1], in the following form:

5.2. Theorem (Stokes). *Let η be an $(n-1)$-form with compact support on N. Let W be a submanifold of N with or without boundary, $W \cap \partial N = \emptyset$. Assume that η is differentiable at every point of $N \setminus W$ and that η is continuous. Then $\mathrm{d}\eta$ is a PU-integrable n-form and*

$$(\mathrm{PU}) \int_N \mathrm{d}\eta = \int_{\partial N} \eta \, .$$

5.3. Remark. If we make use of Theorem 5.1, case (5.5), we may modify the above theorem in the following way: $W = \{w_1, w_2, \ldots, w_m\} \subset N$, $W \cap \partial N = \emptyset$, η is differentiable at every point of $N \setminus W$ and fulfils the growth condition analogous to (5.5) in a neighbourhood of each w_j, $j = 1, \ldots, m$.

5.4. Remark. Let $\bar{B} = \bar{B}(0, 1) \subset \mathbb{R}^n$, $h: \bar{B} \to \mathbb{R}$. Let h have continuous derivatives of the second order on $\bar{B} \setminus \{0\}$ and let

$$\|(\operatorname{grad} h)(x)\| = o\{\|x\|^{1-n}\} \, . \tag{5.11}$$

It can be deduced from Theorem 5.2 that

$$\int_B \operatorname{div} \operatorname{grad} h \, \mathrm{d}x = \int_{\partial B} (\mathbf{v}, \operatorname{grad} h) \, \mathrm{d}S \, , \tag{5.12}$$

$\mathbf{v}$ being the outer normal to the sphere $\partial\bar{B}$ and $\mathrm{d}S$ denoting the $(n-1)$-dimensional Lebesgue integration on $\partial\bar{B}$. If (5.11) is relaxed to

$$\|(\operatorname{grad} h)(x)\| = \mathcal{O}\{\|x\|^{1-n}\} \, , \tag{5.13}$$

then (5.12) need not hold. This can be seen if we put $h(x) = \|x\|^{2-n}$ in case $n \geqq 3$, $h(x) = \ln \|x\|^{-1}$ in case $n = 2$.

References

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