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ALGEBRAIC CONNECTIVITY OF TREES

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I. INTRODUCTION

Let $T = (V, E)$ be a tree with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. We will frequently abuse the language by writing $v \in T$, rather than $v \in V$, to indicate that v is a vertex of T.

Denote by $d(v)$ the degree of v. The *Laplacian matrix* afforded by T (and some particular ordering of its vertices) is $L(T) = (a_{ij})$, where

$$a_{ij} = \begin{cases} d(v_i), & \text{if} \quad i = j \\ -1. & \text{if} \quad \{v_i, v_j\} \in E \\ 0, & \text{otherwise}. \end{cases}$$

It is well known (and easily seen) that $L(T)$ is positive semidefinite symmetric. It is then clear from the sign pattern of its entries that $L(T)$ is a (singular) M-matrix. Moreover, since T is connected, $L(T)$ is irreducible. Let $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n = 0$ be the eigenvalues of $L(T)$. Then $\lambda_{n-1} \neq 0$. We denote this second smallest eigenvalue by $a(T)$. M. Fiedler [2] has called $a(T)$ the *algebraic connectivity* of T. Indeed, $a(T) \leqq 1$ [2], with equality if and only if T is the *star* graph $K_{1,n-1}$ [5].

Denote by $\xi(T)$ the set of eigenvectors of $L(T)$ afforded by $a(T)$. Of course, $\xi(T)$ lacks only the zero vector to be a vector space. For our purposes, it is useful to think of the elements of $\xi(T)$ as real valued functions of V. If, for example, $(x_1, x_2, \ldots, x_n)$ is an eigenvector of $L(T)$ afforded by $a(T)$, then we write $f \in \xi(T)$ for the function defined by $f(v_i) = x_i$, $1 \leqq i \leqq n$. Note that the function f does not depend on the labeling of V. Fiedler has called the elements of $\xi(T)$ *characteristic valuations* of T.

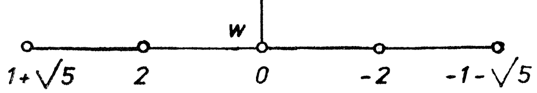

Fig. 1

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Example 1. Let T be the tree in Figure 1. Then the characteristic polynomial of $L(T)$ is $x(x-2)(x^2-3x+1)(x^2-5x+3)$, and $a(T) = (3-\sqrt{5})/2$ is a simple eigenvalue. Apart from nonzero multiples, there is but one $f \in \xi(T)$ and it, too, is illustrated in Figure 1.

In this article, we are interested in trees T for which there exist a $v \in V$ and an $f \in \xi(T)$ such that $f(v) = 0$. These have been called *Type I trees* [5]. It turns out that if T is a Type I tree, there is at least one vertex $w \in V$ such that $f(w) = 0$ for every $f \in \xi(T)$. This is an indirect consequence of the following result of Fielder [4, Theorem (3, 14)].

Theorem A. *Let $T = (V, E)$ be a Type I tree. Suppose $f \in \xi(T)$. If $V_f = \{v \in V \mid f(v) = 0\}$, then the graph $T_f = (V_f, E_f)$ induced by T on V_f is connected and there is exactly one vertex $w_f \in V_f$ which is adjacent (in T) to a vertex not belonging to V_f. Moreover, the values of f along any path in T starting at w_f are strictly increasing, strictly decreasing or identically zero.*

Example 2. In Figure 1, $V_f = \{u, w\}$ and $w_f = w$.

It is proved in [5] that while V_f may depend on f, w_f does not. Specifically, if T is a Type I tree, then there is a unique vertex w_T such that $f(w_T) = 0$ and $w_f = w_T$, for all $f \in \xi(T)$. We call w_T the *characteristic vertex* of T.

Denote by $\Gamma(T)$ the group of automorphisms of T expressed as permutations of V. Since $f\sigma \in \xi(T)$ for all $\sigma \in \Gamma(T)$ and all $f \in \xi(T)$, it follows from the uniqueness of w_T that it is a fixed point of every automorphism of T.

Suppose v is a vertex of a tree T. Denote by T_v the subgraph of T obtained by deleting v and all edges incident with it. A *branch* (of T) *at* v is a connected component of T_v. If T is a Type I tree, $v = w_T, f \in \xi(T)$, and B is a branch at w_T, then (Theorem A) f is uniformly positive, uniformly negative, or identically zero on the vertices of B. Of course, every $f \in \xi(T)$ is orthogonal to the vector each of whose components is 1, i.e., an eigenvector afforded by 0. Thus, there will always be a positive branch and a negative branch at w_T for any characteristic valuation.

If B is a branch at v, we denote by $r(B)$ the vertex of B which is adjacent (in T) to v. It will frequently be convenient to view B as a rooted tree. In such a situation, we will always take $r(B)$ as the root. In particular, if $v = w_T$ and $f \in \xi(T)$, then $f(r(B))$ determines the sign of f throughout B.

Suppose B is a rooted tree with vertex set $\{u_1, u_2, \ldots, u_k\}$. Denote by $\hat{L}(B) = (a_{ij})$ the k-by-k matrix where

$$a_{ij} = \begin{cases} d(u_i) + 1\,, & \text{if } i = j \text{ and } u_i \text{ is the root} \\ d(u_i)\,, & \text{if } i = j \text{ and } u_i \text{ is not the root} \\ -1\,, & \text{if } \{u_i, u_j\} \in E \\ 0\,, & \text{otherwise}\,. \end{cases}$$

For the remainder of this article we will adopt the convention that u_k is the root (if there is one) so that $\hat{L}(B) = L(B) + X_k$, where X_k is the k-by-k matrix whose only nonzero entry is a 1 in position (k, k).

Lemma 1. *If B is a rooted tree, then* $\det(\hat{L}(B)) = 1$. *(In particular, $\hat{L}(B)$ is a positive definite symmetric M-matrix.)*

Proof. $\det(\hat{L}(B))$ differs from $\det(L(B))$ by the determinant of the principal submatrix obtained by deleting the last row and column of $L(B)$. By the Matrix-Tree Theorem, the determinant of this submatrix is 1. □

Let v be a vertex of T of degree $d = d(v)$. Denote the branches at v by $B_1, B_2, \ldots, B_d$. Suppose B_i has n_i vertices. View each B_i as a rooted tree with $r(B_i)$ listed last among the n_i vertices. If we order the vertices of T with those of B_i preceding those of B_{i+1}, $1 \leqq i < d$, and with v coming last of all, then $L(T)$ takes the following partitioned form:

$$(1) \qquad \begin{pmatrix} \hat{L}(B_1) & 0 & \ldots & 0 & C_{n_1}^t \\ 0 & \hat{L}(B_2) & \ldots & 0 & C_{n_2}^t \\ & \ldots & & & \\ 0 & 0 & \ldots & \hat{L}(B_d) & C_{n_d}^t \\ C_{n_1} & C_{n_2} & & C_{n_d} & d \end{pmatrix},$$

where C_{n_i} is the row vector of length n_i whose only nonzero entry is -1 in the last column.

II. THE MULTIPLICITY OF $a(T)$

Let T be a Type I tree. Let B be a branch at w_T. We call B *passive* if $f(r(B)) = 0$ for every $f \in \xi(T)$. A branch at w_T is *active* if it is not passive.

Theorem 1. *Let T be a Type I tree with characteristic vertex $w = w_T$ and algebraic connectivity $a(T)$. Let B be a branch at w with root $r(B)$. Then B is active if and only if $a(T)$ is an eigenvalue of $\hat{L}(B)$. Moreover, if $a(T)$ is an eigenvalue of $\hat{L}(B)$, then it is simple and it is the smallest eigenvalue of $\hat{L}(B)$.*

Proof. Let $B_1, B_2, \ldots, B_k$ be the active branches at w and $B_{k+1}, \ldots, B_d$ the passive branches. Then $L = L(T)$ takes the partitioned form illustrated in (1).

For each B_i, $1 \leqq i \leqq k$, there is a characteristic valuation $f_i \in \xi(T)$ such that $f_i(r(B_i)) \neq 0$. Indeed, by Theorem A, we may assume $f_i(r(B_i)) = 1$ and $f_i(v) > 1$ for all vertices of B_i different from $r(B_i)$. From this it is easy to see that there must exist an eigenvector $f \in \xi(T)$ which is simultaneously nonzero on every B_i, $1 \leqq i \leqq k$, i.e., such that $f(v) = 0$ if and only if $v = w$ or v is a vertex of a passive branch. With respect to the same ordering of the vertices that produced (1), we may partition

$$(2) \qquad f = (x^{(1)}, x^{(2)}, \ldots, x^{(k)}, 0, \ldots, 0),$$

where $x^{(i)}$ is 1-by-n_i, $1 \leqq i \leqq k$. Moreover, by Theorem A, either $x^{(i)}$ or $-x^{(i)}$ is a positive vector, $1 \leqq i \leqq k$. Using block matrix multiplication on the equation $fL = a(T)f$, we see that $x^{(i)}$ is an eigenvector of $\hat{L}(B_i)$ afforded by $a(T)$, $1 \leqq i \leqq k$.

Rescale each $x^{(i)}$ (separately) so that its last component (corresponding to $r(B_i)$)

is 1. Call the rescaled vector $y^{(i)}$. Then $y^{(i)}$ is a positive eigenvector of $\hat{L}(B_i)$ afforded by $a(T)$, $1 \leqq i \leqq k$. Now, since $\hat{L}(B_i)$ is an (irreducible) M-matrix (Lemma 1), its inverse, A_i, is (entrywise) positive. Moreover, $y^{(i)}A_i = a(T)^{-1}\, y^{(i)}$ so that $a(T)^{-1}$ must be the (simple) Perron root of A_i. It follows that $a(T)$ is both simple and the smallest eigenvalue of $\hat{L}(B_i)$.

Suppose, now, that $a(T)$ is an eigenvalue of $\hat{L}(B_j)$ for some $j > k$, and let $z^{(j)}$ be a corresponding (row) eigenvector. There are two cases.

Case 1. The entry in the last column of $z^{(j)}$ is zero. Then the 1-by-n vector $g = (0, \ldots, 0, z^{(j)}, 0, \ldots, 0)$, with $z^{(j)}$ in columns $n_1 + \ldots + n_{j-1} + 1$ through $n_1 + \ldots + n_j$ and zero in every other column, is an eigenvector of L afforded by $a(T)$. But then $g \in \xi(T)$ is nonzero on a vertex of the passive branch B_j.

Case 2. The entry in the last column of $z^{(j)}$ is not zero, in which case we may assume it is -1. Then $g = (y^{(1)}, 0, \ldots, 0, z^{(j)}, 0, \ldots, 0)$, with $y^{(1)}$ in the first n_1 rows and $z^{(j)}$ positioned as in Case 1, is an eigenvector of L afforded by $a(T)$ and we reach the same contradiction. □

Corollary 1. *Let T be a Type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let $L = L(T)$ be a Laplacian matrix for T and denote by L_w the principal submatrix of L obtained by deleting the row and column corresponding to w. Then the number of active branches (of T) at w is equal to the multiplicity of $a(T)$ as an eigenvalue of L_w.*

Proof. We may assume the vertices of T are ordered so that $L(T)$ takes the form (1). Then L_w is the direct sum of the $\hat{L}(B)$ as B ranges over the branches at w. Therefore, the conclusion is immediate from Theorem 1. □

Theorem 2. *Let T be a Type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let m be the multiplicity of $a(T)$ as an eigenvalue of $L(T)$. Then exactly $m + 1$ of the branches at w are active.*

Proof. As in the proof of Theorem 1, there is a characteristic valuation $f \in \xi(T)$ such that $f(v) = 0$ if and only if $v = w$ or v is a vertex of a passive branch at w. Once again, we may assume the vertices of T are ordered so that $L = L(T)$ takes the form (1) where $B_1, B_2, \ldots, B_k$ are the active branches and $B_{k+1}, \ldots, B_d$ are passive. Partition the eigenvector f as in (2) and rescale each $x^{(i)}$ to obtain $y^{(i)}$ with 1 in the last column. For $2 \leqq i \leqq k$, define $f_i = (y^{(1)}, 0, \ldots, 0, -y^{(i)}, 0, \ldots, 0)$, where $-y^{(i)}$ occurs in columns $n_1 + \ldots + n_{i-1} + 1$ through $n_1 + \ldots + n_i$. It follows from block multiplication that $f_i L = a(T) f_i$, $2 \leqq i \leqq k$. For example, the last column of $f_i L$ is $y^{(1)}C^t_{n_1} - y^{(i)}C^t_{n_i} = -1 - (-1) = 0$. In particular $\{f_2, f_3, \ldots, f_k\}$ is a linearly independent set of eigenvectors for $a(T)$.

Suppose $h \in \xi(T)$ is a fixed but arbitrary eigenvector for $a(T)$. Since $h(v) = 0$ for all vertices v of B_j, $k < j \leqq d$, we may partition h conformally with L as $h = (u^{(1)}, u^{(2)}, \ldots, u^{(k)}, 0, \ldots, 0)$, where $u^{(i)}$ is a 1-by-n_i row vector. Now, again by Theorem A, if $u^{(i)} \neq 0$, then $u^{(i)}$ or $-u^{(i)}$ is a positive vector. Moreover, by block multiplication of $hL = a(T)\,h$, each nonzero $u^{(i)}$ is an eigenvector of the corresponding $\hat{L}(B_i)$ afforded by the simple eigenvalue $a(T)$. Thus, there are constants c_i such that $u^{(i)} = c_i y^{(i)}$, $1 \leqq i \leqq k$. (Indeed, $c_i = h(r(B_i))$.) From the last column of $hL = a(T)\,h$, we see that the sum of the c's is 0. Therefore,

$$h = \sum_{i=2}^{k} -c_i f_i \,.$$

So, $\{f_2, f_3, \ldots, f_k\}$ spans $\xi(T)$.

We have shown that $k - 1 = m$. □

Corollary 2. *Let T be a Type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let B be a passive branch at w. Then the smallest eigenvalue of $\hat{L}(B)$ is strictly larger than $a(T)$. In particular, if L_w is the principal submatrix of $L(T)$ obtained by deleting the row and column corresponding to w, then $a(T)$ is the smallest eigenvalue of L_w.*

Proof. We may assume $L(T)$ has the form (1) so that L_w is the direct sum of the $\hat{L}(B)$ as B ranges over the branches at w. By Theorem 1, it suffices to prove that $a(T)$ is the smallest eigenvalue of L_w. Let $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_{n-1} > 0$ be the eigenvalues of L_w. Let m be the multiplicity of $a(T)$ as an eigenvalue of L. Let

$$\lambda_1 \geqq \ldots \geqq \lambda_{n-m-1} > \lambda_{n-m} = \ldots = \lambda_{n-1} = a(T) > \lambda_n = 0$$

be the eigenvalues of L. By the Cauchy interlacing inequalities,

$$\alpha_{n-m-1} \geqq \lambda_{n-m} \geqq \alpha_{n-m} \geqq \ldots \geqq \lambda_{n-1} \geqq \alpha_{n-1} \,.$$

It follows that the $m - 1$ numbers α_i, $n - m \leqq i \leqq n - 2$, are all equal to $a(T)$. By Corollary 1 and Theorem 2, two more of the α's must equal $a(T)$. But, $\alpha_{n-m-2} \geqq \geqq \lambda_{n-m-1} > a(T)$ so the only remaining possibilities are α_{n-m-1} and α_{n-1}. □

III. PRUNING AND GRAFTING PASSIVE BRANCHES

Suppose T is a Type I tree with a branch B at w_T and a characteristic valuation $f \in \xi(T)$ such that $f(r(B)) = 0$. Then, as we have seen, f is identically zero on B. (It may happen that $g(r(B)) \neq 0$ for some $g \in \xi(T)$ different from f. We do not necessarily assume that B is a passive branch.)

Suppose u_1 is a pendant vertex of T which lies in the branch B. Let T_1 be the subtree of T obtained by removing u_1 and the edge incident with it. Let f_1 be the restriction of f to T_1. It was shown in [5] that $a(T_1) = a(T)$ and $f_1 \in \xi(T_1)$. In particular, T_1 is a Type I tree and $w_{T_1} = w_T$. By successive applications of this procedure, zero branches can be "pruned away" leaving a Type I subtree T_* and a characteristic valuation f_* which takes the value 0 only at the characteristic vertex. Our next result shows that new zero pendant vertices can be "grafted on" to the characteristic vertex.

Theorem 3. *Let $T = (V, E)$ be a Type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let $f \in \xi(T)$. Suppose $T' = (V', E')$ is the tree obtained from T by adjoining a new pendant vertex, p, to w. (So $V' = V \cup \{p\}$ and $E' = E \cup \{\{p, w\}\}$.) Extend f to a function f' of V' by defining $f'(p) = 0$. Then $a(T') = a(T)$ and $f' \in \xi(T')$. (In particular, T' is a Type I tree with characteristic vertex w.)*

Proof. We have $V = \{v_1, v_2, \ldots, v_n\}$ and we let $v_n = w$. Thus we may take $V' = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ with $v_{n+1} = p$. Let $L = L(T)$ and $L' = L(T')$ be the Laplacian matrices afforded by these orderings of the vertices of T and T', respectively. (If we temporarily view w as a root of T, then L' is the partitioned matrix

$$L(T') = \begin{pmatrix} \hat{L}(T) & C_n^t \\ C_n & 1 \end{pmatrix}.)$$

Since $f(w) = 0$, we see that $f'L' = a(T)f'$. In other words $a(T)$ is an eigenvalue of L' afforded by the eigenvector f'. Since $a(T) \neq 0$, we must have $a(T) \geqq a(T')$. It remains to show that $a(T') \geqq a(T)$.

By Corollary 2, the minimum eigenvalue of L_w is $\alpha_{n-1} = a(T)$. Since $1 \geqq a(T)$, the minimum eigenvalue of L'_w is $a(T)$, where $L'_w = L_w \dotplus (1)$ is the principal submatrix of L' obtained by deleting row and column n. But, by Cauchy interlacing, this eigenvalue is no larger than $a(T')$, the second smallest eigenvalue of $L(T')$. □

In Theorem 3, it is necessary to assume that w is the characteristic vertex. If w is some other vertex of T for which $f(w)$ is zero, it would still be the case that f' is an eigenvector of T' afforded by $a(T)$, but it might happen that $a(T) > a(T')$, even if w were a vertex of a passive branch.

The following companion of Theorem 3 is an immediate consequence of Theorem 1.

Corollary 3. *Let T be a Type I tree with characteristic vertex w. Let p be a pendant vertex attached to w. If $T \neq K_{1,n-1}$, then $f(p) = 0$ for every $f \in \xi(T)$.*

Proof. Let $B = (\{p\}, \emptyset)$ be the branch at w containing p. Then $\hat{L}(B) = (1)$. Since $T \neq K_{1,n-1}$, we know that $a(T) < 1$. By Theorem 1, B is a passive branch. □

IV. THE ALTERNATING PART OF THE SPECTRUM

Let $G = (V, E)$ be a graph with automorphism group $\Gamma(G)$ regarded as a group of permutations of $V = \{v_1, \ldots, v_n\}$. For any subgroup H of $\Gamma(G)$, denote by $V_1, \ldots, V_t$ the orbits of H in V and let n_i be the cardinality of V_i, $i = 1, \ldots, t$. Denote by $P(\sigma)$, $\sigma \in H$, the n-by-n permutation matrix corresponding to the action of σ on $V = V_1 \cup \ldots \cup V_t$. If we order the vertices such that $V_1 = \{v_1, \ldots, v_{n_1}\}$, $V_2 = \{v_{n_1+1}, \ldots, v_{n_1+n_2}\}$, etc., then

$$P(\sigma) = P_1(\sigma) \dotplus \ldots \dotplus P_t(\sigma)$$

where $P_i(\sigma)$ is the n_i-by-n_i permutation matrix corresponding to the action of σ on V_i.

Since H acts transitively on V_i, the principal representation of H occurs exactly once in the reduction of the representation $\sigma \mapsto P_i(\sigma)$ into irreducible components. Indeed, the constant n_i-tuple $\bar{u}_i$, each of whose components is $1/\sqrt{n_i}$, is a unit eigenvector of $P_i(\sigma)$, $\sigma \in H$, which corresponds to the eigenvalue 1. Define the unit n-tuples u_i by

$$u_i = (0, \ldots, 0, \bar{u}_i, 0, \ldots, 0), \quad 1 \leqq i \leqq t,$$

with $\bar{u}_i$ in positions $n_1 + \ldots + n_{i-1} + 1$ through $n_1 + \ldots + n_i$. Let U be any n-by-n orthogonal matrix with first t rows equal to $u_1, \ldots, u_t$. The matrix U satisfies $U\, P(\sigma)\, U^t = I_t \dotplus Q(\sigma)$, $\sigma \in H$, where I_t is the t-by-t identity matrix and Q is a representation of H which does not contain the principal representation as a component. Partition $U\, L(G)\, U^t$ into a 2-by-2 block matrix (L_{ij}) where L_{11} is t-by-t. Since $P(\sigma)$ commutes with $L(G)$ for all $\sigma \in H$ it follows that $L_{12} = L_{12}\, Q(\sigma)$, $\sigma \in H$. If $L_{12} \neq 0$ then any nonzero row would be an eigenvector of $Q(\sigma)$ corresponding to the eigenvalue 1 for all $\sigma \in H$. Since this would contradict that Q does not contain the principal representation as a component it must be that $L_{12} = L_{21}^t = 0$, so that $U\, L(G)\, U^t$ is the direct sum of L_{11} and L_{22}. The spectrum of L_{11} will be called the *H-symmetric* part of the spectrum of $L(G)$, and the (complementary) spectrum of L_{22} will be referred to as the *H-alternating* part. Clearly, an eigenvalue is in the H-symmetric part of the spectrum if and only if there is a corresponding eigenvector which is constant on the orbits of H, and an eigenvalue is in the H-alternating part of the spectrum if and only if there is a corresponding eigenvector which is not constant on the orbits of H. Both parts of the spectrum are counted according to multiplicities and it can happen that a multiple eigenvalue of $L(G)$ occurs in both the H-symmetric and H-alternating parts of the spectrum. (There is an analogous division of the spectrum of the adjacency matrix [1, Sect. 5.3] which has been generalized by means of so-called "divisors".)

One can easily produce the matrix L_{11} without using group representations. If U_t denotes the t-by-n matrix consisting of the first t rows of U it is evident that $U_t\, L(G)\, U_t^t = L_{11}$. Hence if $L(G)$ is partitioned into a t-by-t block matrix (A_{ij})

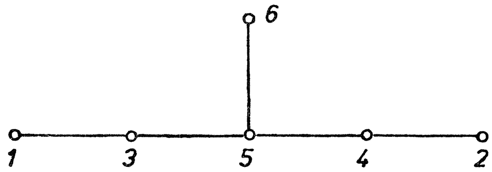

Fig. 2

where A_{ij} is n_i-by-n_j, then the (i, j)-entry of L_{11} is just $1/\sqrt{n_i n_j}$ times the sum of the entries in A_{ij}.

Example 3. Consider the tree T in Figure 2. The automorphism group is $\Gamma(T) = \{\mathrm{id}, (12)(34)\}$ and it yields four orbits: $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, $V_3 = \{5\}$ and

$V_4 = \{6\}$. Letting $H = \Gamma(T)$, the "orbit-partitioned" form of $L(T)$ is

$$L(T) = \left(\begin{array}{cc|cc|c|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ \hline -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & 3 & -1 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right).$$

Therefore

$$L_{11} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial of L_{11} is $p(x) = x(x - 2)(x^2 - 5x + 3)$. The roots of $p(x)$ comprise the H-symmetric part of the spectrum. From Example 1, the characteristic polynomial of $L(T)$ is $(x^2 - 3x + 1)\, p(x)$. The (complementary) H-alternating part of the spectrum consists of the roots of $x^2 - 3x + 1$, namely $(3 + \sqrt{5})/2$ and $a(T) = (3 - \sqrt{5})/2$.

In case $H = \Gamma(G)$, we will use the terms *symmetric* and *alternating* rather than H-symmetric and H-alternating. We will refer to an eigenvalue of $L(G)$ as *symmetric* (*alternating*) if it is in the symmetric (alternating) part of the spectrum. Then, as we have already observed, an eigenvalue λ of $L(G)$ is symmetric if and only if there is a corresponding eigenvector which is constant on the orbits of $\Gamma(G)$, i.e., a vector $x \neq 0$ such that $x\, L(T) = \lambda x$ and $x\, P(\sigma) = x$ for all $\sigma \in \Gamma(G)$.

(When $G = T$ is a tree, a similar justification can be given for "alternating": λ is an alternating eigenvalue of $L(T)$ if and only if there is a vector $x \neq 0$ such that $x\, L(T) = \lambda x$ and $x\, P(\sigma) = -x$ for *some* $\sigma \in \Gamma(T)$. When $\lambda = a(T)$, this observation will be addressed after Theorem 4. In the general case it seems appropriate merely to sketch a proof: Sufficiency is clear. To prove necessity, let y be an eigenvector for λ which is not constant on orbits. Then there is an automorphism τ such that $y_i \neq [y\, P(\tau)]_i$ for some (fixed) i. It is a fact about $\Gamma(T)$, for which we have no other use, that there is an involutary automorphism σ such that $\tau(v_i) = \sigma(v_i)$, for the same fixed i. Let $x = y - y\, P(\sigma)$. Then $x\, L(T) = \lambda x$ and $x\, P(\sigma) = -x$.)

If $\Gamma(G)$ is trivial, the alternating part of the spectrum is empty. On the other hand, if $\Gamma(G)$ is transitive, then the symmetric part consists of the eigenvalue 0 and nothing else. Example 3 shows that $a(T)$ may occur in the alternating part. In Theorem 4 we characterize those Type I trees T for which $a(T)$ is aternating.

Suppose v is a vertex of the tree T. Let B_1 and B_2 be two branches at v. We say that B_1 and B_2 are *isomorphic* if (and only if) they are isomorphic as rooted trees. In particular, B_1 is isomorphic to B_2 if and only if there is a $\sigma \in \Gamma(T)$ such that $\sigma(B_1) = B_2$ with $\sigma(r(B_1)) = \sigma(r(B_2))$, and $\sigma(u) = u$ for every $u \notin B_1 \cup B_2$.

Lemma 2. *Let T be a tree. Suppose $B_1, B_2, \ldots, B_k$ are isomorphic branches at $v \in T$, $k > 1$. Let λ be an eigenvalue, with multiplicity m, of $\hat{L}(B_1)$. Then λ is an eigenvalue, with multiplicity at least $(k-1)m$, of the alternating part of the spectrum of $L(T)$.*

Proof. We may assume without loss of generality that $L(T)$ has the form (1) and that $\hat{L}(B_1) = \ldots = \hat{L}(B_k)$. Let $x_1, x_2, \ldots, x_m$ be a linearly independent set of eigenvectors for $\hat{L}(B_1)$ corresponding to λ. Then the $(k-1)$ vectors $(x_i, -x_i, 0, \ldots, 0)$, $(x_i, 0, -x_i, 0, \ldots, 0)$, $(x_i, 0, 0, -x_i, 0, \ldots, 0)$, etc., for each $i = 1, 2, \ldots, m$, constitute a linearly independent set of $(k-1)m$ eigenvectors of $L(T)$ corresponding to λ. Since every one of them fails to be constant on the orbits of $\Gamma(T)$, the proof is complete. □

Note that the following interesting observation (due to M. Fiedler) emerges from the proof of Lemma 2: Suppose B_1 and B_2 are isomorphic branches at an arbitrary vertex v of an arbitrary tree T. Then there is an eigenvector x of $L(T)$ supported by the vertices $B_1 \cup B_2$ and such that $x\,P(\sigma) = -x$ for an automorphism $\sigma \in \Gamma(T)$ which interchanges B_1 and B_2, and fixes every vertex not belonging to $B_1 \cup B_2$.

Consider next the case that T is a Type I tree and v is chosen to be w_T. If T has two isomorphic *active* branches at w_T, it follows from Theorem 1 and Lemma 2 that $a(T)$ is alternating. Moreover, from the immediately previous discussion, there is a $\sigma \in \Gamma(T)$ and an $f \in \xi(T)$ such that $\sigma(B_1) = B_2$, $\sigma(v) = v$ for all $v \notin B_1 \cup B_2$, and $f(v) = -f\,\sigma(v) > 0$ for all $v \in B_1$. The main result of this section is a converse. Before proceeding to Theorem 4, we establish the following:

Lemma 3. *Let T be a Type I tree with a branch B at the characteristic vertex $w = w_T$. Denote the orbits of $\Gamma(T)$ by $V_1, V_2, \ldots, V_t$. If $f \in \xi(T)$, then f is constant on $V_i \cap B$, for all $i = 1, 2, \ldots, t$.*

Proof. We may assume that B is an active branch for otherwise f is identically zero on all of B and hence constant on each $B \cap V_i$. Let f_1 be the restriction of f to B. As we have seen, f_1 is an eigenvector of $\hat{L}(B)$ corresponding to $a(T)$. Moreover, $a(T)$ is a simple eigenvalue of $\hat{L}(B)$ and the components of f_1 are all positive or all negative.

Suppose that $v_1, v_2 \in B \cap V_i$ and that $\sigma(v_1) = v_2$ for some $\sigma \in \Gamma(T)$. Since T is a tree and since w is a fixed point of every automorphism of T, σ acts as a permutation of the branches of T at w. Since $v_2 \in B \cap \sigma(B)$, it must be that $\sigma(B) = B$. Let σ_1 be the restriction of σ to B. Then f_1 and $f_1\sigma_1$ are two eigenvectors of $\hat{L}(B)$ corresponding to the simple eigenvalue $a(T)$. Since f_1 and $f_1\sigma_1$ have the same (uniformly signed) entries, it must be that $f_1 = f_1\sigma_1$, so $f(v_1) = f(v_2)$. □

Theorem 4. *Suppose that T is a Type I tree with algebraic connectivity $a(T)$ and characteristic vertex w. Then $a(T)$ is alternating if and only if T has two isomorphic active branches, B_1 and B_2, at w.*

Proof. If T has two isomorphic active branches, then $a(T)$ is alternating by Theorem 1 and Lemma 2. Conversely, assume $a(T)$ is alternating. Let $B_1, \ldots, B_k$

be the active branches of T at w. Since w is a fixed point for every automorphism of T, any $\sigma \in \Gamma(T)$ will act as a permutation on $\{B_1, \ldots, B_k\}$. (Since $f\sigma \in \xi(T)$ for all $f \in \xi(T)$ and all $\sigma \in \Gamma(T)$, B and $\sigma(B)$ are both active or both passive for every branch B at w.)

Suppose, for contradiction, that no two branches in $\{B_1, \ldots, B_k\}$ are isomorphic, so that $\sigma(B_i) = B_i$, for all $i = 1, \ldots, k$, and $\sigma \in \Gamma(T)$. This implies that for any orbit V_j of $\Gamma(T)$, either V_j is entirely contained in the passive branches at w, or V_j is entirely contained in a *single* active branch. Together with Lemma 3, this yields that any valuation is constant on orbits, contradicting that $a(T)$ is alternating. □

Let $\tau: B_1 \to B_2$ be an isomorphism of branches at w. Extend τ to an automorphism σ of T by defining $\sigma(v) = \tau(v)$ if $v \in B_1$, $\sigma(v) = \tau^{-1}(v)$ if $v \in B_2$, and $\sigma(v) = v$ otherwise. We may assume that $L(T)$ has the form (1) with $\hat{L}(B_1) = \hat{L}(B_2)$. Let x be an eigenvector of $\hat{L}(B_1)$ corresponding to $a(T)$. Then $f = (x, -x, 0) \in \xi(T)$ and $f\sigma = (-x, x, 0) = -f$.

The discussion in Theorems 1 and 4 makes it clear how Type I trees may be constructed when $a(T)$ is alternating. Simply take several copies of any rooted tree and connect each of the roots to a new vertex w which will be the characteristic vertex. One may wonder whether $a(T)$ must always be alternating when T is a Type I tree. In view of Theorem 4 this amounts to asking whether Type I trees must always have two isomorphic active branches at the characteristic vertex. The answer to this question is no, and we will proceed to construct an example.

Our method of producing a Type I tree having no isomorphic branches at w begins with Theorem 1. Suppose B_1 and B_2 are two nonisomorphic rooted trees where $\hat{L}(B_1)$ and $\hat{L}(B_2)$ have the same smallest eigenvalue a. Let T be the tree obtained by attaching each of the roots to a new vertex w. Then we may write

$$L(T) = \begin{pmatrix} \hat{L}(B_1) & 0 & C_{n_1}^t \\ 0 & \hat{L}(B_2) & C_{n_2}^t \\ C_{n_1} & C_{n_2} & 2 \end{pmatrix}.$$

Let $x^{(i)}$ be an eigenvector of $\hat{L}(B_i)$ affording a, $i = 1, 2$. Since $\hat{L}(B_i)$ is an M-matrix, we can assume $x^{(1)}$ is positive with 1 in its last column and $x^{(2)}$ is negative with -1 in its last column. Then $f = (x^{(1)}, x^{(2)}, 0)$ is an eigenvector of $L(T)$ affording a.

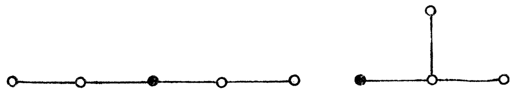

Fig. 3

By Cauchy interlacing with respect to the principal submatrix $\hat{L}(B_1) \dotplus \hat{L}(B_2)$, we see that $a = a(T)$. Therefore, $f \in \xi(T)$ and we are finished.

It is not difficult (using, e.g., [6]) to locate rooted trees with the desired common eigenvalue. For example the $\hat{L}$ matrices for the rooted trees in Figure 3 share the minimum eigenvalue $a \doteq 0.139194$, the smallest root of $p(x) = x^3 - 6x^2 + 8x - 1$. This leads to the Type I tree T in Figure 4 with (simple) $a(T) = a$. The (apart from nonzero multiples) unique characteristic valuation of T also appears in Figure 4 (rounded to four decimal places).

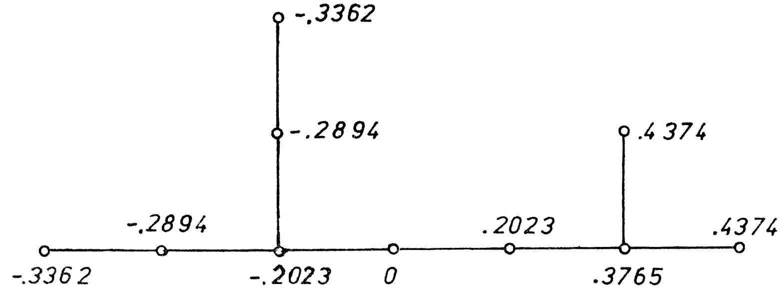

Fig. 4

It is possible to modify the example in Figure 4 to produce a Type I tree T with $a(T)$ belonging both to the symmetric and alternating parts of the spectrum. Simply attach a second copy of either branch to w_T.

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