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Shinji Yamashita

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SUBHARMONICITY FOR AREAL BLOCH-FUNCTION CRITERIA

SHINJI YAMASHITA, Tokyo

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1. INTRODUCTION

A holomorphic function f in $D = \{|z| < 1\}$ is called *Bloch*, $f \in B$ in notation, if

$$K = K(f) = \sup_{z \in D} (1 - |z|^2) |f'(z)| < \infty .$$

We shall prove some necessary and sufficient conditions for a holomorphic function in D to be in B in terms of areal means.

Let u be subharmonic in an open disk δ with the area $m(\delta)$. The areal mean of u in δ is then defined by

$$A[u, \delta] = m(\delta)^{-1} \iint_\delta u(z) \, dx \, dy \quad (z = x + iy) .$$

It is familiar that $f \in B$ if and only if f is areally of BMO (bounded mean oscillation), namely,

$$\sup A[|f - f(a)|, \delta] < \infty ,$$

where δ ranges over all open disks contained in D and a is the center of δ; see [2, p. 632].

For u subharmonic in D we set

$$L(u, r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{it}) \, dt , \quad r \in (0, 1) ,$$

and

$$L[u] \equiv \sup_{0 < r < 1} L(u, r) = \lim_{r \to 1} L(u, r) .$$

Let

$$\phi_a(w) = (w + a)/(1 + \bar{a}w) , \quad w \in D , \quad a \in D .$$

Then BMOA consists of f holomorphic in D with

$$\sup_{a \in D} L[|f \circ \phi_a - f(a)|] < \infty .$$

The celebrated "exponential" and "logarithmic" criteria for f to be in BMOA, due to A. Baernstein II [1, Corollaries 2 and 3(d), pp. 15–16], then read:

(E) *There exists* $c > 0$ *such that*

$$\sup_{a\in D} L[\exp\{c|f\circ\phi_a - f(a)|\}] < \infty .$$

(L) $$\sup_{a\in D} L[\log^+|f\circ\phi_a - f(a)|] < \infty .$$

As usual, $\log^+ x = \max(\log x, 0)$, $x \geqq 0$. Each of (E) and (L) is equivalent to f being in BMOA. We could regard (E) and (L) to be the strongest and the weakest conditions, respectively, in the familiar "linear" integrals.

It appears to be natural to find areal analogues. Our typical result is the following:

Theorem 1. *The following are mutually equivalent for f holomorphic in D.*

(B) $$f \in B .$$

(BE) *There exists* $c > 0$ *such that*

$$\sup_{a\in D} A[\exp\{c|f\circ\phi_a - f(a)|\}, D] < \infty .$$

(BL) $$\sup_{a\in D} A[\log|f\circ\phi_a - f(a)|, D] < \infty .$$

In (BL), log instead *of* $\log^+$ is considered.

Let B_0 be the family of functions f holomorphic in D such that

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0 .$$

Then $B_0 \subset B$. Some criteria for B_0 will be considered in contrast with those for B.

2. STATEMENT OF THE MAIN RESULT

Theorem 2. *For f holomorphic in D, each of the following nine conditions,* (Sj), (Ej), (Lk) *for* j = 1, 2; k = 1, 2, 3, 4, 5, *is equivalent to* (B).

Each open disk $\delta \subset D$ can be expressed as $\delta = D(a, \varrho)$, where

$$D(a, \varrho) = \{z; |z - a| < \varrho(1 - |a|)\}, \quad a \in D, \quad 0 < \varrho \leqq 1 .$$

In particular, $D(a, 1)$ $(a \neq 0)$ is tangent to the unit circle. Less generally, D itself or each open disk $\delta \subset D$ with $\bar{\delta} \subset D$ can be expressed as $\delta = \Delta(a, \varrho)$, where

$$\Delta(a, \varrho) = \{z; |\phi_{-a}(z)| < \varrho\}$$

is the non-Euclidean disk of center a and radius $\tanh^{-1}\varrho$, $a \in D$, $0 < \varrho \leqq 1$. We shall denote

$$D(\varrho) = D(0, \varrho) = \Delta(0, \varrho), \quad 0 < \varrho \leqq 1 .$$

If u is subharmonic in $\delta \subset D$ we set

$$A^*[u, \delta] = \frac{1}{m^*(\delta)} \iint_\delta u(z) \frac{dxdy}{(1 - |z|^2)^2},$$

where

$$m^*(\delta) = \iint_\delta (1 - |z|^2)^{-2}\, dx\, dy$$

is the non-Euclidean area of δ. This is the non-Euclidean areal mean of u in δ.

We are now ready to propose cited conditions.

(S1) *For each* $\beta > 0$, *there exists* $\varrho \in (0, 1)$ *such that*

$$\sup_{a\in D} \sup_{z\in D(a,\varrho)} |f(z) - f(a)| < \beta\,.$$

(S2) *For each* $\beta > 0$, *there exists* $\varrho \in (0, 1)$ *such that*

$$\sup_{a\in D} \sup_{z\in \Delta(a,\varrho)} |f(z) - f(a)| < \beta\,.$$

(E1) *For each* $\beta > 1$, *there exists* $c > 0$ *such that*

$$\sup_{a\in D} A[\exp\{c|f - f(a)|\},\ D(a, 1)] < \beta\,.$$

(E2) *For each* $\beta > 1$, *there exists* $c > 0$ *such that*

$$\sup_{a\in D} A[\exp\{c|f \circ \phi_a - f(a)|\},\ D] < \beta\,.$$

(L1) *There exist* $\varrho, \lambda \in (0,1)$ *such that*

$$\sup_{\lambda<|a|<1} A[\log|f - f(a)|,\ D(a, \varrho)] < \infty\,.$$

(L2) *There exist* $\varrho, \lambda \in (0, 1)$ *such that*

$$\sup_{\lambda<|a|<1} A[\log|f \circ \phi_a - f(a)|,\ D(\varrho)] < \infty\,.$$

(L3) *There exist* $\varrho, \lambda \in (0, 1)$ *such that*

$$\sup_{\lambda<|a|<1} A[\log^+|f - f(a)|,\ \Delta(a, \varrho)] < \infty\,.$$

(L4) *There exist* $\varrho, \lambda \in (0, 1)$ *such that*

$$\sup_{\lambda<|a|<1} A^*[\log^+|f - f(a)|,\ D(a, \varrho)] < \infty\,.$$

(L5) *There exist* $\varrho, \lambda \in (0, 1)$ *such that*

$$\sup_{\lambda<|a|<1} A^*[\log|f - f(a)|,\ \Delta(a, \varrho)] < \infty\,.$$

We note that

$$m(D(a, \varrho)) = \pi\varrho^2(1 - |a|)^2\,, \quad m(\Delta(a, \varrho)) = \frac{\pi\varrho^2(1 - |a|^2)^2}{(1 - \varrho^2|a|^2)^2}\,,$$

$$m^*(\Delta(a, \varrho)) = \frac{\pi\varrho^2}{1 - \varrho^2}\,.$$

Although $m^*(D(a, \varrho))$ has the expression in a and ϱ [4, Lemma 1 and its proof], we shall not need the detailed form.

As will be observed,

$$D(a, \varrho) \subset \Delta(a, \varrho), \quad a \in D, \quad \varrho \in (0, 1), \tag{2.1}$$

so that (S2) $\Rightarrow$ (S1) is immediate. Since $\phi_a(w) \in \Delta(a, \varrho)$ if and only if $w \in D(\varrho)$, $\varrho \in (0, 1)$, we easily observe that (S2) $\Rightarrow$ (L2). Rather trivial are (S1) $\Rightarrow$ (L1) and (L4); (S2) $\Rightarrow$ (L3) and (L5).

Theorem 1 is a consequence of Theorem 2 because

$$(\mathrm{B}) \Rightarrow (\mathrm{E2}) \Rightarrow (\mathrm{BE}) \Rightarrow (\mathrm{BL}) \Rightarrow (\mathrm{L2}) \Rightarrow (\mathrm{B}).$$

Here we note that by the celebrated F. Riesz theorem on areal mean of subharmonic functions, $A[u, D(a, \varrho)]$ is a non-decreasing function of $\varrho \in (0, 1]$ for u subharmonic in D; see [3, p. 8]. This proves (BL) $\Rightarrow$ (L2).

Finally we note that (Ej) $\Rightarrow$ (Lj) for $j = 1, 2$. Therefore, the rest we should prove is

$$(\mathrm{B}) \Rightarrow (\mathrm{S2}) \text{ and } (\mathrm{Ej}), \quad j = 1, 2.$$

and

$$(\mathrm{Lk}) \Rightarrow (\mathrm{B}), \quad k = 1, 2, 3, 4, 5.$$

Sections 3 and 4 are devoted to the proof of them.

The following figure would be of use for the deduction.

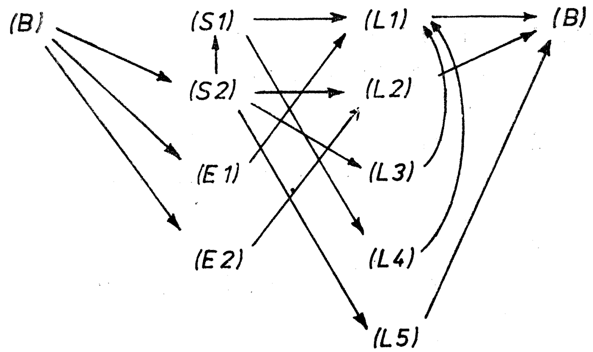

Figure 1

Remark. As will be apparent it is not difficult to show that each of the following is equivalent to (B).

(S1′) *For each* $\varrho \in (0, 1)$,

$$\sup_{a \in D} \sup_{z \in D(a,\varrho)} |f(z) - f(a)| < \infty.$$

(S2′) *For each* $\varrho \in (0, 1)$,

$$\sup_{a \in D} \sup_{z \in \Delta(a,\varrho)} |f(z) - f(a)| < \infty.$$

On replacing log by $\log^+$ in (Lk) we obtain the conditions (Lk$^+$), $k = 1, 2, 5$. It is easy to observe that

$$(Sk) \Rightarrow (Lk^+) \Rightarrow (Lk), \quad k = 1, 2;$$
$$(S2) \Rightarrow (L5^+) \Rightarrow (L5).$$

3. CONDITIONS (Sj) AND (Ej)

Suppose (B). Then

$$|f(z) - f(w)| \leqq K\sigma(z, w), \quad z, w \in D, \tag{3.1}$$

where

$$\sigma(z, w) = \tanh^{-1}|\phi_{-w}(z)|$$

is the non-Euclidean distance.

To prove (B) $\Rightarrow$ (S2), we choose $\varrho \in (0, 1)$ such that $K\sigma(\varrho, 0) < \beta$. Then

$$|f(z) - f(a)| \leqq K\sigma(z, a) \leqq K\sigma(\varrho, 0) < \beta$$

for $z \in \Delta(a, \varrho)$, or we have (S2).

If $a \in D$ and if $z \in D(a, 1)$, then for

$$z = \psi_a(w) \equiv a + (1 - |a|)\, w, \quad w \in D,$$

we have

$$|\phi_{-a}(z)| \leqq |w|. \tag{3.2}$$

In particular, if $z \in D(a, \varrho)$, then $w \in D(\varrho)$. This yields (2.1).

It is apparent that the integral

$$P \equiv \pi^{-1} \iint_D \left(\frac{1 + |w|}{1 - |w|}\right)^{1/2} du\, dv \quad (w = u + iv)$$

satisfies $1 < P < \infty$.

For the proof of (B) $\Rightarrow$ (E1), we choose $c > 0$ such that

$$cK < 1 \quad \text{and} \quad P^{cK} < \beta. \tag{3.3}$$

It then follows from (3.1) and (3.2) that

$$|f(z) - f(a)| \leqq K \tanh^{-1}|w|,$$

so that

$$\exp\{c|f(z) - f(a)|\} \leqq \left(\frac{1 + |w|}{1 - |w|}\right)^{cK/2}$$

for $z = \psi_a(w) \in D(a, 1)$. Since $dx\, dy = (1 - |a|)^2\, du\, dv$, it follows that

$$A[\exp\{c|f - f(a)|\}, D(a, 1)] \leqq$$
$$\leqq \pi^{-1} \iint_D \left(\frac{1 + |w|}{1 - |w|}\right)^{cK/2} du\, dv \leqq P^{cK} < \beta$$

because $cK < 1$.

For the proof of (B) ⇒ (E2) we first note that for $w \in D$,

$$|f \circ \phi_a(w) - f(a)| \leqq K\sigma(\phi_a(w), \phi_a(0)) = K \tanh^{-1} |w| .$$

Given $\beta > 1$ we choose $c > 0$ with (3.3). By the similar argument as above we have

$$A[\exp \{c|f \circ \phi_a - f(a)|\}, D] < \beta .$$

4. CONDITIONS (Lk)

We shall denote the ring domain by

$$R(\lambda) = \{\lambda < |z| < 1\}, \quad \lambda \in (0, 1) .$$

For the proof of (L1) ⇒ (B) we set $z = \psi_a(w)$ for $a \in R(\lambda)$, $w \in D(\varrho)$. Then

$$w\, g(w) = f \circ \psi_a(w) - f(a), \quad w \in D(\varrho),$$

where $g \equiv g_a$ is holomorphic in $D(\varrho)$ with

$$|g(0)| = (1 - |a|)\, |f'(a)| .$$

Since $\log |g|$ is subharmonic in $D(\varrho)$ it follows that

$$\log |g(0)| \leqq A[\log |g|, D(\varrho)] =$$

$$= A[\log |f \circ \psi_a - f(a)|, D(\varrho)] + A\left[\log \frac{1}{|w|}, D(\varrho)\right] =$$

$$= A[\log |f - f(a)|, D(a, \varrho)] - \log \varrho + 1/2 .$$

Therefore,

$$(1 - |a|^2)\, |f'(a)| \leqq 2\varrho^{-1}\, e^{1/2} \exp A[\log |f - f(a)|, D(a, \varrho)] ,$$

whence $(1 - |a|^2)\, |f'(a)|$ is bounded in $R(\lambda)$. Since f' is continuous, it follows that $f \in B$.

The proof of (L2) ⇒ (B) is similar on considering $\log |g|$ in $D(\varrho)$, where, in this case,

$$w\, g(w) = f \circ \phi_a(w) - f(a) \tag{4.1}$$

in $D(\varrho)$ with

$$|g(0)| = (1 - |a|^2)\, |f'(a)| .$$

We then have for $a \in R(\lambda)$,

$$(1 - |a|^2)\, |f'(a)| \leqq \varrho^{-1}\, e^{1/2} \exp A[\log |f \circ \phi_a - f(a)|, D(\varrho)] .$$

We next prove (L3) ⇒ (L1). Then (L3) ⇒ (B). In view of (2.1) we obtain for $u \geqq 0$ in $\Delta(a, \varrho)$, $\varrho \in (0, 1)$, that

$$A[u, \Delta(a, \varrho)] \geqq \frac{1}{m(\Delta(a, \varrho))} \iint_{D(a,\varrho)} u(z)\, dx\, dy =$$

$$= \frac{(1 - \varrho^2|a|^2)^2}{(1 + |a|)^2} A[u, D(a, \varrho)] \geqq c_1 A[u, D(a, \varrho)] , \quad c_1 = (1 - \varrho^2)^2/4 .$$

On letting $u = \log^+|f - f(a)|$, we have

$$A[\log^+|f - f(a)|, \Delta(a, \varrho)] \geqq c_1 A[\log|f - f(a)|, D(a, \varrho)] .$$

Therefore, (L3) $\Rightarrow$ (L1).

We next prove (L4) $\Rightarrow$ (L1). First we remember (see [4, Lemma 1]) that

$$m^*(D(a, \varrho)) \leqq \frac{\pi\varrho^2}{\sqrt{(|a|)}\,(1 + |a|)\sqrt{(1 - \varrho^2)}}$$

for $a \neq 0$, $\varrho \in (0, 1)$, so that for $a \in R(\lambda)$,

$$m^*(D(a, \varrho))^{-1} \geqq c_2/\pi\varrho^2 , \quad c_2 = \{\lambda(1 - \varrho^2)\}^{1/2} .$$

For $z = \psi_a(w)$, $w \in D(\varrho)$, we have $2|a| - 1 \leqq |z|$, so that

$$1 - |z|^2 \leqq 4(1 - |a|) .$$

Therefore, for our ϱ, λ with $a \in R(\lambda)$ we have

$$A^*[\log^+|f - f(a)|, D(a, \varrho)] =$$

$$= m^*(D(a, \varrho))^{-1} \iint_{D(\varrho)} \{\log^+|f \circ \psi_a(w) - f(a)|\} \frac{(1 - |a|)^2}{(1 - |z|^2)^2}\, du\, dv \geqq$$

$$\geqq (c_2/16)\, A[\log^+|f \circ \psi_a - f(a)|, D(\varrho)] =$$

$$= (c_2/16)\, A[\log^+|f - f(a)|, D(a, \varrho)] .$$

Thus, (L4) $\Rightarrow$ (L1).

For the proof of (L5) $\Rightarrow$ (B) we note that if u is subharmonic in $D(\varrho)$, $\varrho \in (0, 1)$, then

$$u(0) \leqq A^*[u, D(\varrho)] .$$

Actually,

$$A^*[u, D(\varrho)] = \frac{1 - \varrho^2}{\varrho^2} \int_0^\varrho \frac{2r}{(1 - r^2)^2} L(u, r)\, dr \geqq \tag{4.2}$$

$$\geqq \frac{1 - \varrho^2}{\varrho^2} \int_0^\varrho \frac{2r\, dr}{(1 - r^2)^2} u(0) = u(0) .$$

We consider g of (4.1) again. Then,

$$\log|g(0)| \leqq A^*[\log|g|, D(\varrho)] = A^*[\log|f \circ \phi_a - f(a)|, D(\varrho)] + Q(\varrho) ,$$

where

$$Q(\varrho) = \frac{1 - \varrho^2}{2\varrho^2} \log \frac{1 + \varrho}{1 - \varrho} - \log \varrho .$$

Therefore, we have

$$(1 - |a|^2)\, |f'(a)| \leqq e^{Q(\varrho)} \exp A^*[\log|f - f(a)|, \Delta(a, \varrho)] .$$

Theorem 3. *For f holomorphic in D, the condition*

(B0) $$f \in B_0$$

is equivalent to one of the following:

(S10) *There exists $\varrho \in (0, 1)$ such that*

$$\lim_{|a| \to 1} \sup_{z \in D(a,\varrho)} |f(z) - f(a)| = 0 .$$

(S20) *There exists $\varrho \in (0, 1)$ such that*

$$\lim_{|a| \to 1} \sup_{z \in \Delta(a,\varrho)} |f(z) - f(a)| = 0 .$$

(E10) *For each $c > 0$,*

$$\lim_{|a| \to 1} A[\exp\{c|f - f(a)|\}, D(a, 1)] = 1 .$$

(L10) *There exists $\varrho \in (0, 1)$ such that*

$$\lim_{|a| \to 1} A[\log |f - f(a)|, D(a, \varrho)] = -\infty .$$

(L20) *There exists $\varrho \in (0, 1)$ such that*

$$\lim_{|a| \to 1} A[\log |f \circ \phi_a - f(a)|, D(\varrho)] = -\infty .$$

(L50) *There exists $\varrho \in (0, 1)$ such that*

$$\lim_{|a| \to 1} A^*[\log |f - f(a)|, \Delta(a, \varrho)] = -\infty .$$

The conditions (E2), (L3), and (L4) appear to have no analogues.

In the figure

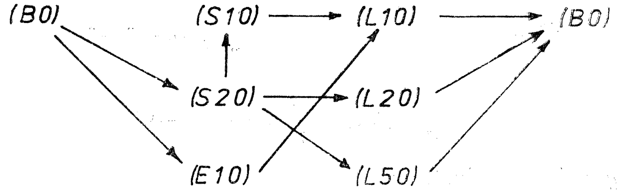

Figure 2

trivial are: (S10) $\Rightarrow$ (L10); (S20) $\Rightarrow$ (L50). Since $\phi_a(w) \in \Delta(a, \varrho)$ if $w \in D(\varrho)$, it follows that (S20) $\Rightarrow$ (L20). The inclusion formula (2.1) yields that (S20) $\Rightarrow$ (S10).

Leaving (B0) $\Rightarrow$ (S20); (Lk0) $\Rightarrow$ (B0), $k = 2, 5$, as exercises, we shall give the detailed proofs of

$$\text{(B0)} \Rightarrow \text{(E10)} \Rightarrow \text{(L10)} \Rightarrow \text{(B0)} .$$

Proof of (B0) $\Rightarrow$ (E10). Fix $c > 0$, and let $\varepsilon > 0$. Then, we may find $\delta \in (0, 1/c)$

such that $P^{c\delta} < 1 + \varepsilon$. We next find $\lambda \in (0, 1)$ such that

$$(1 - |z|^2) |f'(z)| \leqq \delta \quad \text{for} \quad z \in R(\lambda).$$

Since

$$D(a, 1) \subset R(\lambda) \quad \text{for} \quad a \in R(\lambda + 1)/2,$$

and since $D(a, 1)$ is convex in the non-Euclidean hyperbolic sense, we now have

$$|f(z) - f(a)| \leqq \delta\sigma(z, a) \leqq \delta \tanh^{-1} |w|$$

for $z = \psi_a(w) \in D(a, 1)$ with $w \in D$ by (3.2).

Therefore, for $a \in R((\lambda + 1)/2)$,

$$1 \leqq A[\exp\{c|f - f(a)|\}, D(a, 1)] \leqq$$
$$\leqq \pi^{-1} \iint_D \left(\frac{1 + |w|}{1 - |w|}\right)^{c\delta/2} du\, dv \leqq P^{c\delta} < 1 + \varepsilon,$$

which proves (E10).

Proof of (E10) $\Rightarrow$ (L10). Since

$$\log x = \log \log e^x \quad (x \geqq 0),$$

it follows that, for each $\varrho \in (0, 1)$,

$$A[\log |f - f(a)|, D(a, \varrho)] \leqq \log \log A[\exp |f - f(a)|, D(a, \varrho)] \leqq$$
$$\leqq \log \log A[\exp |f - f(a)|, D(a, 1)],$$

so that (L10) follows.

Proof of (L10) $\Rightarrow$ (B0). For each $x \in (0, \infty)$ there exists $\lambda \in (0, 1)$ such that

$$A[\log |f - f(a)|, D(a, \varrho)] < -x \quad \text{for each} \quad a \in R(\lambda).$$

By the estimate of $(1 - |a|^2) |f'(a)|$ in the proof of (L1) $\Rightarrow$ (B) we observe that

$$\sup_{a \in R(\lambda)} (1 - |a|^2) |f'(a)| \leqq 2\varrho^{-1} e^{1/2 - x}.$$

Hence (B0).

6. A REMARK ON BMOA

For u subharmonic in D we set

$$A^*[u] = \sup_{0 < \varrho < 1} A^*[u, D(\varrho)].$$

In connection with Baernstein's criteria cited in Section 1 we show that each of the following is equivalent to $f \in$ BMOA for f holomorphic in D.

(E*) *There exists* $c > 0$ *such that*

$$\sup_{a \in D} A^*[\exp\{c|f \circ \phi_a - f(a)|\}] < \infty.$$

(L*) $$\sup_{a \in D} A^*[\log^+ |f \circ \phi_a - f(a)|] < \infty.$$

Actually we can prove much more.

(6.1) *If u is subharmonic in D, then*

$$A^*[u] \leqq L[u].$$

(6.2) *If u is subharmonic in D with*

$$u(0) \geqq 0,$$

then

$$A^*[u] \geqq L[u].$$

Therefore, if u is subharmonic in D with $u(0) \geqq 0$, then $A^*[u] = L[u]$. We thus obtain

$$A^*[\exp\{c|f \circ \phi_a - f(a)|\}] = L[\exp\{c|f \circ \phi_a - f(a)|\}]$$

and

$$A^*[\log^+|f \circ \phi_a - f(a)|] = L[\log^+|f \circ \phi_a - f(a)|]$$

for $a \in D$, $c > 0$. The suprema in (E) and (L) are, therefore, the same as those in (E*) and (L*), respectively.

First, it follows from (4.2) that

$$A^*[u, D(\varrho)] \leqq \sup_{0<r<1} L(u, r) = L[u],$$

whence (6.1). For the proof of (6.2) we note that $0 \leqq L(u, r) \uparrow$ as $r \uparrow 1$. Given $\varepsilon \in (0, 1)$, we set

$$\varrho = \varepsilon R + 1 - \varepsilon \quad \text{for} \quad R \in (0, 1).$$

Then

$$A^*[u] \geqq A^*[u, D(\varrho)] = \frac{1-\varrho^2}{\varrho^2} \int_0^\varrho \frac{2r}{(1-r^2)^2} L(u, r)\, dr \geqq$$

$$\geqq \frac{1-\varrho^2}{\varrho^2} \int_R^\varrho \frac{2r\, dr}{(1-r^2)^2} L(u, R) \equiv \Phi(R)\, L(u, R),$$

where

$$\Phi(R) = \frac{(1-\varepsilon)\{(1+\varepsilon)R + (1-\varepsilon)\}}{(\varepsilon R + 1 - \varepsilon)^2 (1+R)}$$

is a decreasing function of R with

$$\lim_{R\to 1} \Phi(R) = 1 - \varepsilon \leqq \Phi(R) \leqq 1.$$

We thus have $A^*[u] \geqq (1-\varepsilon)\, L[u]$. Since ε is arbitrary, we have (6.2).

Remark. (i) We also have

$$\sup_{a\in D} L[|f \circ \phi_a - f(a)|] = \sup_{a \in D} A^*[|f \circ \phi_a - f(a)|].$$

(ii) We prove that $A^*[u, D(\varrho)]$ for u subharmonic in D is a nondecreasing function of $\varrho \in (0, 1)$. Actually,

$$L(u, r) \leqq L(u, \varrho) \quad \text{for} \quad r \in (0, \varrho),$$

so that

$$(\mathrm{d}/\mathrm{d}\varrho)\, A^*[u, D(\varrho)] = -2\varrho^{-3} \int_0^\varrho \frac{2r}{(1-r^2)^2} L(u, r)\, \mathrm{d}r +$$

$$+ \frac{1-\varrho^2}{\varrho^2} \frac{2\varrho}{(1-\varrho^2)^2} L(u, \varrho) \geqq - \frac{2}{\varrho(1-\varrho^2)} L(u, \varrho) + \frac{2}{\varrho(1-\varrho^2)} L(u, \varrho) = 0 .$$

We thus have

$$A^*[u] = \lim_{\varrho \to 1} A^*[u, D(\varrho)] .$$

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Author's address: Department of Mathematics, Tokyo Metropolitan University, Fukasawa, Setagaya, Tokyo 158, Japan.