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CARDINALITY OF THE SYSTEM OF ALL SEQUENTIAL CONVERGENCES ON AN ABELIAN LATTICE ORDERED GROUP

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E. Čech and B. Pospíšil investigated the number of topologies fulfilling certain conditions which can be defined on a set of a given cardinality (cf. [3], Thms. IV, V and VI). In this paper an analogous question concerning the number of convergence structures which can be defined on a given abelian lattice ordered group will be dealt with.

Let G be an abelian lattice ordered group. We denote by L_G the set of all compatible convergence structures which can be introduced on G (for definitions, cf. § 1 below). Let D be the system of all orthogonal subsets of G. Put $\sup \{\text{card } A: A \in D\} = b(G)$. The following results will be proved:

(A) If $b(G) \geqq \aleph_0$, then card $L_G \geqq 2^{2^{\aleph_0}}$.

(A′) The estimate given in (A) cannot be improved.

(B) If $b(G) = n$, n a positive integer, then there exists an integer k with $0 \leqq k \leqq n$, such that card $L_G = 2^k$.

(C) Let n be a positive integer and let k be an integer such that $0 \leqq k \leqq n$. Then there exists a proper class $\{G(i): i \in I\}$ of nonisomorphic abelian lattice ordered groups $G(i)$ such that for each $i \in I$ the relations $b(G(i)) = n$ and card $L_{G(i)} = 2^k$ are valid.

Some further results on the sequential convergence on an abelian lattice ordered group will also be established.

1. CONVERGENCE

In this section the notion of a compatible sequential convergence on an abelian lattice ordered group will be introduced and it will be shown that it suffices to consider positive sequences which converge to the neutral element of the given group.

Recall that an abelian lattice ordered group is a triple $(G, +, \leqq)$ such that

G is an abelian group with respect to $+$,
G is a lattice with respect to $\leqq$, and
$a \leqq b$ implies $a + g \leqq b + g$ whenever a, b, g are elements of G.

We use the following notation:

N — the set of all positive integers;
Z — the set of all integers;
G — the underlying set of an abelian lattice ordered group;
G^N — the set of sequences with elements belonging to G;
0 — the neutral element of a group;
G^+ — the positive cone of G;
$(G^+)^N$ — the set of sequences with elements belonging to G^+; this set is viewed as an ordered semigroup with respect to the induced $+$ and $\leqq$ (see $R + S$ and $R \leqq S$ below);
$\mathrm{id}(A)$ — the identical mapping on the set A;
$A \times B$ — the cartesian product of the sets A and B;
u, v, w — denote monotone mappings of N into N;
$S(n)$ denotes the n-th term of the sequence S;
$S \circ u$ ($S \circ u \circ v$) denotes the subsequence of the sequence S whose n-th term is $S(u(n))$ ($S(u(v(n)))$, respectively);
$\mathrm{const}\,(g)$ denotes the constant sequence $(g, g, g, \ldots)$;
$\leqq$ denotes the order on G or the induced pointwise order on G^N or any other order in the text;
$\wedge, \vee$ denote the lattice operations induced by $\leqq$;
$R + S$ denotes the sequence whose n-th term is $R(n) + S(n)$;
$S + g$ denotes the sequence whose n-th term is $S(n) + g$;
$S \leqq g$ means that $S(n) \leqq g$ for each $n \in N$.

The symbols $R \wedge S$, $R \vee S$, $-S$, $|S|$, $S - g$, $S \wedge g$, $S \vee g$ and $R \leqq S$ are to be understood analogously.

In the following definition the axioms (i)–(vi) define a FLUSH-convergence structure for G, i.e., a convergence group (cf. P. Mikusiński [9], J. Novák [10]); the remaining three axioms concern relations between the convergence and the order on G.

1.1. Definition. Let $(G, +, \leqq)$ be an abelian lattice ordered group. A set $\mathscr{L} \subset G^N \times G$ is said to be *a convergence on* G, if the following nine axioms are satisfied ($S \in G^N$):

(i) $(S, s) \in \mathscr{L}$ implies $(S \circ u, s) \in \mathscr{L}$ for every subsequence $S \circ u$ of S.
(ii) If for each u there exists v such that $(S \circ u \circ v, s) \in \mathscr{L}$, then $(S, s) \in \mathscr{L}$.
(iii) $(\mathrm{const}\,(s), s) \in \mathscr{L}$ for each $s \in G$.
(iv) $(S, a) \in \mathscr{L}$ and $(S, b) \in \mathscr{L}$ imply $a = b$.
(v) $(S, s) \in \mathscr{L}$ implies $(-S, -s) \in \mathscr{L}$.
(vi) $(R, r) \in \mathscr{L}$ and $(S, s) \in \mathscr{L}$ imply $(R + S, r + s) \in \mathscr{L}$.
(vii) $(R, r) \in \mathscr{L}$ and $(S, s) \in \mathscr{L}$ imply $(R \wedge S, r \wedge s) \in \mathscr{L}$.
(viii) $(R, r) \in \mathscr{L}$ and $(S, s) \in \mathscr{L}$ imply $(R \vee S, r \vee s) \in \mathscr{L}$.
(ix) If $(R, g) \in \mathscr{L}$ and $(T, g) \in \mathscr{L}$ and $R \leqq S \leqq T$, then $(S, g) \in \mathscr{L}$.

1.2. Example. Let G be an abelian tl-group (topological lattice ordered group), cf. B. Šmarda [11, 12], and let the convergence of sequences on G be induced by the interval topology. Then the conditions (i)–(ix) are fulfilled.

Denote by L_G the set of all convergences on G.

1.3. Lemma. *Let $\mathscr{L} \in L_G$. Then $(S, s) \in \mathscr{L}$ if and only if $(|S - s|, 0) \in \mathscr{L}$.*

Proof. Since $|S - s| = (S - s) \vee -(S - s)$ and $-|S - s| \leqq S - s \leqq |S - s|$, the assertion follows.

Lemma 1.3 above indicates that a convergence is determined by the system of its positive sequences which converge to zero. For a more detailed discussion of this situation cf. Theorem 1.6 below.

We denote by $P(\mathscr{L})$ the set of all $S \in (G^+)^N$ such that $(S, 0) \in \mathscr{L}$.

1.4. Lemma. *Let $\mathscr{L} \in L_G$. Then $P(\mathscr{L})$ is a convex subsemigroup of $(G^+)^N$ and the following conditions are fulfilled:*

(I) *$P(\mathscr{L})$ is closed under taking subsequences.*

(II) *Let $S \in (G^+)^N$. If for each u there exists v such that $S \circ u \circ v \in P(\mathscr{L})$, then $S \in P(\mathscr{L})$.*

(III) $\operatorname{const}(s) \in P(\mathscr{L})$ *if and only if* $s = 0$.

The proof is straightforward and will be omitted. The converse proposition is also valid (see Theorem 1.6 below).

1.5. Lemma. *Let P be a convex semigroup of $(G^+)^N$ and let the conditions (I)–(III) be fulfilled (with $P(L)$ replaced by P). Let $S \in G^N$ and $s \in G$. Then $|S - s| \in P$ if and only if there exist $A \in P$ and $B \in P$ such that $S = A - B + s$.*

Proof. Assume that $|S - s| \in P$. We put $A = (S - s) \vee 0$ and $B = (-S + s) \vee \vee 0$. Then $A - B = ((S - s) \vee 0) - (-(S - s) \vee 0) = ((S - s) \vee 0) + + ((S - s) \wedge 0) = S - s$ and thus $S = A - B + s$. Since $0 \leqq A = (S - s) \vee 0 \leqq \leqq ((S - s) \vee 0) - ((S - s) \wedge 0) = |S - s|$ and P is the convex semigroup of $(G^+)^N$ containing the sequence $\operatorname{const}(0)$, we have $A \in P$. Similarly, $0 \leqq B = = (-S + s) \vee 0 \leqq ((-S + s) \vee 0) - ((-S + s) \wedge 0) = |-S + s| = |S - s|$, therefore $B \in P$.

Conversely, assume that $S = A - B + s$, where A and B are elements of P. Then $|S - s| = |A - B|$ and the inequalities $0 \leqq |A - B| \leqq |A| + |-B| = A + B$ together with the convexity of the semigroup P imply $|S - s| \in P$.

1.6. Theorem. *Let P_1 be a convex semigroup of $(G^+)^N$ fulfilling the conditions (I)–(III). Then there exists $\mathscr{L}_1 \in L_G$ such that $P(\mathscr{L}_1) = P_1$. Moreover, if $P(\mathscr{L}_2) = = P_1$ for some $\mathscr{L}_2 \in L_G$, then $\mathscr{L}_2 = \mathscr{L}_1$.*

Proof. Put $\mathscr{L}_1 = \{(S, s) \in G^N \times G : |S - s| \in P_1\}$. We will verify that $\mathscr{L}_1 \in L_G$. The verification of the conditions (ii), (iii) and (v) is simple; in view of Lemma 1.5 also (i) and (vi) are evident. It remains to show the validity of the conditions (iv), (vii), (viii) and (ix).

(iv): Let $(S, a) \in \mathscr{L}_1$ and $(S, b) \in \mathscr{L}_1$. Then $|S - a| \in P_1$ and $|S - b| \in P_1$. The triangle inequality (cf. [1]) implies $0 \leqq |a - b| = |a - S + S - b| \leqq |a - S| + |S - b| = |S - a| + |S - b|$. Since P_1 is a convex semigroup, we have $\text{const}(|a - b|) \in P_1$ and the condition (III) gives $|a - b| = 0$ and hence $a = b$.

(ix): Let $(R, g) \in \mathscr{L}_1$ and $(T, g) \in \mathscr{L}_1$ and $R \leqq S \leqq T$. By Lemma 1.5 there exist elements A_R, B_R, A_T, B_T of P_1 such that $R = A_R - B_R + g$ and $T = A_T - B_T + g$. Denote $A_S = B_R + S - g$ and let $B_S = B_R$. Then evidently $B_S \in P_1$ and in view of the convexity of the semigroup P_1 and the relations $0 \leqq A_R = B_R + R - g \leqq B_R + S - g = A_S \leqq B_R + T - g = B_R + A_T - B_T \leqq B_R + A_T$, we obtain $A_S \in P_1$. So $A_S \in P_1$, $B_S \in P_1$ and $S = A_S - B_S + g$, thus $(S, g) \in \mathscr{L}_1$.

(vii): Let $(R, r) \in \mathscr{L}_1$ and $(S, s) \in \mathscr{L}_1$. Consider the relations $-|R - r| - |S - s| = -(|R - r| + |S - s|) = -((|R - r| \vee |S - s|) + (|R - r| \wedge |S - s|)) \leqq -(|R - r| \vee |S - s|) = (-|R - r|) \wedge (-|S - s|) \leqq (R - r) \wedge (S - s) \leqq R - r$. In view of the definition of $\mathscr{L}_1$ and by (v), (vi) and (ix) we have $((R - r) \wedge (S - s), 0) \in \mathscr{L}_1$. Hence *(iii)* and (vi) infer $((R - r) + (S - s), 0) \in \mathscr{L}_1$. Since $(R - r) \vee (S - s) = ((R - r) + (S - s)) - ((R - r) \wedge (S - s))$, by applying (v) and (vi) we obtain $((R - r) \vee (S - s), 0) \in \mathscr{L}_1$.

We have

$$\begin{aligned}(R \wedge S) - (r \wedge s) &= (R \wedge S) + (-r \vee -s) = \\ &= ((R \wedge S) - r) \vee ((R \wedge S) - s) = \\ &= ((R - r) \wedge (S - r)) \vee ((R - s) \wedge (S - s)) \leqq (R - r) \vee (S - s)\end{aligned}$$

and

$$\begin{aligned}(R \wedge S) - (r \wedge s) &= (R - (r \wedge s)) \wedge (S - (r \wedge s)) = \\ &= (R + (-r \vee -s)) \wedge (S + (-r \vee -s)) = \\ &= ((R - r) \vee (R - s)) \wedge ((S - r) \vee (S - s)) \geqq (R - r) \wedge (S - s)\,.\end{aligned}$$

Because of the convexity of $\mathscr{L}_1$ we conclude that $((R \wedge S) - (r \wedge s), 0) \in \mathscr{L}_1$, and by (iii) and (vi) we finally have $(R \wedge S, r \wedge s) \in \mathscr{L}_1$.

(viii): Let $(R, r) \in \mathscr{L}_1$ and $(S, s) \in \mathscr{L}_1$. By (vi) and (vii) we have $(R + S, r + s) \in \mathscr{L}_1$ and $(R \wedge S, r \wedge s) \in \mathscr{L}_1$. Since $R \vee S = (R + S) - (R \wedge S)$ and $r \vee s = (r + s) - (r \wedge s)$, the conditions (v) and (vi) imply $(R \vee S, r \vee s) \in \mathscr{L}_1$.

We have verified that $\mathscr{L}_1 \in L_G$. Using this fact, it is easy to prove that $P(\mathscr{L}_1) = P_1$; we omit the proof.

For completing the proof of the theorem is suffices to show that if $\mathscr{L}_2 \in L_G$ and $P(\mathscr{L}_2) = P_1$, then $\mathscr{L}_2 = \mathscr{L}_1$. But $(S, s) \in \mathscr{L}_2$ implies $|S - s| \in P(\mathscr{L}_2)$ and in accordance with $P(\mathscr{L}_2) = P_1 = P(\mathscr{L}_1)$ we have $(S, s) \in \mathscr{L}_1$. The converse implication is also true and thus $\mathscr{L}_1 = \mathscr{L}_2$.

1.7. Lemma. *Let $\mathscr{L} \in L_G$, $(R, r) \in \mathscr{L}$, $(S, s) \in \mathscr{L}$ and $R \leqq S$. Then $r \leqq s$.*

Proof. The condition (vii) implies $(R \wedge S, r \wedge s) \in \mathscr{L}$. Since $R \wedge S = R$, the relation $(R, r \wedge s) \in \mathscr{L}$ is valid. But also $(R, r) \in \mathscr{L}$, thus by (iv) we have $r \wedge s = r$ and hence $r \leqq s$.

1.8. Lemma. *Let* $\mathscr{L} \in L_G$ *and* $(R, 0) \in \mathscr{L}$. *Let* $p \in G$ *such that* $p > 0$. *Then the set* $\{n \in N: |R(n)| \geqq p\}$ *is finite.*

Proof. By way of contradiction, assume that there exists u such that $|R \circ u| \geqq p$. Then we have $0 \leqq \mathrm{const}\,(p) \leqq |R \circ u|$ and from (i), (iii), Lemma 1.3 and (ix) we infer that $(\mathrm{const}\,(p), 0) \in \mathscr{L}$. Since $(\mathrm{const}\,(p), p) \in \mathscr{L}$, the condition (iv) implies $p = 0$.

2. SPECIAL CONVERGENCES

Denote $\mathrm{dc}(G) = \{(S, s) \in G^N \times G: S(n) = s$ for all but finitely many $n \in N\}$. Then $\mathrm{dc}(G)$ is the well-known discrete convergence on G; it is easy to see that the following lemma is valid.

2.1. Lemma. $\mathrm{dc}(G) \in L_G$, *and if* $\mathscr{L} \in L_G$, *then* $\mathrm{dc}(G) \subset \mathscr{L}$. So, $\mathrm{dc}(G)$ is the smallest element of the set L_G partially ordered by the inclusion.

Let us recall the definition of the order convergence on a lattice ordered group G (cf. [5]); we denote it by $\mathrm{oc}(G)$:

2.2. Definition. $(S, s) \in \mathrm{oc}(G)$ if $(S, s) \in G^N \times G$ and there are $R \in G^N$ and $T \in G^N$ such that the following conditions are fulfilled:

(a) $R(n) \leqq R(n + 1)$ *for each* $n \in N$;
(b) $\sup\{R(n): n \in N\} = s$;
(c) $T(n) \geqq T(n + 1)$ *for each* $n \in N$;
(d) $\inf\{T(n): n \in N\} = s$;
(e) $R \leqq S \leqq T$.

The order convergence on a lattice ordered group G was studied by C. J. Everett and S. Ulam in [4]; it is clear from their results that $\mathrm{oc}(G)$ fulfils the conditions (i) and (iii) – (ix). Thus $\mathrm{oc}(G)$ is a convergence on G in the sense of Definition 1.1 if and only if the condition (ii) is fulfilled. We shall see (cf. Lemma 3.6 below) that if G is an abelian linearly ordered group, then $\mathrm{oc}(G) \in L_G$.

Since the condition (ii) was not applied in the proofs of Lemmas 1.3 and 1.8, the same lemmas concerning $\mathrm{oc}(G)$ are valid. We will apply them below (see § 3).

2.3. Lemma. $(S, s) \in \mathrm{oc}(G)$ *if and only if* $(|S - s|, 0) \in \mathrm{oc}(G)$.

2.4. Lemma. *Let* $(S, 0) \in \mathrm{oc}(G)$ *and let* $p \in G$ *be such that* $p > 0$. *Then the set* $\{n \in N: |S(n)| \geqq p\}$ *is finite.*

3. LINEARLY ORDERED GROUPS

The aim of this section is to prove that an arbitrary abelian linearly ordered group G has at most two convergences, namely $\mathrm{dc}(G)$ and $\mathrm{oc}(G)$.

Let $(G, +, \leqq)$ be an abelian linearly ordered group.

3.1. Lemma. *Let* $\mathscr{L} \in L_G$, $S \in P(\mathscr{L})$ *and* $n \in N$. *Then* $\inf\{S(i): i \in N \text{ and } i \geqq n\} = 0$.

Proof. For $i \in N$, define $u(i) = n + i - 1$. Since $S \in P(\mathscr{L})$, we have $S \circ u \geqq 0$. Let c be a lower bound of this set, i.e., $S \circ u \geqq c$. In view of linearity, either $c > 0$ or $c \leqq 0$ holds. But if $c > 0$, then from $0 \leqq c \leqq S \circ u$ and from the properties of $P(\mathscr{L})$ (cf. Lemma 1.4) we get $c = 0$. Thus $\inf\{S(i): i \in N \text{ and } i \geqq n\} = 0$.

3.2. Lemma. *Let* $\mathscr{L} \in L_G$ *and* $S \in P(\mathscr{L})$. *Then* $\inf\{\sup\{S(i): i \in N \text{ and } i \geqq n\}: n \in N\} = 0$.

Proof. First we show that whenever $n \in N$ then $\sup\{S(i): i \in N \text{ and } i \geqq n\}$ exists. If $\{S(i): i \in N \text{ and } i \geqq n\} = \{0\}$, then $\sup\{S(i): i \in N \text{ and } i \geqq n\} = 0 = S(n)$. On the other hand, if there exists $n_0 \in N$ such that $n_0 \geqq n$ and $S(n_0) \neq 0$ (i.e. $S(n_0) > 0$), then the set $\{i \in N: S(i) \geqq S(n_0)\}$ is finite (cf. Lemma 1.8) and hence $\sup\{S(i): i \in N \text{ and } i \geqq n\} = \max\{S(i): i \in N \text{ and } i \geqq n\} = S(j)$ for some $j \in N$. Since $\{\sup\{S(i): i \in N \text{ and } i \geqq n\}: n \in N\} \subset \{S(n): n \in N\}$, the suprema of the final segments of S exist. Next, we shall show that the infimum of these suprema is zero. Consider the set $\{n \in N: S(n) \neq 0\}$. Clearly, if it is finite, then the assertion of the lemma is valid. Assume that the set is infinite. By Lemma 1.8, for each element $p \in G$, $p > 0$, there are infinitely many members of the sequence S belonging to the open interval from zero to p. Since $S \in P(\mathscr{L})$, the relation $0 \leqq \sup\{S(i): i \in N \text{ and } i \geqq n\}$ is valid for each $n \in N$. By way of contradiction we shall prove that there exists no strictly positive lower bound of all these suprema. So, assume that for some $c \in G$, $c > 0$, we have $c \leqq \sup\{S(i): i \in N \text{ and } i \geqq n\}$ for all $n \in N$. By Lemma 1.8, there is $m \in N$ such that $0 < S(m) < c$. Apply Lemma 1.8 again, with $p = S(m)$. Then the set of all those members of the sequence S which are not contained in the closed interval from zero to $S(m)$ is finite. Hence there exists $k \in N$ such that $\sup\{S(i): i \in N \text{ and } i \geqq k\} \leqq S(m)$. For each $n \in N$ we have $\sup\{S(i): i \in N \text{ and } i \geqq n\} \geqq c$, thus $S(m) \geqq c$, a contradiction.

3.3. Lemma. *Let* $\mathscr{L} \in L_G$ *and* $T \in P(\mathscr{L})$. *Then either* $T \in P(\mathrm{dc}(G))$ *or there exists decreasing subsequence* $T \circ w$ *of* T.

Proof. If $T \notin P(\mathrm{dc}(G))$, then there is u such that $T(u(n)) > 0$ for each $n \in N$. By Lemma 1.8, the set $\{n \in N: T(u(n)) \geqq p\}$ is finite for each $p \in G$, $p > 0$, and the assertion follows by induction.

3.4. Lemma. *Let* $\mathscr{L}_1, \mathscr{L}_2 \in L_G$ *such that* $\mathscr{L}_2 \neq \mathrm{dc}(G)$. *Then* $\mathscr{L}_1 \subset \mathscr{L}_2$.

Proof. In view of Lemma 1.3 and Theorem 1.6 it suffices to prove that $P(\mathscr{L}_1) \subset P(\mathscr{L}_2)$. Let $S \in P(\mathscr{L}_1)$. We shall show that for each u there is v such that $S \circ u \circ v \in P(\mathscr{L}_2)$ and thus $S \in P(\mathscr{L}_2)$. For an arbitrary u we have $S \circ u \in P(\mathscr{L}_1)$. If $S \circ u \in P(\mathrm{dc}(G))$, then Lemma 2.1 implies $S \circ u \circ \mathrm{id}(N) \in P(\mathscr{L}_2)$. On the other hand, if $S \circ u \notin P(\mathrm{dc}(G))$, then there is a subsequence $S \circ u \circ u_1$ of the sequence S such that $S \circ u \circ u_1(n) > 0$ for each $n \in N$. The assumption $\mathscr{L}_2 \neq \mathrm{dc}(G)$ and Lemma 2.1 imply that there is $T \in P(\mathscr{L}_2)$ such that $T \notin P(\mathrm{dc}(G))$. By Lemma 3.3 there exists w

such that $T \circ w$ is a decreasing subsequence of T. Since $T \circ w \in P(\mathscr{L}_2)$, there is $n_1 \in N$ such that $S \circ u \circ u_1(n_1) < T \circ w(1)$, by Lemma 1.8. The relation $\inf\{T \circ w(n): n \in N\} = 0$ (cf. Lemma 3.1) implies that there is $m_1 \in N$ such that $T \circ w(m_1) < S \circ u \circ u_1(n_1)$. By induction we obtain mappings v_1 and u_1 such that $S \circ u \circ u_1 \circ u_2 \leqq T \circ w \circ v_1$ and, since $T \in P(\mathscr{L}_2)$, we have $S \circ u \circ u_1 \circ u_2 \in P(\mathscr{L}_2)$. So, to complete the proof, it suffices to take $v = u_1 \circ u_2$.

3.5. Lemma. *Let $S \in (G^+)^N$. If for each u there exists v such that $(S \circ u \circ v, 0) \in \mathrm{oc}(G)$, then $(S, 0) \in \mathrm{oc}(G)$.*

Proof. Denote $M_0 = \{n \in N: S(n) > 0\}$. If M_0 is the empty set, then $(S, 0) \in \mathrm{oc}(G)$. If M_0 is a finite non-empty set, then put $T(i) = \max\{S(n): n \in N\}$ for $i \in N$, $i \leqq \max M_0$ and $T(i) = 0$ for $i > \max M_0$. The sequence T is non-increasing and such that $S \leqq T$ and $\inf\{T(n): n \in N\} = 0$; thus $(S, 0) \in \mathrm{oc}(G)$. Now, let M_0 be an infinite set. Then there is u such that $S \circ u > 0$. By the assumption of the lemma and by Lemma 2.4 there is v such that $S \circ u \circ v$ is a decreasing sequence. By the same assumptions the sets $M_1 = \{n \in N: S \circ u \circ v(1) \leqq S(n)\}$ and $M_{k+1} = \{n \in N: S \circ u \circ v(k + 1) \leqq S(n) < S \circ u \circ v(k)\}$, $k \in N$, are non-empty and finite. Denote $m_k = \max M_k$ for each $k \in N$ and define $T(i) = \max\{S(n): n \in M_1\}$ for $i \in N$, $i \leqq m_1$, and $T(i) = \max\{S(n): n \in M_{k+1}\}$ for $i \in N$, $m_1 + \ldots + m_k < i \leqq m_1 + \ldots + m_k + m_{k+1}$. Then T is a non-increasing sequence such that $S \leqq T$ and $\inf\{T(i): i \in N\} = 0$. Thus $(S, 0) \in \mathrm{oc}(G)$.

3.6. Lemma. $\mathrm{oc}(G) \in L_G$.

Proof. In view of the results of C. J. Everett and S. Ulam [4], it suffices to prove the condition (ii) of Definition 1.1 (cf. § 2). Assume that $(S, s) \in G^N \times G$ and for each u there is v such that $(S \circ u \circ v, s) \in \mathrm{oc}(G)$. Consequently, applying Lemmas 2.3, 3.5 and again 2.3 we get $(S, s) \in \mathrm{oc}(G)$.

3.7. Lemma. *If $\mathscr{L} \in L_G$, then $\mathscr{L} \subset \mathrm{oc}(G)$.*

Proof. Let $S \in P(\mathscr{L})$. Then by Lemma 3.2 we have $0 \leqq S \leqq T$ and $\inf\{T(n): n \in N\} = 0$ where $T(n) = \sup\{S(i): i \in N \text{ and } i \geqq n\}$. Hence $S \in P(\mathrm{oc}(G))$ by Definition 2.2 and therefore $\mathscr{L} \subset \mathrm{oc}(G)$ by Theorem 1.6.

3.8. Lemma. *Let $\mathscr{L} \in L_G$ and $\mathscr{L} \neq \mathrm{dc}(G)$. Then $\mathscr{L} = \mathrm{oc}(G)$.*

Proof. In view of Lemmas 3.6 and 3.4 we have $\mathrm{oc}(G) \subset \mathscr{L}$. But, in view of Lemma 3.7, also $\mathscr{L} \subset \mathrm{oc}(G)$.

Lemmas 2.1, 3.6 and 3.8 yield the following result.

3.9. Theorem. *Let G be an abelian linearly ordered group. Then $L_G = \{\mathrm{dc}(G), \mathrm{oc}(G)\}$.*

Recall that the sequence $S \in (G^+)^N$ is said to be coinitial in G^+ if $\inf\{S(n): n \in N\} = 0$ (cf. [7]). As a corollary we get the following result.

3.10. Corollary. *Let G be an abelian linearly ordered group. Then either* card $L_G = 1$ *or* card $L_G = 2$. *Further,* card $L_G = 2$ *if and only if there is a decreasing coinitial sequence in* G^+.

4. DIRECT PRODUCTS

Let us recall the definition of the direct product of two lattice ordered groups.

4.1. Definition. We say that $(G, +, \leqq)$ is the *direct product* of lattice ordered groups $(H, +_H, \leqq_H)$ and $(K, +_K, \leqq_K)$ if the following three conditions are fulfilled:
(1) $G = H \times K$.
(2) $(h_1, k_1) + (h_2, k_2) = (h_1 +_H h_2, k_1 +_K k_2)$ for each $h_1, h_2 \in H$ and $k_1, k_2 \in K$.
(3) $(h_1, k_1) \leqq (h_2, k_2)$ whenever $h_1 \leqq_H h_2$ and $k_1 \leqq_K k_2$ hold.

The direct product of $(H, +_H, \leqq_H)$ and $(K, +_K, \leqq_K)$ will be denoted by $H \otimes K$.

Throughout this section let H and K be abelian lattice ordered groups and $G = H \otimes K$. Denote by φ the natural isomorphism between lattice ordered groups $H^N \otimes K^N$ and $(H \otimes K)^N$ defined as follows:
$\varphi((A, B))(n) = (A(n), B(n))$ whenever $(A, B) \in H^N \times K^N$ (φ maps a pair (A, B) of sequences into the corresponding sequence of pairs $(A(n), B(n))$. We denote $\varphi(M) = \{\varphi(m): m \in M\}$ for $M \subset H^N \times K^N$.

4.2. Lemma. *Let $\mathscr{L}_H \in L_H$ and $\mathscr{L}_K \in L_K$. Then $P = \varphi(P(\mathscr{L}_H) \times P(\mathscr{L}_K))$ is a convex semigroup of $(G^+)^N$ fulfilling conditions* (I)–(III) *of Lemma* 1.4.

Proof. The inclusions $P(\mathscr{L}_H) \subset (H^+)^N$ and $P(\mathscr{L}_K) \subset (K^+)^N$ imply that $P \subset \varphi((H^+)^N \times (K^+)^N) = (G^+)^N$. It is easy to verify that P is a convex semigroup of $(G^+)^N$.

(I): Let $S \in P$. Then there are $A \in P(\mathscr{L}_H)$ and $B \in P(\mathscr{L}_K)$ such that $S = \varphi(A, B)$. Then $S \circ u = \varphi(A \circ u, B \circ u)$ for each u and, since $A \circ u \in P(\mathscr{L}_H)$ and $B \circ u \in P(\mathscr{L}_K)$, we have $S \circ u \in P$.

(II): Let $S \in (G^+)^N$, i.e., $S = \varphi(A, B)$ for some $A \in (H^+)^N$ and $B \in (K^+)^N$. Suppose that for each u there exists v such that $S \circ u \circ v \in P$. Since $S \circ u \circ v = \varphi(A \circ u \circ v, B \circ u \circ v)$, we have $A \circ u \circ v \in P(\mathscr{L}_H)$ and $B \circ u \circ v \in P(\mathscr{L}_K)$. Therefore $A \in P(\mathscr{L}_H)$ and $B \in P(\mathscr{L}_K)$, and thus $S \in P$.

(III): Let $\text{const}(s_G) = \varphi(\text{const}(s_H), \text{const}(s_K)) \in P$. Hence $\text{const}(s_H) \in P(\mathscr{L}_H)$ and $\text{const}(s_K) \in P(\mathscr{L}_K)$. Thus $s_H = 0_H$ and $s_K = 0_K$ and therefore $s_G = 0_G$. The converse implication is trivially true.

If $\mathscr{L} \in L_G$, then instead of $(S, s) \in \mathscr{L}$ we write also $s = \mathscr{L}\text{-lim } S$.

4.3. Lemma. *Let $\mathscr{L} \in L_G$. Then the following two conditions are equivalent:*
(1) $(a, b) = \mathscr{L}\text{-lim } \varphi(A, B)$;
(2) $(a, 0_K) = \mathscr{L}\text{-lim } \varphi(A, \text{const}(0_K))$ *and* $(0_H, b) = \mathscr{L}\text{-lim } \varphi(\text{const}(0_H), B)$.

Proof. Assume that (1) holds. Since $(a, b) = \mathscr{L}\text{-lim } \varphi(\text{const}(a), \text{const}(b))$, we have $(a, b) = \mathscr{L}\text{-lim } \varphi(A \wedge a, B \wedge b)$. The inequalities $(A \wedge a, B \wedge b) \leqq (A \wedge a, \text{const}(b)) \leqq (\text{const}(a), \text{const}(b))$ imply that $\mathscr{L}\text{-lim } \varphi(A \wedge a, \text{const}(b)) = (a, b)$.

Similarly, we obtain $\mathscr{L}$-lim $\varphi(A \vee a, \operatorname{const}(b)) = (a, b)$. Hence $\mathscr{L}$-lim $\varphi(A + a, \operatorname{const}(2b)) = (2a, 2b)$. But $\mathscr{L}$-lim $\varphi(\operatorname{const}(-a), \operatorname{const}(-2b)) = (-a, -2b)$ and thus $\mathscr{L}$-lim $\varphi(A, \operatorname{const}(0_K)) = (a, 0_K)$. The identity $(0_H, b) = \mathscr{L}$-lim $\varphi(\operatorname{const}(0_H), B)$ can be proved in the same way.

Conversely, assume that (2) holds. Then, adding $(a, 0_K)$ and $(0_H, b)$, we get the condition (1).

4.4. Lemma. *Let $\mathscr{L}_G \in L_G$. Then there are $\mathscr{L}_H \in L_H$ and $\mathscr{L}_K \in L_K$ such that $\varphi(P(\mathscr{L}_H) \times (P(\mathscr{L}_K)) = P(\mathscr{L}_G)$.*

Proof. Define $P_1 = \{A \in (H^+)^N: \mathscr{L}_G\text{-lim } \varphi(A, \operatorname{const}(0_K)) = 0_G\}$ and $P_2 = \{B \in (K^+)^N: \mathscr{L}_G\text{-lim } \varphi(\operatorname{const}(0_H), B) = 0_G\}$. Easily, P_1 is a convex semigroup of $(H^+)^N$ fulfilling the conditions (I)–(III); thus P_1 defines $\mathscr{L}_H \in L_H$ with $P(\mathscr{L}_H) = P_1$. Analogously, P_2 defines $\mathscr{L}_K \in L_K$ such that $P(\mathscr{L}_K) = P_2$. Then $\varphi(P(\mathscr{L}_H) \times P(\mathscr{L}_K)) = \varphi(P_1 \times P_2)$ and, by Lemma 4.3, we have $\varphi(P_1 \times P_2) = P(\mathscr{L}_G)$.

4.5. Theorem. *If $G = H \otimes K$, then* $\operatorname{card} L_G = \operatorname{card} L_H \,.\, \operatorname{card} L_K$.

Proof. Define a mapping ξ from $L_H \times L_K$ into L_G in the following way: for $\mathscr{L}_H \in L_H$ and $\mathscr{L}_K \in L_K$ let $\xi(\mathscr{L}_H, \mathscr{L}_K) = \mathscr{L}_G$, where $\mathscr{L}_G$ is defined by $P(\mathscr{L}_G) = \varphi(P(\mathscr{L}_H) \times P(\mathscr{L}_K))$. In view of Lemma 4.2 and Theorem 1.6, ξ is well defined. It is easy to see that ξ is an injective mapping and Lemma 4.4 implies that each element of L_G has its counterimage in $L_H \times L_K$. Thus ξ is a bijection between $L_H \times L_K$ and L_G and the proof is complete.

5. LEX-EXTENSIONS

Throughout this section let G and H be abelian lattice ordered groups such that $H \neq \{0\}$ and let C be a linearly ordered group. Let us recall the notion of a lex-extension (cf. [2]).

5.1. Definition. G is said to be a *lex-extension* of H by means of C, in symbols $G = \operatorname{lex}_C H$, if the following conditions are fulfilled:

(A) H is an l-ideal of G;

(B) the factor group G/H equipped with the induced operation and order is linearly ordered group C;

(C) if $g \in G^+$ and $h \in H$, then $g \notin H$ implies $g > h$.

5.2. Lemma. *Let $G = \operatorname{lex}_C H$ and $\mathscr{L} \in L_H$. Let $P_1(\mathscr{L}) = \{S \in (G^+)^N$: for each u there is v such that $S \circ u \circ v \in P(\mathscr{L})\}$. Then $P_1(\mathscr{L})$ is a convex subsemigroup of $(G^+)^N$ fulfilling the conditions* (I)–(III).

Proof. If $R, S \in P_1(\mathscr{L})$, then $R + S \in (G^+)^N$. Since $R \in P_1(\mathscr{L})$ then for an arbitrary u there is v_1 such that $R \circ u \circ v_1 \in P(\mathscr{L})$. Similarly, $S \in P_1(\mathscr{L})$ infers the existence of v_2 such that $S \circ u \circ v_1 \circ v_2 \in P(\mathscr{L})$. Then, due to the properties of $P(\mathscr{L})$, the relation $R \circ u \circ v_1 \circ v_2 + S \circ u \circ v_1 \circ v_2 \in P(\mathscr{L})$ holds. So, for a given u we have found $v = v_1 \circ v_2$ such that $(R + S) \circ u \circ v \in P(\mathscr{L})$, i.e., $R + S \in P_1(\mathscr{L})$. Thus $P_1(\mathscr{L})$ is a subsemigroup of $(G^+)^N$. The convexity of $P_1(\mathscr{L})$ can be proved analogously. Finally, the properties (I)–(III) of $P_1(\mathscr{L})$ are implied by the corresponding properties of $P(\mathscr{L})$; we omit the obvious proof.

It is easy to verify that $P_1(\mathscr{L}) = \{S \in (G^+)^N$: for some $n \in N$, the sequence $S(n)$, $S(n + 1), \ldots, S(n + i), \ldots$ belongs to $P(\mathscr{L})\}$.

The proof of the following lemma is straightforward and is omitted.

5.3. Lemma. *Let* $G = \text{lex}_C H$ *and* $\mathscr{L} \in L_G$. *Let* $P_2(\mathscr{L}) = P(\mathscr{L}) \cap (H^+)^N$. *Then* $P_2(\mathscr{L})$ *is a convex subsemigroup of* $(H^+)^N$ *fulfilling the conditions* (I)–(III).

5.4. Theorem. *If* $G = \text{lex}_C H$ *with* $H \neq \{0\}$, *then* card L_G = card L_H.

Proof. For $\mathscr{L} \in L_G$ denote by $\xi(\mathscr{L})$ the element of L_H defined by $P_2(\mathscr{L})$ (cf. Lemma 5.3 and Theorem 1.6). We shall show that ξ is a bijection between L_G and L_H. According to Lemma 5.3, ξ is a mapping from L_G into L_H. If $\mathscr{L} \in L_H$, then the convergence defined by $P_1(\mathscr{L})$ (cf. Lemma 5.2) is a counterimage of $\mathscr{L}$ with respect to ξ; thus ξ is surjective. To complete the proof we need to show that ξ is also injective. So, let $\mathscr{L}$ and $\mathscr{K}$ be elements of L_G. Since $H \neq \emptyset$, it is easy to see that $S \in P(\mathscr{L})$ if and only if there is a final segment of S belonging to $P_2(\mathscr{L})$. Then the relation $\xi(\mathscr{L}) = \xi(\mathscr{K})$ implies $\mathscr{L} = \mathscr{K}$.

6. FINITE CASE

Now we will describe the cardinality of L_G in case when G is of finite breadth.

6.1. Definition. A subset A of G will be said to be *orthogonal* if $a > 0$ for each $a \in A$, and $a_1 \wedge a_2 = 0$ for each pair of distinct elements $a_1, a_2 \in A$.

6.2. Definition. An abelian lattice ordered group G is said to be of *breadth n*, in symbols $b(G) = n$, if G contains an orthogonal subset with n elements but no orthogonal subset with $n + 1$ elements.

6.3. Lemma. *Let C be an abelian linearly ordered group and let G be an abelian lattice ordered group of breadth n. Then* $b(\text{lex}_C G) = n$.

Proof. We have $b(G) = n$, and hence there is an orthogonal subset $\{a_1, a_2, \ldots, a_n\}$ of G. The inclusion $G \subset \text{lex}_C G$ and the strict positivity of the elements $a_1, a_2, \ldots, a_n$ in $\text{lex}_C G$ are obvious. Since G is a convex subset of $\text{lex}_C G$, $a_1, a_2, \ldots, a_n$ are pairwise disjoint elements in the $\text{lex}_C G$ as well. It suffices to show that there is no orthogonal subset of $\text{lex}_C G$ with $n + 1$ elements. By way of contradiction, let $\{b_1, b_2, \ldots, b_{n+1}\}$ be such a set. Since $b(G) = n$ there is $k \in \{1, 2, \ldots, n + 1\}$ such that $b_k \notin G$. But if $b_i \in G$, $i \in \{1, 2, \ldots, n + 1\}$, then we have $b_k > b_i$ (cf. Def. 5.1 (C)), and therefore $b_k \wedge b_i = b_i \neq 0$. Thus $b_i \notin G$ for each $i \in \{1, 2, \ldots, n + 1\}$. Since $n + 1 \geqq 2$, there exist b_1 and b_2. From $b_1 \notin G$, $b_2 \notin G$ and $a_1 \in G$ we infer that $b_1 > a_1$ and $b_2 > a_1$. Then $b_1 \wedge b_2 \geqq a_1 > 0$, a contradiction.

6.4. Lemma. *Let G and H be abelian lattice ordered groups such that $b(G) = m$ and $b(H) = n$. Then $b(G \otimes H) = m + n$.*

Proof. Let $\{(a_j, b_j): j \in I\}$ be an orthogonal subset of $G \otimes H$. In this set there are at most m elements with $a_j \neq 0_G$. Otherwise, we could easily construct an orthogonal subset of G with $m + 1$ elements, which contradicts $b(G) = m$. Similarly, there are at most n elements of $\{(a_j, b_j): j \in I\}$ with $b_j \neq 0_H$. Thus each orthogonal subset of $G \otimes H$ can contain at most $m + n$ elements. Now, let $\{g_1, g_2, \ldots, g_m\}$ be an orthogonal subset of G and let $\{h_1, h_2, \ldots, h_n\}$ be an orthogonal subset of H. It is easy to verify that $\{(g_1, 0_H), (g_2, 0_H), \ldots, (g_m, 0_H), (0_G, h_1), (0_G, h_2), \ldots, (0_G, h_n)\}$ is an orthogonal subset of $G \otimes H$ with $m + n$ elements.

Note that if $G = \{0_G\}$, then card $L_G = 1$.

6.5. Theorem. *Let G be an abelian lattice ordered group of breadth n, $n \in N$. Then there is $k \in N \cup \{0\}$ such that $k \leqq n$ and* card $L_G = 2^k$.

Proof. If $n = 1$, then G is a linearly ordered group and, in view of Theorem 3.9, there is $k \in \{0, 1\}$ such that card $L_G = 2^k$. Suppose $n \geqq 2$. By induction, assume that for each abelian lattice ordered group H with $b(H) < n$ there is $m \in N \cup \{0\}$ such that $m \leqq b(H)$ and card $L_H = 2^m$. In view of [2], there are lattice ordered groups K, $H(1), H(2), \ldots, H(r)$ $(r \in N)$, and a linearly ordered group C such that

$G = \text{lex}_C K$,
$K = H(1) \otimes H(2) \otimes \ldots \otimes H(r)$ and
$b(H(j)) < n$ for each $j \in \{1, 2, \ldots, r\}$.

Clearly, $K \neq 0_G$ because, by Lemma 6.3, $b(K) = n \geqq 2$. By Lemmas 6.3 and 6.4 we get $n = b(G) = b(\text{lex}_C K) = b(K) = b(H(1) \otimes H(2) \otimes \ldots \otimes H(r)) = b(H(1)) + b(H(2)) + \ldots + b(H(r))$. By induction, there are $k(1), k(2), \ldots, k(r) \in N \cup \{0\}$ such that $k(j) \leqq b(H(j))$ and card $L_{H(j)} = 2^{k(j)}$ for each $j \in \{1, 2, \ldots, r\}$. If we denote $k = k(1) + k(2) + \ldots + k(r)$, then $k \leqq n$. By Theorems 5.4 and 4.5, card $L_G =$ card $L_K =$ card $L_{H(1)}$. card $L_{H(2)}$ . ... . card $L_{H(r)} = 2^{k(1)} . 2^{k(2)} . \ldots . 2^{k(r)} = 2^k$.

A question arises whether, for $n \in N$ and $k \in N \cup \{0\}$ such that $k \leqq n$, there exists an abelian lattice ordered group of breadth n with 2^k convergences. The answer is affirmative and we prove a bit more in the following theorem. For the terminology cf. L. Fuchs [5] and T. Jech [8].

6.6. Theorem. *If $n \in N$ and $k \in N \cup \{0\}$ such that $k \leqq n$, then the system of abelian lattice ordered groups of breadth n with 2^k convergences is a proper class.*

Proof. For a given ordinal number α we can construct an abelian lattice ordered group G such that $b(G) = n$, card $L_G = 2^k$ and card $G = 2^{\aleph_\alpha}$, in the following way: Put $H(\alpha) = Z^{\aleph_\alpha}$. Then $H(\alpha)$ can be viewed as a linearly ordered group with respect to the lexicographic order and the pointwise operation induced by the order and the obvious operation of the additive linearly ordered group Z of integers. For $H(\alpha)$ we have $b(H(\alpha)) = 1$, card $L_{H(\alpha)} = 1$ and card $H(\alpha) = 2^{\aleph_\alpha}$. There are two possibilities: either $k = n$ or $k < n$. If $k = n$, denote by $G(\alpha)$ the lexicographic product $H(\alpha)$ with Q (the group of rationals with the obvious additive operation and the obvious order) and put

$$G = \underbrace{G(\alpha) \otimes G(\alpha) \otimes \ldots \otimes G(\alpha)}_{n\text{-copies}} .$$

If $k < n$, denote

$$G = (\underbrace{Q \otimes Q \otimes \ldots \otimes Q}_{k\text{-copies}}) \otimes (\underbrace{Z \otimes Z \otimes \ldots \otimes Z}_{(n-k-1)\text{-copies}}) \otimes H(\alpha) .$$

In both cases we obtain a lattice ordered group G such that card $G = 2^{\aleph_\alpha}$, $b(G) = n$ and card $L_G = 2^k$. These equalities are implied by Lemmas 6.3, 6.4, Theorems 5.4, 4.5 and by the properties of $H(\alpha)$. The construction is complete. The system of cardinals $\{2^{\aleph_\alpha}: \alpha$ is an ordinal$\}$ is a proper class. It is easy to see that all lattice ordered groups G with $b(G) = n$ and card $L_G = 2^k$ constructed as above form a proper class as well.

7. INFINITE CASE

In the preceding section we have investigated the class of lattice ordered groups of finite breadth. Now we will consider the infinite case.

7.1. Lemma. *Let G be a lattice ordered group such that for each $n \in N$ there is an orthogonal subset of G with n elements. Then G has an infinite orthogonal subset.*

Proof. Assume, by way of contradiction, that there is an orthogonal subset of G with n elements whenever $n \in N$ and that all orthogonal subsets of G are finite. Then, by P. Conrad [2] (p. 3.26, Corollary IV, and p. 3.33, Remarks a) and d)), there is $k \in N$ such that G is of breadth k. Therefore there is no orthogonal subset of G with $k + 1$ elements, a contradiction.

7.2. Construction. Let D be an infinite orthogonal subset of G. By [8] (Lemma 23.9) there exists an almost (the intersections are finite) system $\mathcal{M}$ of subsets of D such that card $Y = 2^{\aleph_0}$ where $Y = \{M \in \mathcal{M}:$ card $M = \aleph_0\}$. Arrange each element of Y into a one-to-one sequence; then Y may be viewed as a set of sequences, $Y \subset \subset (G^+)^N$.

As in [6], we define for $X \subset (G^+)^N$:

$\delta X \ = \{S \in (G^+)^N$: there exists $R \in X$ such that $S = R \circ u$ for some $u\}$;

$\langle X \rangle = \{S \in (G^+)^N$: there exist $n \in N$ and elements $S_1, S_2, \ldots, S_n$ of X such that $S = S_1 + S_2 + \ldots + S_n\}$;

$[X] \ = \{S \in (G^+)^N$: there exists $T \in X$ such that const $(0) \leqq S \leqq T\}$;

$X^* \ = \{S \in (G^+)^N$: for each u there exists v such that $S \circ u \circ v \in X\}$.

Now, we can establish the following results concerning Y.

7.3. Theorem. *Let $A \subset Y$. Then $[\langle \delta A \rangle]^*$ is a convex subsemigroup of $(G^+)^N$ fulfilling the conditions* (I)–(III).

Proof. According to [6], it suffices to prove that if const(s) belongs to $[\langle\delta A\rangle]$, then $s = 0$. Let const(s) be an element of $[\langle\delta A\rangle]$, i.e., there is $T \in \langle\delta A\rangle$ such that $0 \leqq s \leqq T$. Thus there are elements $S_1, S_2, \ldots, S_n$ of A and monotone mappings $u_1, u_2, \ldots, u_n$ of N into N such that $T = S_1 \circ u_1 + S_2 \circ u_2 + \ldots + S_n \circ u_n$. Since every sequence belonging to Y is a one-to-one sequence, there is $k \in N$ such that $S_j \circ u_j(k) \neq S_i \circ u_i(1)$ whenever $i, j \in \{1, 2, \ldots, n\}$. Then for each $i, j \in \{1, 2, \ldots, n\}$ we have $S_j \circ u_j(k) \wedge S_i \circ u_i(1) = 0$. Consider $T(1) = S_1 \circ u_1(1) + \ldots + S_n \circ u_n(1)$ and $T(k) = S_1 \circ u_1(k) + \ldots + S_n \circ u_n(k)$. It is easy to express these elements in the form $T(1) = p_1 x_1 + \ldots + p_r x_r$, $T(k) = m_1 y_1 + \ldots + m_q y_q$ where $\{r, q, p_1, \ldots \ldots, p_r, m_1, \ldots, m_q\} \subset N$, $r \leqq n$, $q \leqq n$ and $x_1, \ldots, x_r, y_1, \ldots, y_q$ are pairwise distinct elements of the orthogonal subset D of G. From $s \leqq T(1)$ and $s \leqq T(k)$ we have $s \leqq T(1) \wedge T(k) = 0$; hence $s = 0$.

7.4. Lemma. *Let B be a subset of Y. If $S \in Y$ and $S \notin B$, then $S \notin [\langle\delta B\rangle]^*$.*

Proof. By way of contradiction, let $S \in [\langle\delta B\rangle]^*$. Take u, an arbitrary monotone mapping of N into N, $u \neq \mathrm{id}(N)$. Then there is w such that $S \circ u \circ w \in [\langle\delta B\rangle]$. Obviously, $S \circ u \circ w \neq S$. Since $\mathcal{M}$ is an almost disjoint system of subsets of D (see 7.2) and $S \in Y$, we have $S \circ u \circ w \notin Y$. Thus $S \circ u \circ w \notin B$.

As $S \circ u \circ w \in [\langle\delta B\rangle]$, there are elements $S_1, S_2, \ldots, S_n$ of B and monotone mappings $u_1, u_2, \ldots, u_n$ of N into N such that $S \circ u \circ w \leqq \sum_{i=1}^{n} S_i \circ u_i$. Since $S, S_1, \ldots, S_n$ are elements of Y, S has at most finitely many common members with S_i for each $i \in \{1, 2, \ldots, n\}$. Therefore there exists $k \in N$ such that $S(k) \wedge S_i \circ u_i(k) = 0$ for each $i \in \{1, 2, \ldots, n\}$. It is easy to express the sum $\sum_{i=1}^{n} S_i \circ u_i(k)$ in the form $\sum_{j=1}^{q} m_j y_j$, where $\{q, m_1, \ldots, m_q\} \subset N$, $q \leqq n$ and $y_1, \ldots, y_q$ are pairwise distinct elements of the set $\{S_i \circ u_i(k) : i \in N \text{ and } i \leqq n\}$. Then $S \circ u \circ w(k) = S \circ u \circ w(k) \wedge \sum_{i=1}^{n} S_i \circ u_i(k) =$
$= S \circ u \circ w(k) \wedge \sum_{j=1}^{q} m_j y_j = S \circ u \circ w(k) \wedge \bigvee_{j=1}^{q} m_j y_j = \bigvee_{j=1}^{q} (S \circ u \circ w(k) \wedge m_j y_j) = 0$, which contradicts the fact that $S \circ u \circ w(k)$ is a strictly positive element of G (an element of D).

7.5. Corollary. *If $B \subset Y$, then $B = Y \cap [\langle\delta B\rangle]^*$.*

Proof. If $S \in B$, then $S \in Y$ and by virtue of the obvious inclusion $B \subset [\langle\delta B\rangle]^*$ we obtain $B \subset Y \cap [\langle\delta B\rangle]^*$. Conversely, if $S \in Y \cap [\langle\delta B\rangle]^*$, then, by 7.4, we have $S \in B$.

7.6. Corollary. *Let A and B be subsets of Y such that $A \neq B$. Then $[\langle\delta A\rangle]^* \neq$ $\neq [\langle\delta B\rangle]^*$.*

7.7. Theorem. *Let G be an abelian lattice ordered group such that for each $n \in N$, G is not of breadth n and $G \neq \{0\}$. Then* card $L_G \geqq 2^{2^{\aleph_0}}$. *Moreover, there is an abelian lattice ordered group with precisely $2^{2^{\aleph_0}}$ convergences.*

Proof. There is an infinite orthogonal subset D in G. Construct Y in the same way as in 7.2 and define a mapping φ of the system of all subsets of Y into L_G in the following way: for $A \subset Y$, let $\varphi(A)$ be the convergence $\mathscr{L} \in L_G$ for which $P(\mathscr{L}) = [\langle \delta A \rangle]^*$ (cf. 7.3 and 1.6). According to Corollary 7.6, φ is injective. Thus card $L_G \geqq$ card $2^Y = 2^{2^{\aleph_0}}$. To complete the proof, we present an example of a known abelian lattice ordered group with precisely $2^{2^{\aleph_0}}$ convergences. Let H be the multiplicative group of strictly positive rational numbers with $a \leqq b$ whenever $na = b$ for some $n \in N$ (i.e. $H^+ = N$). Then card $(H^+)^N = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ and thus there are $2^{2^{\aleph_0}}$ subsets of $(H^+)^N$. Hence card $L_H \leqq 2^{2^{\aleph_0}}$ (cf. 1.6). On the other hand, the set of prime numbers is an infinite orthogonal subset of H. Therefore H is of breadth n for no $n \in N$ and because of the first part of the theorem we obtain card $L_H = 2^{2^{\aleph_0}}$.

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