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ON A PROBLEM OF P. VESTERGAARD CONCERNING CIRCUITS IN GRAPHS

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At the colloquium "Graphs and Orders" held in Banff (Canada) in 1984 [1], P. Vestergaard proposed a problem which came from his work with C. Hoede (Problem 5.18):

Let G be the union of two circuits C_1 and C_2 with nonempty intersection ($C_1 \cap C_2 \neq \emptyset$) plus an edge F joining a vertex of C_1 to a vertex of C_2. Assume further that $G = C_1 \cup C_2 \cup F$ has the maximal valency 3.

Are there distinct circuits $\{D_1, D_2, \ldots, D_q\}$ in G, $q \geqq 2$, with the property that
(1) *F belongs to exactly two distinct circuits D_i, D_j,*
(2) *each edge in $E(C_1) \cap E(C_2)$ appears in exactly two circuits from the list, and*
(3) *each edge in the symmetric difference of $E(C_1)$ and $E(C_2)$ appears in exactly one circuit from the list?*

We shall solve this problem.

Theorem. *The answer to the above mentioned problem is negative.*

Proof. A counterexample is the Petersen graph. In Fig. 1 it is drawn in such a way that the edges of C_1 are drawn as full lines and the edges of C_2 as dashed lines. Edges from $E(C_1) \cap E(C_2)$ are drawn in both the ways. The edge F is drawn as a dotted line. Now suppose that the answer to the question is affirmative, i.e. that there exists a system of circuits $\mathscr{D} = \{D_1, D_2, \ldots, D_q\}$ with the required properties. Denote $E_1 = (E(C_1) - E(C_2)) \cup (E(C_2) - E(C_1))$, $E_2 = (E(C_1) \cap E(C_2)) \cup \{F\}$. In any of the circuits from $\mathscr{D}$ there can be no pair of neighbouring edges belonging to E_1, because in the opposite case the edge of E_2 incident to their common end vertex could not belong to any circuit of $\mathscr{D}$. No two edges of E_2 are adjacent in the graph G and hence they cannot be adjacent in any circuit from $\mathscr{D}$. Therefore, if two edges are neighbouring in a circuit from $\mathscr{D}$, then one of them is in E_1 and the other in E_2. Let D_i be the circuit from $\mathscr{D}$ which contains the edge u_1u_2 (see Fig. 1). Then it must contain the edges u_1v_1, u_2v_2. Further, it contains either v_1v_3, or v_1v_4. In the first case it contains the edges v_3u_3, u_3u_4, u_4v_4, v_4v_2, in the second case the edges v_2v_5, v_5u_5, u_5u_4, u_4v_4. If we delete the edges of $E_1 \cap E(D_i)$ (i.e. the edges which belong only to D_i and not to any other circuit from $\mathscr{D}$) from G, in both the cases we obtain a graph consisting of two circuits of the length 5 and a bridge joining them. The bridge cannot be contained in any circuit from $\mathscr{D}$, which is a contradiction with our assumption.

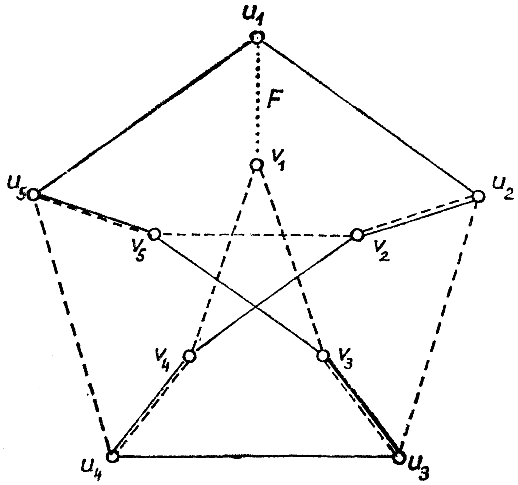

Fig. 1

Reference

[1] Problem Sessions. In: Graphs and Orders. Proc. Coll. Banff 1984.

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