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BUTLER GROUPS OF INFINITE RANK AND AXIOM 3

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1. INTRODUCTION

The structure of completely decomposable torsion-free groups was determined by R. Baer [3] in 1937. However, their pure subgroups can be so complex that no complete classification of them has been approached. In the case of finite rank, though, significant progress has been made recently. A pure subgroup of a completely decomposable torsion-free group G of finite rank is now generally referred to as a Butler group. These groups have been studied in detail by Butler, Lady, Arnold, Bican, and others; see, for example, [1], [4], [6], and [13]. Moreover, F. Richman has determined the structure of a special class of Butler groups in [14].

There are several known characterizations of Butler groups among torsion-free groups of finite rank, and these different characterizations naturally lead to various classes of torsion-free groups of infinite rank. One of these classes is the so-called class of B_2-groups. A torsion-free group G is called a *B_2-group* if G is the union of a smooth chain

$$0 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots$$

of pure subgroups G_α such that, for each α, $G_{\alpha+1} = G_\alpha + B_\alpha$ where B_α is a Butler group of finite rank. Another one of these classes consists of those torsion-free groups G such that $\operatorname{Bext}(G, T) = 0$ for all torsion T. These two classes actually coincide for countable groups [5], and it is apparently not known whether they differ in general or not although the first class of groups is contained in the second. Either class, depending on one's preference, might be taken to be (by definition) the Butler groups of arbitrary rank. To be definite, we shall call a B_2-group a *Butler group* (of infinite rank). In this paper we show that there is another approach to Butler groups of infinite rank. They have a natural description in terms of the third axiom of countability. We use this approach to study Butler groups, as well as pure subgroups of completely decomposable groups, of arbitrary rank. We are thereby able to establish various criteria for a group to be a Butler group of infinite rank. As we have indicated, these criteria on G automatically establish sufficient conditions for $\operatorname{Bext}(G, T) = 0$ whenever T is torsion. A side result of some interest is that the continuum hypothesis implies that $\mathrm{Bext}^3(G, T) = 0$ whenever G is a torsion-free group and T is torsion.

Recall that if $(*)$ is a subgroup property then G is said to *satisfy the third axiom of countability with respect to* $(*)$ provided there exists a collection $\mathscr{C}$ of $(*)$-subgroups of G such that the following conditions are satisfied.

(i) $0 \in \mathscr{C}$.

(ii) If $N_i \in \mathscr{C}$ for each $i \in I$, then $\langle N_i \rangle_{i \in I} \in \mathscr{C}$,

(iii) If K is a countable subgroup of G, there exists $N \in \mathscr{C}$ with $N \supseteq K$ and N countable.

It is to be understood throughout the paper that by a group we mean an abelian group. Unless otherwise explained, our terminology and notation should be in general agreement with [7]. If $x \in G$, the (global) height of x in G is denoted by $|x|$ or, if necessary for clarity, by $|x|_G$. The height of x in G at the prime p is denoted by $|x|^p$. Thus $|x|$ simply denotes the sequence $|x|^p$, where p ranges over the primes.

By a *smooth* ascending chain of subgroups of a group, we mean a collection of subgroups H_α indexed by an initial segment of ordinals with the property that $H_\alpha \subseteq H_\beta$ when $\alpha < \beta$ and $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$ whenever β is a limit ordinal.

2. SEPARABILITY AND BALANCED COVERS

Recall that a pure subgroup H of the torsion-free group G is said to be a *balanced subgroup* of G if, for each $g \in G$, there exists $h_0 \in H$ such that $|g + H|_G = |g + h_0|$, where $|g + H|_G$ stands for $\sup_{h \in H} \{|g + h| : h \in H\}$. It is easily verified that if H is balanced in G then the height of $g + H$ in G/H, which we designate by $|g + H|_{G/H}$, is in fact equal to $|g + H|_G$. It is well known that balanced subgroups now play an important role in the general theory of Abelian groups. A related concept that is weaker and therefore even more applicable in certain situations is the notion of what one of the authors has called a separable subgroup [10]; this is not to be confused with R. Baer's notion of separability (which we call Baer separability). Heretofore, our notion of separability has been employed primarily for torsion groups [10], [8], [12]. However, as we shall soon see, it is also relevant to the theory of torsion-free groups and in particular to pure subgroups of completely decomposable groups.

Definition 2.1. A subgroup H of G is *separable* if for each $g \in G$ there is a countable subset $\{h_n\}_{n<\omega}$ of H satisfying the following condition: for $h \in H$, there is a corresponding $n < \omega$ such that $|g + h| \leqq |g + h_n|$.

Trivially, all countable subgroups and all balanced subgroups are separable. We shall demonstrate that groups satisfying certain structural properties are always separable whenever they appear as pure subgroups. In this connection, we introduce the concept of a balanced cover for the countable subgroups of a given group G.

Definition 2.2. A collection $\mathscr{C}$ of countable, balanced subgroups of G is called a *balanced cover of the countable subgroups of* G provided $\mathscr{C}$ satisfies the following conditions.

(i) $\mathscr{C}$ is closed with respect to countable ascending chains.

(ii) If H is a countable subgroup of G there is a $K \in \mathscr{C}$ containing H.

Observe that if $|G| = \aleph_1$ then G has a balanced cover for its countable subgroups precisely when G satisfies the third axiom of countability with respect to balanced subgroups. However, for larger groups the property of having a balanced cover is weaker than the third axiom for balanced subgroups.

It is convenient and natural to call a torsion-free group H *absolutely separable* if it is separable in any torsion-free group G in which it appears as a pure subgroup.

Theorem 2.3. *Let H be a torsion-free group. If H has a balanced cover for its countable subgroups, then H is absolutely separable.*

Proof. Suppose that H is contained as a pure subgroup in a torsion-free group G, but H is not a separable subgroup. Let $\mathscr{C}$ be a balanced cover for the countable subgroup of H. Without loss of generality, we may assume that $0 \in \mathscr{C}$. Since H is not separable in G, there exists an element $g \in G$ such that for each countable subset K of H there is a corresponding element $h^* \in H$ with the property that $|g + x| \geqq |g + h^*|$ fails for every $x \in K$.

Assume that we have already constructed a finite ascending chain $0 = H_0 \subseteq \subseteq H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n$ in $\mathscr{C}$ that satisfies the following condition:

(c) if $h_i \in H_i$ and the inequality $|g + h_i|^p \nleqq |g + h|^p$ holds for some $h \in H$ and a prime p, there exists $h_{i+1} \in H_{i+1}$ (if $i < n$) such that $|g + h_i|^p \nleqq |g + h_{i+1}|^p$.

It is easy to see that the chain can be extended another link to $H_{n+1} \supseteq H_n$ because H_n is countable, and therefore there are only a countable number of elements $g + h_n$ to consider and a countable number of primes p. For each $h_n \in H_n$ and each prime p for which $|g + h_n|^p \nleqq |g + h|^p$ holds for some $h \in H$, choose exactly one such h. Finally, choose $H_{n+1} \in \mathscr{C}$ so that H_{n+1} contains H_n and all the designated h's. Clearly, H_{n+1} provides the desired extension.

We have shown that there exists an infinite ascending chain $\{H_n\}$ in $\mathscr{C}$ that satisfies condition (c). Let $H_\omega = \bigcup_{n<\omega} H_n$ and observe that $H_\omega \in \mathscr{C}$. Since H_ω is countable, there exists $h^* \in H$ such that $|g + x| \geqq |g + h^*|$ fails for every $x \in H_\omega$. The fact that H_ω is balanced in H (by virtue of its membership in $\mathscr{C}$) enables us to choose $h' \in H_\omega$ so that $|h^* - h'| \geqq |h^* + x|$ for each $x \in H_\omega$. Having chosen $h' + H_\omega$ in this way, we utilize the fact that $|g + h'| \geqq |g + h^*|$ must fail (as we have already observed). Therefore, $|g + h'|^p \nleqq |g + h^*|^p$ for some prime p. Since h' must in fact belong to H_n for some $n < \omega$, we conclude that $|g + h'|^p \nleqq |g + h''|^p$ for some $h'' \in H_{n+1}$. This, however, quickly leads to the following contradiction

$$|g + h'|_G^p = |h' - h''|_G^p = |h' - h''|_H^p \nleqq |h^* - h''|_H^p \leqq |h^* - h'|_H^p = \\ = |h^* - h'|_G^p = |g + h'|_G^p .$$

Consequently, we conclude that H must be separable in G if it is pure and has a balanced cover for its countable subgroups.

Corollary 2.4. *Let G be a torsion-free group that has a balanced cover for its countable subgroups and let H be a pure subgroup of G. If H is separable in G, then H is in fact absolutely separable.*

Proof. Suppose that H is a pure subgroup of an arbitrary torsion-free group G'. Consider the push-out diagram

$$\begin{array}{ccccc} H & \longrightarrow & G' & \twoheadrightarrow & G'/H \\ \downarrow & & \downarrow & & \downarrow \\ G & \rightarrowtail & P & \twoheadrightarrow & G'/H \, . \\ \downarrow & & \downarrow & & \downarrow \\ G/H & \rightarrowtail & G/H & \longrightarrow & 0 \end{array}$$

Since H is pure in G', G'/H is torsion free. Therefore, G must be pure in P. According to Theorem 2.3, G is separable in P. Since separability is transitive for pure subgroups, H is separable in P. Therefore, since G' is pure in P, it quickly follows that H is separable in G'.

It should be observed that the preceding corollary implies that a pure and separable subgroup of a completely decomposable group is absolutely separable. This motivates a study in the next section of pure and separable subgroups of completely decomposable groups. We actually study these subgroups in a slightly more general setting.

3. SEPARABLE SUBGROUPS OF COMPLETELY DECOMPOSABLE GROUPS

Let $G = \bigoplus_{i \in I} G_i$ be a direct sum of countable, torsion-free groups G_i with i ranging over an arbitrary index set I. If J is a subset of I, we let $G(J)$ denote the subgroup $\bigoplus_{j \in J} G_j$. Likewise, if H is a subgroup of G the short notation for $H \cap G(J)$ will be simply $H(J)$. Now, in preparation for the following lemma, we let H be a fixed subgroup of G that is both pure and separable. We call a subset J of I *special* if it satisfies the following conditions.

(i) $H \parallel G(J)$, that is, for each $h \in H$ and $g \in G(J)$, there exists c in $H(J)$ such that $|h + c| \geqq |h + g|$.

(ii) $H + G(J)$ is pure in G.

Lemma 3.1. *Let H be a pure and separable subgroup of $G = \bigoplus_{i \in I} G_i$, where G_i is a countable torsion-free group for each $i \in I$. If A is a subgroup of G and I_0 is a subset of I such that $|A| \leqq |I_0| \aleph_0$, there exists a special subset J of I containing I_0 that satisfies the following conditions*:

(a) $|J| \leqq |I_0| \aleph_0$,
(b) $A + H \subseteq G(J) + H$.

Proof. There is no loss of generality in assuming that I_0 is infinite and that $A \subseteq G(I_0)$ because $|A| \leqq |I_0| \aleph_0$. Thus condition (b) is satisfied for any subset J of I containing I_0. It remains only to verify that there is a special subset $J \supseteq I_0$ that satisfies (a).

Set $J_0 = I_0$ and $B_0 = A$. Construct ascending sequences of subsets J_n of I and subgroups B_n of G inductively as follows. If J_n and B_n have been constructed of cardinality not exceeding $|I_0| \aleph_0$, choose for each $x \in G(J_n)$ a countable number of elements $h_{x,m}$, $m < \omega$, in H with the property that for each $h \in H$ there is an $m < \omega$ for which $|x + h_{x,m}| \geqq |x + h|$. Clearly, the subgroup

$$A_{n+1} = G(J_n) + \langle h_{x,m} \colon x \in G(J_n),\ m < \omega \rangle$$

has cardinality not exceeding $|I_0| \aleph_0$ since $|J_n| \leqq |I_0| \aleph_0$. Moreover, there is a subgroup $B_{n+1} \supseteq B_n$ whose cardinality still does not exceed $|I_0| \aleph_0$ such that $B_{n+1} \supseteq A_{n+1}$ and $H + B_{n+1}$ is pure in G; this is an immediate consequence of H being pure in G, which means that we can purify $H + A_{n+1}$ by purifying $(A_{n+1} + H)/H$ in G/H. Let $J(n + 1)$ be the smallest subset of I that contains $J(n)$ with the property that $G(J_{n+1}) \supseteq B_{n+1}$. Observe that $G(J_n) \subseteq B_{n+1} \subseteq G(J_{n+1})$. Hence, if $J = \bigcup_{n<\omega} J(n)$ and $B = \bigcup_{n<\omega} B_n$, we conclude that $G(J) = B$. It is easily verified that $H \parallel G(J)$, for if $x \in G(J)$ and $h \in H$ then $x \in G(J_n)$ for some n. Therefore, $|x + h| \leqq |x + h_{x,m}|$ where

$$h_{x,m} \in H \cap B_{m+1} \subseteq H \cap B = H \cap G(J)\,.$$

Since we have constructed the ascending sequence of subgroups B_n so that $H + B_n$ is pure in G, it follows at once that $H + G(J) = H + B$ is pure. Therefore, J is a special subset of I satisfying condition (a), and the lemma is proved.

The significance of special subsets is revealed by the following lemma. As before, let H be a pure and separable subgroup of the direct sum G of countable torsion-free groups G_i over an arbitrary index set I. It is understood that all special subsets J of I are relative to H in G.

Lemma 3.2. *If J is a special subset of I then $H(J) = H \cap G(J)$ is a balanced subgroup of H.*

Proof. Let h be an arbitrary element of H. Since $H + G(J)$ is pure in G and $G(J)$ is balanced, we conclude that, for some $g \in G(J)$,

$$|h + H(J)|_{H/H(J)} = |h + G(J)|_{(H+G(J))/G(J)} = |h + G(J)|_{G/G(J)} = |h + g|_G\,.$$

However, $H \parallel G(J)$, which means that for some $c \in H \cap G(J) = H(J)$ the inequality $|h + c|_G \geqq |h + g|_G$ holds. Therefore,

$$|h + c|_H = |h + c|_G \geqq |h + g|_G = |h + H(J)|_{H/H(J)}\,,$$

and $H(J)$ is balanced in H.

We are now prepared to prove one of our main results.

Theorem 3.3. *Let $G = \bigoplus_{i \in I} G_i$ be a direct sum of countable torsion-free groups G_i over an arbitrary index set I. If H is any pure and separable subgroup of G, then H has a balanced cover for its countable subgroups.*

Proof. Consider the following collection of subgroups of H.

$$\mathscr{C} = \{H(J): |J| \leqq \aleph_0 \text{ and } J \text{ is special in } I\} .$$

Lemma 3.2 implies that the members of $\mathscr{C}$ are balanced in H. Since conditions (i) and (ii), in the definition of a special subset J of I, are both inductive we conclude that $\mathscr{C}$ is closed with respect to countable ascending chains. An immediate consequence of Lemma 3.1 is that any countable subgroup of H is contained in a member of $\mathscr{C}$. Therefore, $\mathscr{C}$ is a balanced cover for the countable subgroups of H, and the theorem is proved.

Although pure subgroups of completely decomposable groups can be very complicated, the preceding theorem does give some structure to the separable ones as the following corollaries demonstrate.

Corollary 3.4. *If H is a pure and spearable subgroup of a completely decomposable torsion-free group G, then H has a balanced cover for its countable subgroups.*

Corollary 3.5. *If H is a pure and spearable subgroup of a completely decomposable torsion-free group G and H has cardinality not exceeding $\aleph_1$, then H satisfies the third axiom of countability with respect to balanced subgroups.*

Following Bican and Salce [5], we call a torsion-free group H a B_2-group if it is the union of a smooth chain $\{H_\alpha\}_{\alpha<\sigma}$ of pure subgroups H_α beginning with $H_0 = 0$ such that $H_{\alpha+1} = H_\alpha + B_\alpha$ for each α, where B_α is a finite rank Butler group. It is convenient to have available the following criterion for B_2-groups.

Lemma 3.6. *If the torsion-free group G is the union of a smooth chain*

$$0 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots$$

of balanced subgroups G_α, then G is a B_2-group provided that $G_{\alpha+1}/G_\alpha$ is a B_2-group for each α.

Proof. The key to the proof is the observation that if H is a balanced subgroup of G and the subgroup $\bar{B} = A/H$ of G/H is a Butler group of finite rank, there is a subgroup B of G that is a Butler group of finite rank such that $(B + H)/H = \bar{B}$. Indeed, if $\bar{B} = \langle\langle \bar{b}_i\rangle_*: 1 \leqq i \leqq N\rangle$ then $\langle\langle b_i\rangle_*: 1 \leqq i \leqq N\rangle$ is the desired subgroup B provided we choose b_i so that $b_i + H = \bar{b}_i$ and $|b_i| = |\bar{b}_i|$ for each i. The point here is that $|nb_i| = |n\bar{b}_i|$ for any positive integer n, so $\langle b_i\rangle_*$ maps naturally onto $\langle \bar{b}_i\rangle_*$.

To finish the proof of the lemma, we observe that since the quotient $G_{\alpha+1}/G_\alpha$ is a B_2-group it is the union of an ascending chain of pure subgroups $\bar{H}_\lambda$, $\lambda < \mu(\alpha)$, with $\bar{H}_0 = 0$ and $\bar{H}_{\lambda+1} = \bar{H}_\lambda + \bar{B}_\lambda$ where $\bar{B}_\lambda$ is a finite rank Butler group. Let $G_{\alpha,\lambda}$ be the subgroup of $G_{\alpha+1}$ containing G_α defined by $G_{\alpha,\lambda}/G_\alpha = \bar{H}_\lambda$. According to the preceding observation, we can choose a subgroup $B_{\alpha,\lambda}$ of $G_{\alpha+1}$ so that $(G_\alpha + B_{\alpha,\lambda})/G_\alpha = \bar{B}_\lambda$ and so that $B_{\alpha,\lambda}$ is a Butler group of finite rank. We conclude that $G_{\alpha,\lambda+1} = G_{\alpha,\lambda} + B_{\alpha,\lambda}$, and that the refined chain $\{G_{\alpha,\lambda}\}$ demonstrates that G is a B_2-group.

The next theorem shows that if G is a direct sum of countable B_2-groups then the following conditions are closely related for a pure subgroup H of G.

(1) H is separable in G.

(2) H has a balanced cover for its countable subgroups.

(3) H is a B_2-group.

Theorem 3.7. *Let H be a pure subgroup of a direct sum of countable B_2-groups.*

(a) *Conditions (1) and (2) are equivalent.*

(b) *If H has cardinality not exceeding $\aleph_1$, either condition (1) or (2) implies (3).*

Proof. Part (a) is a direct consequence of Theorem 2.3 and Theorem 3.3. In order to prove part (b), let $G = \oplus_{i\in I} G_i$ be a direct sum of countable B_2-groups and let H be a pure subgroup of G of cardinality not exceeding $\aleph_1$. Without loss of generality we may assume that the index set I has cardinality $\aleph_1$ and that H has a balanced cover for its countable subgroups. Thus H satisfies the third axiom of countability with respect to balanced subgroups. Indeed, according to Lemma 3.1, there exists a smooth ascending chain

$$\emptyset = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_\alpha \subseteq \ldots$$

of special, countable subsets J_α of I whose union is I. Not only is $H_\alpha = H \cap G(J_\alpha)$ balanced in H by Lemma 3.2, but $H_{\alpha+1}/H_\alpha$ is isomorphic to a pure subgroup of $G(J_{\alpha+1})/G(J_\alpha)$. Moreover, $G(J_{\alpha+1})/G(J_\alpha)$ is a countable B_2-group by virtue of being a countable direct sum of B_2-groups. In [5] it is shown that a pure subgroups of a countable B_2-group is itself a B_2-group. Therefore $H_{\alpha+1}/H_\alpha$ is a B_2-group. Since the smooth ascending chain $0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots$ leads up to H, Lemma 3.6 implies that H is a B_2-group.

The second part of the preceding theorem can be generalized as follows.

Corollary 3.8. *Let G be a direct sum of countable B_2-groups and let H be a pure and separable subgroup of G. If H is a direct sum of groups of cardinality not exceeding $\aleph_1$, then H is a B_2-group.*

Proof. Let $H = \oplus_{i\in I} H_i$ where $|H_i| \leqq \aleph_1$ for each i. Since H is separable in G and since the property of being pure and separable is transitive, we conclude that H_i is pure and separable in G. Hence H_i is B_2-group according to Theorem 3.7, and therefore $H = \oplus_{i\in I} H_i$ is a B_2-group.

Specializing even further, we obtain the following result.

Corollary 3.9. *Let G be a completely decomposable torsion-free group and let H*

be a pure subgroup of G having cardinality not exceeding $\aleph_1$. *If H is separable then H is a* B_2*-group. In particular, a balanced subgroup of a completely decomposable torsion-free group of cardinality not exceeding* $\aleph_1$ *must be a* B_2*-group.*

4. THE THIRD AXIOM OF COUNTABILITY OVER A SUBGROUP

We begin here with a precise formulation of the third axiom of countability over a subgroup.

Definition 4.1. Let H be a subgroup of a group G. We say that G *satisfies the third axiom of countability over H with respect to a condition* $(*)$ *on subgroups* of G if there is a collection $\mathscr{C}$ of subgroups of G satisfying $(*)$ and containing H for which the following conditions hold.

(i) $H \in \mathscr{C}$.

(ii) If $N_i \in \mathscr{C}$ for each $i \in I$, then $\langle N_i \rangle_{i \in I} \in \mathscr{C}$.

(iii) If $H \subseteq K \subseteq G$ and K/H is countable, there exists $N \in \mathscr{C}$ with $N \supseteq K$ such that N/H is countable.

We note that G satisfies the third axiom of countability with respect to $(*)$ if G satisfies the third axiom of countability with respect to $(*)$ over the trivial subgroup 0.

The main result of this section is the following.

Theorem 4.2. *Let H be a pure subgroup of a direct sum G of countable* B_2*-groups. If G satisfies the third axiom of countability over H with respect to separable subgroups, then H is a* B_2*-group. Moreover, in this case, H is the union of a smooth ascending chain*

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots$$

of balanced subgroups of H such that $H_{\alpha+1}/H_\alpha$ *is a countable* B_2*-group for each* α.

Proof. Let $\mathscr{C}$ be a family of separable subgroups of G containing H that satisfies conditions (i)–(iii). It follows from [11] that G satisfies the third axiom of countability over H with respect to pure subgroups. If $\mathscr{D}$ is a family of pure subgroups of G containing H that satisfies conditions (i)–(iii), then $\mathscr{C} \cap \mathscr{D}$ is a family of pure and separable subgroups of G containing H that also satisfies (i)–(iii). Therefore, without loss of generality, we may assume that the members of $\mathscr{C}$ are all pure in G. Let $G = \bigoplus_{i \in I} G_i$ where G_i is a countable B_2-group for each i.

The hypothesis of the theorem implies that H is separable in G since $H \in \mathscr{C}$. The proof that H is a B_2-group is similar to the proof that an isotype subgroup A of a totally projective group C is itself totally projective provided C satisfies the third axiom of countability over A with respect to separable subgroups (see Theorem 4 in [10]). However, the situation here is sufficiently different to warrant some details.

Thus, let $\mathscr{S}$ be the family of summands of G defined by

$$\mathscr{S} = \{\oplus_{j\in J} G_j \colon J \text{ is arbitrary subset of } I\} .$$

Suppose that B is a subgroup of G that satisfies the conditions:

(1) $B \in \mathscr{S}$.

(2) $H + B \in \mathscr{C}$.

(3) $H \parallel B$.

Recall that we write $H \parallel B$ if, for any $h \in H$ and $b \in B$, there exists $c \in H \cap B$ such that $|h + b| \leqq |h + c|$. It should be observed that $B = 0$ satisfies conditions (1)–(3). Moreover, if B is any subgroup of G that satisfies (1)–(3) and K is an arbitrary countable subgroup of G, there exists a countable extension C of B containing K that still satisfies conditions (1)–(3). Due to the character of the third axiom of countability, it is essentially trivial to find such a subgroup C that satisfies (1) and (2) since both $\mathscr{C}$ and $\mathscr{S}$ are axiom 3 collections. Since the relation $H \parallel B$ is inductive (in the variable B) and $H + B$ is separable in G, we can obtain C satisfying condition (3) along with (1) and (2). At this point, it follows that G is the union of a smooth ascending chain

$$0 = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_\alpha \subseteq \ldots,$$

where for each α the subgroup B_α satisfies conditions (1)–(3) and where $B_{\alpha+1}/B_\alpha$ is countable. Set $H_\alpha = H \cap B_\alpha$ and consider the smooth ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots .$$

Clearly, H is the union of this chain. Let $B_\alpha = \oplus_{j\in J_\alpha} G_j$; since $B_\alpha \in \mathscr{S}$, this is possible. Observe that J_α is special in I (relative to the pure and separable subgroup H) because $H \parallel G(J_\alpha)$ in view of condition (3) and $H + G(J_\alpha)$ is pure in G in view of condition (2) since the members of $\mathscr{C}$ are pure in G. By Lemma 3.2, H_α is balanced in H. Finally,

$$H_{\alpha+1}/H_\alpha \simeq \langle B_{\alpha+1} \cap H, B_\alpha\rangle/B_\alpha = ((H + B_\alpha) \cap B_{\alpha+1})/B_\alpha ,$$

a pure subgroup of the countable B_2-group $B_{\alpha+1}/B_\alpha$. Therefore, $H_{\alpha+1}/H_\alpha$ is a coutable B_2-group. By Lemma 3.6, H is a B_2-group, and the theorem is proved.

The main interest in Theorem 4.2 may be the following corollary; compare with Corollary 3.9.

Corollary 4.3. *Let G be a completely decomposable torsion-free group and let H be a pure subgroup of G. If G satisfies the third axiom of countability over H with respect to separable subgroups, then H is a B_2-group.*

Recall that a torsion-free group G is said to be *almost completely decomposable* if G is quasi-isomorphic to a completely decomposable group. Hence G is almost completely decomposable if $nG \subseteq \oplus_{i\in I} A_i \subseteq G$, where A_i is a subgroup of Q. It is well known that if G has finite rank then G is a Butler group if it is almost completely decomposable. We close this section by observing that this remains true for B_2-groups even in the case of infinite rank.

Proposition 4.4. *Any almost completely decomposable torsion-free group is a B_2-group.*

Proof. Let $nG \subseteq \oplus_{\alpha<\mu} A_\alpha \subseteq G$, where $A_\alpha \subseteq Q$. Without loss of generality, we may assume that G has infinite rank and that μ is a limit ordinal. For each $\lambda < \mu$, let H_λ denote the pure closure in G of $\oplus_{\alpha<\lambda} A_\alpha$. Observe that G is the union of the smooth ascending chain of pure subgroups

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\lambda \subseteq \ldots, \quad \lambda < \mu .$$

Moreover,

$$n(H_{\lambda+1}/H_\lambda) \subseteq (H_\lambda + A_\lambda)/H_\lambda \subseteq H_{\lambda+1}/H_\lambda \subseteq Q .$$

Hence, $H_{\lambda+1}/(H_\lambda + A_\lambda)$ is finite, so $H_{\lambda+1} = H_\lambda + B_\lambda$ where B_λ is a Butler group of finite rank. This demonstrates that G is a B_2-group.

5. AN AXIOM 3 CHARACTERIZATION OF B_2-GROUPS

Call a subgroup H of a torsion-free group G *decent* if for any finite subset S of G there exists a finite number of rank 1 pure subgroups A_i of G such that $H + \sum_{i\leqq N} A_i$ is pure in G and contains S. It easily follows from the known characterizations of finite rank Butler groups (see, for example, [1]) that the following is yet another characterization.

Proposition 5.1. *A torsion-free group G of finite rank is a Butler group if and only if the zero subgroup is decent in G.*

Proof. Let S be a maximal independent set of elements of G. Since G has finite rank, S must be finite. If $H = 0$ is decent in G, we conclude that there are a finite number of rank 1 pure subgroup A_i of G such that $G = \sum_{i\leqq N} A_i$. Therefore, G is a Butler group. Conversely, if G is a Butler group (of finite rank) the A_i's exist for which $G = \sum_{i\leqq N} A_i$, and the proposition is established.

Motivated by Proposition 5.1 one could define a Butler group of infinite rank to be a torsion-free group that has "plenty" of decent subgroups. More precisely, we introduce the following definition. Keep in mind that other approaches to Butler groups of infinite rank include the B_1-groups, B_2-groups, and the B-groups of [5].

Definition 5.2. A torsion-free group G is called a B_3-*group* if G satisfies the third axiom of countability with respect to decent subgroups.

It is our purpose in this section to show that the class of B_3-groups that we have just introduced is actually not a new class but coincides with the class of B_2-groups. This gives further justification for using the third axiom of countability (with respect to decent subgroups) as the definition of a Butler group of infinite rank. We remark, however, that Arnold [2] has shown that these groups are far away from the original concept of a Butler group since a B_2-group need not even be a pure subgroup of a completely decomposable group. At any rate, these interesting classes of torsion-free groups have been discovered through attempts to generalize classical Butler groups.

In proving that B_3-groups are the same as B_2-groups, we start with the direction that is most direct.

Theorem 5.3. *Every B_3-group is a B_2-group.*

Proof. Let G be a B_3-group and let $\mathscr{C}$ be a collection of decent subgroups satisfying the third axiom of countability. Since G satisfies the third axiom of countability with respect to pure subgroups [11], there is no loss of generality in assuming that the members of $\mathscr{C}$ are pure in G, for we only need to take the intersection of $\mathscr{C}$ with a collection $\mathscr{D}$ of pure subbgroups of G that satisfies the third axiom. Thus we shall assume that the members of $\mathscr{C}$ are pure in G.

Suppose that $G_\alpha \in \mathscr{C}$ and that $A_{\alpha,0}$ is an arbitrary subgroup of G of finite rank. Since G_α is decent in G there exists a finite rank Butler group $B_{\alpha,1}$ in G such that $G_\alpha + B_{\alpha,1}$ is pure in G and contains $A_{\alpha,0}$. Likewise, there is a finite rank Butler group $B_{\alpha,2}$ in G such that $G_\alpha + B_{\alpha,2}$ is pure and contains $B_{\alpha,1} + A_{\alpha,1}$ where $A_{\alpha,1}$ is an arbitrarily chosen finite rank subgroup of G. Consider the chain

$$G_\alpha \subseteq G_\alpha + B_{\alpha,1} \subseteq G_\alpha + B_{\alpha,2} \subseteq \ldots \subseteq G_\alpha + B_{\alpha,n} \subseteq \ldots, \quad n < \omega,$$

of pure subgroups of G. If we can choose the $B_{\alpha,n}$'s so that $\bigcup_{n<\omega}(G_\alpha + B_{\alpha,n}) = G_\alpha + C$ where $C \in \mathscr{C}$, we can set $G_{\alpha+1} = G_\alpha + C$ and continue the process until we exhaust G in view of the fact that $G_{\alpha+1}$ will belong again to $\mathscr{C}$. However, we know that $B_{\alpha,n}$ is contained in some countable C_n that belongs to $\mathscr{C}$. Let $C_n = \{x_{n,1}, x_{n,2}, \ldots, x_{n,i}, \ldots\}$. Now, we can exhaust all the C_n's with a judicious choice of the A_n's. In fact, for $n \geqq 1$, we can let $A_n = \langle x_{i,j} \rangle_{1 \leqq i,j \leqq n}$ and obtain the desired result where $C = \sum_{n<\omega} C_n$ belongs to $\mathscr{C}$.

As we have indicated, the preceding leads to a smooth ascending chain of pure subgroups

$$0 \subseteq B_{0,1} \subseteq \ldots \subseteq B_{0,n} \subseteq \ldots \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \subseteq G_\alpha + B_{\alpha,n} \subseteq \ldots$$

that leads up to G, where $B_{\alpha,n}$ is a Butler group of finite rank. Thus G is a B_2-group, and the theorem is proved.

Now, we consider the other direction. In order to show that every B_2-group is a B_3-group, we let G be the union of a smooth ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots, \quad \alpha < \mu,$$

of pure subgroups H_α of G such that $H_{\alpha+1} = H_\alpha + B_\alpha$ for each α (whenever $\alpha + 1 < \mu$), where B_α is a Butler group of finite rank. Let $T = \{\alpha: \alpha < \mu\}$ and call a subset S of T *closed* if S satisfies the following conditions.

(a) If $\lambda \in S$, then $B_\lambda \cap H_\lambda \subseteq \langle B_\alpha: \alpha \in S, \alpha < \lambda \rangle$.

(b) If $\lambda \in S$, then $(B_\lambda)_* \subseteq \langle B_\alpha: \alpha \in S, \alpha \leqq \lambda \rangle$, where A_* denotes the pure closure of A in G.

Lemma 5.4. *If S is closed in T, then $G(S) = \langle B_\lambda: \lambda \in S \rangle$ is pure in G.*

Proof. Assume the lemma is false. Let $x \in G(S)$, and write $x = b_{\lambda(1)} + b_{\lambda(2)} + \dots + b_{\lambda(m)}$ where $\lambda(i) \in S$ with $\lambda(1) < \lambda(2) < \dots < \lambda(m)$ and $b_{\lambda(i)} \in B_{\lambda(i)}$ for each i. For all $x \in G(S)$ with $ny = x$ for $y \in G$ but not in $G(S)$, choose one that yields a minimal $\lambda(m)$. For n a positive integer and the chosen x, consider $ny = x$ where $y \in G \setminus G(S)$. Since $H_{\lambda(m)+1}$ is pure in G and $x \in H_{\lambda(m)+1}$, we conclude that $y \in H_{\lambda(m)+1}$. Hence, we can write $y = z + b'_{\lambda(m)}$, where $z \in H_{\lambda(m)}$ and $b'_{\lambda(m)} \in B_{\lambda(m)}$. Observe that

$$nz = ny - nb'_{\lambda(m)} = b_{\lambda(1)} + b_{\lambda(2)} + \dots + b_{\lambda(m-1)} + \left(b_{\lambda(m)} - nb'_{\lambda(m)}\right).$$

Hence,

$$b_{\lambda(m)} - nb'_{\lambda(m)} \in B_{\lambda(m)} \cap H_{\lambda(m)} \subseteq \langle B_\alpha\colon \alpha \in S,\ \alpha < \lambda(m)\rangle .$$

By the choice of x and $\lambda(m)$, it follows that $z \in G(S)$. Therefore $y = z + b'_{\lambda(m)}$ must belong to $G(S)$, and we have obtained a contradiction (on the assumption that the lemma is false).

Lemma 5.5. *If S and S' are both closed in T, then $S \cup S'$ is also closed in T.*

Proof. Let $\lambda \in S \cup S'$. Without loss of generality, $\lambda \in S$. Hence,

$$B_\lambda \cap H_\lambda \subseteq \langle B_\alpha\colon \alpha \in S,\ \alpha < \lambda\rangle \subseteq \langle B_\alpha\colon \alpha \in S \cup S',\ \alpha < \lambda\rangle ,$$

and condition (a) for closure is satisfied. Condition (b) is established in the same way, so $S \cup S'$ is closed in T.

Lemma 5.6. *If R is any countable subset of T, there is a countable, closed subset S of T that contains R.*

Proof. Let $S_0 = R$. Obviously, we can choose a countable subset $S_1 \supseteq S_0$ so that

(i) if $\lambda \in S_0$, then $B_\lambda \cap H_\lambda \subseteq \langle B_\alpha\colon \alpha \in S_1, \alpha < \lambda\rangle$

and

(ii) if $\lambda \in S_0$, then $(B_\lambda)_* \subseteq \langle B_\alpha\colon \alpha \in S_1, \alpha \leqq \lambda\rangle$.

We can replace S_i by S_{i+1} and obtain an ascending sequence of countable subsets S_i such that conditions (i) and (ii) hold when S_i replaces S_0 and S_{i+1} replaces S_1. Let $S = \bigcup_{n<\omega} S_n$ and observe that S satisfies conditions (a) and (b). Thus S is a countable subset of T that is closed and contains $S_0 = R$.

Lemma 5.7. *If S is closed in T, then $G(S) = \langle B_\lambda\colon \lambda \in S\rangle$ is a decent subgroup of G.*

Proof. Let S be closed in T and suppose that S' is a countable closed subset of T. List the elements of S' as $\lambda(1) < \lambda(2) < \dots < \lambda(n) < \dots$. For a positive integer N, let $S'_N = \{\lambda(i)\colon i \leqq N\}$. We claim that $G(S \cup S'_N)$ is pure in G. Suppose not and let $ny = x \in G(S \cup S'_N)$ where $y \notin G(S \cup S'_N)$. Write $x = b_{\lambda(1)} + b_{\lambda(2)} + \dots + b_{\lambda(m)}$, where $\lambda(1) < \lambda(2) < \dots < \lambda(m)$ with $b_{\lambda(i)} \in B_{\lambda(i)}$ and $\lambda_{(i)} \in S \cup S'_N$. Let $\lambda(m)$ be chosen as small as possible (to produce a counterexample of the lemma). Write $y = z + b'_{\lambda(m)}$ where $z \in H_{\lambda(m)}$ and $b'_{\lambda(m)} \in B_{\lambda(m)}$. As in the proof of Lemma 5.4, we observe that

$$nz = ny - nb'_{\lambda(m)} = b_{\lambda(1)} + b_{\lambda(2)} + \dots + b_{\lambda(m-1)} + \left(b_{\lambda(m)} - nb'_{\lambda(m)}\right).$$

Therefore,

$$b_{\lambda(m)} - nb'_{\lambda(m)} \in H_{\lambda(m)} \cap B_{\lambda(m)} \subseteq \langle B_\alpha : \alpha \in S \cup S', \alpha < \lambda(m) \rangle .$$

It is clear, however, that

$$\langle B_\alpha : \alpha \in S \cup S', \alpha < \lambda(m) \rangle \subseteq \langle B_\alpha : \alpha \in S \cup S'_{N-1} \rangle .$$

Since $G(S)$ is pure by Lemma 5.4, we can assume by induction that $G(S \cup S'_{N-1})$ is pure. Thus $z \in G(S \cup S'_{N-1})$ and $y = z + b'_{\lambda(m)} \in G(S \cup S'_N)$ after all, so the claim that $G(S \cup S'_N)$ is pure in G has been verified.

To complete the proof of the lemma, let $H = G(S)$ where S is an arbitrary closed subset of T. If $\{x_1, x_2, \ldots, x_k\}$ is a finite subset of G, there is a countable, closed subset S' of T such that $x_i \in G(S')$ if $i \leqq k$. Indeed, if we list the elements of S' as $\lambda(1) < \lambda(2) < \ldots < \lambda(n) < \ldots$, then for some positive integer N each x_i is contained in $G(S'_N)$. Since $G(S \cup S'_N)$ is pure and $G(S \cup S'_N) = G(S) + G(S'_N)$, we conclude that $G(S$ is decent in G because $G(S'_N)$ is a Butler group of finite rank; in fact, $G(S'_N) = \langle B_{\lambda(i)} : i \leqq N \rangle$.

Theorem 5.8. *A torsion-free group G is a B_2-group if and only if it is a B_3-group.*

Proof. The "if" part is Theorem 5.3. Thus we need only prove that a B_2-group is always a B_3-group. Let G be a B_2-group. In more detail, let G be the union of a smooth ascending chain

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_\alpha \subseteq \ldots, \quad \alpha < \mu ,$$

of pure subgroups H_α of G such that $H_{\alpha+1} = H_\alpha + B_\alpha$ for each α, where B_α is a Butler group of finite rank. As before, let $T = \{\alpha : \alpha < \mu\}$ be the index set for the preceding chain. Let

$$\mathscr{C} = \{G(S) : S \text{ is a closed subset of } T\} .$$

Observe that $S = \emptyset$ is closed, so $0 \in \mathscr{C}$. Lemma 5.5 states that the union of two closed subsets is again closed, but the argument trivially generalizes to an arbitrary number. Therefore, if S_i is closed in T, then $\langle G(S_i) \rangle = G(\bigcup S_i)$ belongs to $\mathscr{C}$. If $G(S)$ belongs to $\mathscr{C}$ and K is a countable subgroup of G, by Lemma 5.6 there is a countable, closed subset S' of T such that $K \subseteq G(S')$. Hence $G(S) + K \subseteq G(S \cup S')$, and the collection $\mathscr{C}$ satisfies the conditions of the third axiom of countability. Lemma 5.7 ensures that the members of $\mathscr{C}$ are decent subgroups of G. Consequently, G satisfies the third axiom of countability with respect to decent subgroups. In other words, G is a B_3-group.

6. THE VANISHING OF BEXT

As usual, $\mathrm{Bext}(G, T)$ denotes those elements in $\mathrm{Ext}(G, T)$ that are represented by balanced sequences $T \rightarrowtail X \twoheadrightarrow G$. Bican [4] found the following homological characterization of Butler groups of finite rank. If G is a torsion-free group of finite rank, then G is a Butler group if and only if $\text{Bext}(G, T) = 0$ for all torsion groups T. This led Bican and Salce [5] to investigate torsion-free groups G of arbitrary rank for which $\text{Bext}(G, T)$ vanishes for all torsion T, and they proved the following.

Theorem (Bican and Salce [5]). $\text{Bext}(G, T) = 0$ *for all torsion T whenever G is a B_2-group.*

The next two results are a direct consequence of the preceding theorem together with Theorem 5.3 and Corollary 4.3, respectively.

Corollary 6.1. $\text{Bext}(G, T) = 0$ *for all torsion T provided that G satisfies the third axiom of countability with respect to decent subgroups.*

Corollary 6.2. *If G is a pure subgroup of a completely decomposable torsion-free group A that satisfies the third axiom of countability over G with respect to separable subgroups, then* $\text{Bext}(G, T) = 0$ *for all torsion groups T.*

Griffith [9] solved a famous problem of Baer by proving that for G torsion free $\text{Ext}(G, T) = 0$ for all torsion T only if G is free. Likewise, there are many torsion-free groups G, even of finite rank, such that $\text{Bext}(G, T)$ does not vanish for all torsion T, but we know of no such example where $\text{Bext}^2(G, T) \neq 0$. In this connection, our next result shows that if there is such an example G, it must have cardinality larger than $\aleph_1$.

Theorem 6.3. *If the torsion-free group G is the union of a smooth ascending chain*

$$0 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots$$

of pure and separable subgroups G_α with $G_{\alpha+1}/G_\alpha$ countable for each α, then $\text{Bext}^2(G, T) = 0$ *for every torsion group T.*

Proof. Suppose that we have already constructed a balanced resolution

$$K_\alpha \rightarrowtail A_\alpha \twoheadrightarrow G_\alpha$$

of G_α, where A_α is completely decomposable. Since $G_{\alpha+1}/G_\alpha$ is countable and G_α is separable in $G_{\alpha+1}$, there is a countable subset C of $G_{\alpha+1}$ such that for each $x \in G_{\alpha+1}$ there is an element c in $C \cap (x + G_\alpha)$ with the property that $|c| \geqq |x|$. Let $B_\alpha = \oplus_{c\in C} \langle c\rangle_*$ be the direct sum of the pure subgroups in $G_{\alpha+1}$ generated by the elements of C. Since C is countable, so is B_α. Consider the natural map from B_α into $G_{\alpha+1}$ (that maps c onto c) and the corresponding natural map of the direct sum $A_\alpha \oplus B_\alpha$ onto $G_{\alpha+1}$. Let $A_{\alpha+1} = A_\alpha \oplus B_\alpha$ and observe that the kernel, $K_{\alpha+1}$, of the natural map $A_{\alpha+1} \twoheadrightarrow G_{\alpha+1}$ is a balanced subgroup of $A_{\alpha+1}$. Indeed, if $x \in G_{\alpha+1}$ there exists $c = x - g_\alpha$ in C with the property that $|c| \geqq |x|$, where $g_\alpha \in G_\alpha$. If $a_\alpha \in A_\alpha$ is a preimage of g_α with $|a_\alpha| = |g_\alpha|$, then $a_\alpha + c$ maps onto x and $|a_\alpha + c| = |x|$.

It follows from the preceding that there is a commutative diagram

$$\begin{array}{ccccc} \cdot & & \cdot & & \cdot \\ K_{\alpha+1} & \rightarrowtail & A_{\alpha+1} & \twoheadrightarrow & G_{\alpha+1} \\ \uparrow & & \uparrow & & \uparrow \\ K_{\alpha} & \rightarrowtail & A_{\alpha} & \twoheadrightarrow & G_{\alpha} \\ \cdot & & \cdot & & \cdot \end{array}$$

where the rows are balanced resolutions and the completely decomposable group $A_{\alpha+1} = A_\alpha \oplus B_\alpha$ where B_α is countable. This yields a balanced-projective resolution

$$K \rightarrowtail A \twoheadrightarrow G$$

of G, where K is the union of the smooth chain

$$0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_\alpha \subseteq \ldots .$$

It is easily demonstrated that K_α is balanced in K. Moreover, $K_{\alpha+1}/K_\alpha$ is isomorphic to a pure subgroup of $A_{\alpha+1}/A_\alpha \simeq B_\alpha$. But a pure subgroup of a countable completely decomposable torsion-free group is a B_2-group (by Corollary 3.9). Thus K is a B_2-group by Lemma 3.6. Hence $\mathrm{Bext}(K, T) = 0$ for all torsion T, and therefore $\mathrm{Bext}^2(G, T) = 0$ since $K \rightarrowtail A \twoheadrightarrow G$ is a balanced resolution, for

$$0 = \mathrm{Bext}(K, T) \to \mathrm{Bext}^2(G, T) \to \mathrm{Bext}^2(A, T) = 0$$

is exact.

Using an argument similar to that used in the proof of Theorem 6.3, we can show under the continuum hypothesis that $\mathrm{Bext}^3(G, T) = 0$ whenever G is torsion free and T is torsion.

Theorem 6.4. (C.H.) $\mathrm{Bext}^3(G, T) = 0$ *if G is torsion free and T is torsion.*

Proof. We know that G is the union of a smooth ascending chain

$$0 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots$$

of pure subgroups G_α of G where $|G_{\alpha+1}/G_\alpha| \leqq \aleph_1$ for each α. The idea is to use the fact that the continuum hypothesis implies that every subgroup H of a torsion-free group G is $\aleph_1$-separable. This means that if $g \in G$ there is a subset S of H having cardinality not exceeding $\aleph_1$ with the property that if $x \in H$ then $|g + x| \leqq |g + y|$ for some $y \in S$. Therefore, using the same kind of argument as that used in the proof of Theorem 6.3, we conclude that there are balanced resolutions

$$K_\alpha \rightarrowtail A_\alpha \twoheadrightarrow G_\alpha ,$$

where the diagram

$$\begin{array}{ccccc} K_{\alpha+1} & \rightarrowtail & A_{\alpha+1} & \twoheadrightarrow & G_{\alpha+1} \\ \uparrow & & \uparrow & & \uparrow \\ K_{\alpha} & \rightarrowtail & A_{\alpha} & \twoheadrightarrow & G_{\alpha} \end{array}$$

is commutative and where the completely decomposable torsion-free group $A_{\alpha+1} =$

$= A_\alpha \oplus B_\alpha$ with $|B_\alpha| \leqq \aleph_1$. As before, the limits yield a balanced resolution

$$K \rightarrowtail A \twoheadrightarrow G$$

of G, where K is the union of the smooth ascending chain

$$0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_\alpha \subseteq \ldots$$

of balanced subgroups of K with $|K_{\alpha+1}/K_\alpha| \leqq \aleph_1$. Since the K_α's are balanced and $|K_{\alpha+1}/K_\alpha| \leqq \aleph_1$, the chain of the K_α's can be refined to produce a smooth chain of pure and separable subgroups with countable quotients satisfying the hypothesis of Theorem 6.3. Hence, $\mathrm{Bext}^2(K, T) = 0$ if T is torsion, and consequently $\mathrm{Bext}^3(G, T) = 0$.

Corollary 6.5. (C.H.). *Let G be a torsion free and T torsion. If* $\mathrm{Bext}^2(G, T) = 0$ *then* $\mathrm{Bext}^2(H, T) = 0$ *for any balanced subgroup H of G.*

We conclude with one more result about the vanishing of Bext that does not involve the Continuum Hypothesis.

Proposition 6.6. *Suppose that H is a balanced subgroup of a B_2-group G. If G/H satisfies the third axiom of countability for separable subgroups, then* $\mathrm{Bext}(H, T) = 0$ *for all torsion T.*

Proof. From Theorem 6.3, we conclude that $\mathrm{Bext}^2(G/H, T) = 0$. Hence the result follows from the exactness of the sequence

$$0 = \mathrm{Bext}(G, T) \to \mathrm{Bext}(H, T) \to \mathrm{Bext}^2(G/H, T) = 0\,.$$

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