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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF DELAY EQUATIONS WITH CONSTANT COEFFICIENTS

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1. INTRODUCTION

Our aim in this paper is to obtain a necessary and sufficient condition under which all solutions of the delay differential equation (DDE)

$$x'(t) + px(t - \tau) + qx(t - \sigma) = 0\,, \tag{1}$$

oscillate. Here the coefficients p and q are assumed to be real numbers and the delays τ and σ are nonnegative real numbers.

For delay differential equations with positive coefficients of the form

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0\,,$$

necessary and sufficient conditions were obtained by Tramov [3]. See also Ladas, Sficas and Stavroulakis [2].

Our main result is the following theorem.

Theorem. *Consider the DDE* (1). *Assume that the coefficients p and q are real numbers and the delays τ and σ are nonnegative real numbers. Then the following statements are equivalent*:

(a) *All solutions of Eq.* (1) *oscillate.*

(b) *The characteristic equation*

$$\lambda + pe^{-\lambda\tau} + qe^{-\lambda\sigma} = 0 \tag{2}$$

of Eq. (1) *has no real roots.*

The importance in the present result is that the coefficients of Eq. (1) are not restricted to be positive.

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large t.

2. PRELIMINARY RESULTS

When $\tau = \sigma$, Eq. (1) reduces to an equation with one delay. Also, when $\sigma = 0$, the transformation

$$x(t) = y(t)\,\mathrm{e}^{-qt}$$

reduces Eq. (1) to an equation with one delay. Now, for equations with one delay of the form

$$y'(t) + ry(t - \mu) = 0\,, \tag{1'}$$

where r is a real number and μ is a positive real number, it is known that every solution of Eq. (1') oscillates if and only if its characteristic equation

$$\lambda + r\mathrm{e}^{-\lambda\mu} = 0$$

has no real roots. This result follows from [2] or by observing that

$$\min_{\lambda \in \mathbf{R}} (\lambda + r\mathrm{e}^{-\lambda\mu}) = \frac{1}{\mu} \ln (r\mu \mathrm{e})$$

and that, as it is known from [1], the condition

$$r\mu\mathrm{e} > 1$$

is sufficient for all solutions of Eq. (1') to oscillate.

Hence, in the sequel, without loss of generality, we will assume that the delays τ and σ are such that

$$\tau > \sigma > 0\,. \tag{3}$$

Set

$$F(\lambda) = \lambda + p\mathrm{e}^{-\lambda\tau} + q\mathrm{e}^{-\lambda\sigma}$$

and assume that $F(\lambda)$ has no real roots. As $F(+\infty) = +\infty$, it follows that

$$F(\lambda) > 0 \quad \text{for every} \quad \lambda \in \mathbb{R}\,.$$

In particular,

$$F(0) = p + q > 0\,. \tag{4}$$

Also

$$F(-\infty) = +\infty\,,$$

which implies that

$$p > 0\,. \tag{5}$$

Finally,

$$m \equiv \min_{\lambda \in \mathbf{R}} (\lambda + p\mathrm{e}^{-\lambda t} + q\mathrm{e}^{-\lambda\sigma}) > 0\,. \tag{6}$$

The following lemma summarizes the above observations.

Lemma 1. *Assume that* (3) *holds. Then the inequalities* (4), (5), *and* (6) *are necessary conditions for Eq.* (2) *to have no real roots.*

Let $x(t)$ be a solution of Eq. (1) and set

(7) $$z(t) = x(t) - p \int_{t-\tau}^{t-\sigma} x(s)\,ds\,.$$

Then we have the following result.

Lemma 2. *Assume that $x(t)$ is a solution of Eq. (1). Then $z(t)$ is also a solution of Eq. (1).*

Proof. From (7) and Eq. (1) we find that

(8) $$z'(t) = x'(t) - p[x(t-\sigma) - x(t-\tau)] = -(p+q)\,x(t-\sigma)$$

and so

$$pz'(s+\sigma) = -p(p+q)\,x(s)\,.$$

Integrating this equation from $t-\tau$ to $t-\sigma$ and using (7) and (8), we obtain

$$p[z(t) - z(t+\sigma-\tau)] = (p+q)\,[z(t) - x(t)] = (p+q)\,z(t) + z'(t+\sigma)$$

or equivalently

$$z'(t) + pz(t-\tau) + qz(t-\sigma) = 0\,.$$

The proof is complete.

The following lemma describes the asymptotic behavior of the function $z(t)$ as $t \to \infty$.

Lemma 3. *Consider the DDE (1) and assume that*

(9) $$\tau > \sigma > 0\,, \quad p + q > 0\,, \quad \text{and} \quad p > 0\,.$$

Let $x(t)$ be an eventually positive solution of Eq. (1) and define $z(t)$ as given by (7). Then the following statements are true:

(a) *Assume that*

$$p(\tau - \sigma) \leqq 1\,.$$

Then $z(t)$ is an eventually positive and decreasing solution of Eq. (1).

(b) *Assume that*

$$p(\tau - \sigma) > 1\,.$$

Set

$$w(t) = -z(t)\,.$$

Then $w(t)$ is an eventually positive and increasing solution of Eq. (1).

Proof. Let t_0 be such that $x(t) > 0$ for $t \geqq t_0$. Then:

(a) From Lemma 2 we know that $z(t)$ is a solution of Eq. (1) and from (8) we see that $z(t)$ is a decreasing function of t. To prove that $z(t)$ is positive, it suffices to show that

(10) $$\lim_{t\to\infty} z(t) = 0\,.$$

First, we claim that $\lim_{t\to\infty} z(t)$ is a finite number. Otherwise,

$$\lim_{t\to\infty} z(t) = -\infty$$

which implies that $z(t)$ is eventually negative and $x(t)$ is unbounded. Hence, there exists a $t_1 \geqq t_0 + \max\{\tau, \sigma\}$ such that

$$z(t_1) < 0 \quad \text{and} \quad x(t_1) = \max_{t_0 \leqq s \leqq t_1} x(s)$$

It follows, from (7), that

$$0 > z(t_1) = x(t_1) - p \int_{t_1-\tau}^{t_1-\sigma} x(s)\,\mathrm{d}s \geqq x(t_1)\,[1 - p(\tau - \sigma)] \geqq 0\,.$$

This contradiction establishes our claim that

$$l \equiv \lim_{t\to\infty} z(t)$$

is finite.

Integrating both sides of (8) from t_0 to t and letting $t \to \infty$, we see that

$$l - z(t_0) = -(p + q) \int_{t_0}^{\infty} x(s - \sigma)\,\mathrm{d}s$$

which shows that $x \in L^1[t_0, \infty)$. From Eq. (1), it follows that $x' \in L^1[t_0, \infty)$. Hence, $\lim_{t\to\infty} x(t)$ exists and it has to be zero (because $x \in L^1[t_0, \infty)$). Thus, $\lim_{t\to\infty} x(t) = 0$ and, from (7), we conclude that (10) holds.

(b) From Lemma 2 and the linearity of Eq. (1), it follows that $w(t)$ is a solution of Eq. (1). From (7), we also see that

$$w'(t) = (p + q)\,x(t - \sigma) > 0 \tag{11}$$

and so $w(t)$ is an increasing function of t. To show that $w(t)$ is eventually positive, it suffices to prove that

$$\lim_{t\to\infty} w(t) = +\infty\,. \tag{12}$$

Otherwise,

$$\lim_{t\to\infty} w(t) \equiv l$$

exists and is finite. Integrating both sides of (11) from t_0 to t and letting $t \to \infty$, we find

$$l - w(t_0) = (p + q) \int_{t_0}^{\infty} x(s - \sigma)\,\mathrm{d}s$$

which shows that $x \in L^1[t_0, \infty)$. As in the proof of part (a), we conclude that $l = 0$. Therefore, $w(t)$ increases to zero as $t \to \infty$, which implies that $w(t) < 0$. Thus, there exists a $t_1 \geqq t_0 + \max\{\tau, \sigma\}$ such that

$$w(t_1) < 0 \quad \text{and} \quad x(t_1) = \min_{t_0 \leqq s \leqq t_1} x(s)\,.$$

It follows, from (7), that

$$0 > w(t_1) = -z(t_1) = -x(t_1) + p\int_{t_1-\tau}^{t_1-\sigma} x(s)\,ds \geqq x(t_1)\left[-1 + p(\tau - \sigma)\right] > 0\,.$$

This contradiction establishes (12) and the proof is complete.

3. PROOF OF MAIN RESULT

(a) $\Rightarrow$ (b). Otherwise, Eq. (2) has a real root λ_0. Then $x(t) = e^{\lambda_0 t}$ is a nonoscillatory solution of Eq. (1) which contradicts our assumption.

(b) $\Rightarrow$ (a). We may (and do) assume that (3) holds. Since Eq. (2) has no real roots, the inequalities (4), (5), and (6) are satisfied.

Next, we distinguish the following cases.

Case 1. $p(\tau - \sigma) \leqq 1$. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $x(t)$. Setting

$$z(t) = x(t) - p\int_{t-\tau}^{t-\sigma} x(s)\,ds\,,$$

we know, by Lemma 3(a), that $z(t)$ is a positive and decreasing solution of Eq. (1). Furthermore, from (8), we have

$$z'(t) + (p + q)\,x(t - \sigma) = 0\,, \quad t \geqq t_0\,.$$

Set

$$z_0(t) = z(t)$$

and

$$z_n(t) = z_{n-1}(t) - p\int_{t-\tau}^{t-\sigma} z_{n-1}(s)\,ds\,, \quad n = 1, 2, \ldots\,. \tag{13}$$

Then, for each $n = 1, 2, \ldots$, the function $z_n(t)$ is also a positive and decreasing solution of Eq. (1) such that

$$z_n'(t) + (p + q)\,z_{n-1}(t - \sigma) = 0\,, \quad t \geqq t_0\,. \tag{14}$$

For each $n = 1, 2, \ldots$, we define the set

$$A(z_n) = \{\lambda > 0\colon z_n'(t) + \lambda\, z_n(t) \leqq 0\}\,.$$

The proof will be accomplished by proving that $A(z_n)$ has the following contradictory properties:

(i) For each $n = 1, 2, \ldots$, the set $A(z_n)$ is nonempty and bounded above by a number independent of n.

(ii) $\lambda \in A(z_n) \Rightarrow (\lambda + m) \in A(z_{n+1})$, $n = 1, 2, \ldots$, where m is the positive number defined by (6).

Clearly,

$$z_n(t) \leqq z_{n-1}(t) \leqq z_{n-1}(t - \sigma)$$

and (14) yields

$$z_n'(t) + (p + q)\, z_n(t) \leqq 0\,.$$

Therefore,

$$(p + q) \in \Lambda(z_n)$$

which proves that $\Lambda(z_n) \neq \emptyset$.

Next, we prove that $\Lambda(z_n)$ is bounded above by a number independent of n. Indeed, integrating both sides of (14) from $t - \sigma$ to t, we find

$$z_n(t) - z_n(t - \sigma) + (p + q) \int_{t-\sigma}^{t} z_{n-1}(s - \sigma)\, ds = 0$$

which implies that

$$- z_n(t - \sigma) + \sigma(p + q)\, z_{n-1}(t - \sigma) < 0$$

and hence

(15) $$z_{n-1}(t) < \frac{1}{\sigma(p + q)}\, z_n(t)\,, \quad n = 1, 2. \ldots.$$

From (7) we find that

$$z_n(t) \leqq z_{n-1}(t) - p(\tau - \sigma)\, z_{n-1}(t - \sigma)$$

and so, using (15), we obtain

$$p(\tau - \sigma)\, z_{n-1}(t - \sigma) \leqq z_{n-1}(t) - z_n(t) < \frac{1}{\sigma(p + q)}\, z_n(t)\,.$$

Therefore,

(16) $$(p + q)\, z_{n-1}(t - \sigma) < A\, z_n(t)\,, \quad n = 1, 2, \ldots,$$

where

$$A = \frac{1}{\sigma p(\tau - \sigma)}\,.$$

Using (16) in (14), we conclude that

$$z_n'(t) + A\, z_n(t) > 0$$

which proves that

$$A \notin \Lambda(z_n)$$

and so $\Lambda(z_n)$ is bounded from above by A.

Next, we will prove (ii). Let $\lambda \in \Lambda(z_n)$ and set

$$z_n(t) = e^{-\lambda t} \phi_n(t)\,.$$

Then

$$\phi_n'(t) = e^{\lambda t}[z_n'(t) + \lambda\, z_n(t)] \leqq 0$$

and, from (7), we see that

(17) $$z_{n+1}(t) = e^{-\lambda t} \phi_n(t) - p \int_{t-\tau}^{t-\sigma} e^{-\lambda s} \phi_n(s)\, ds \leqq$$

$$\leqq e^{-\lambda t}\phi_n(t) - \frac{p}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\, e^{-\lambda t}\phi_n(t-\sigma) \leqq$$

$$\leqq \left[1 - \frac{p}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\right] e^{-\lambda t}\phi_n(t-\sigma)\,.$$

Also, from (14), we find

(18) $$z'_{n+1}(t) = -(p+q)\, e^{\lambda\tau} e^{-\lambda t}\phi_n(t-\sigma)\,.$$

Hence, from (17) and (18), we have

$$z'_{n+1}(t) + (\lambda + m)\, z_{n+1}(t) \leqq$$

$$\leqq \left[-(p+q)\, e^{\lambda\tau} + (\lambda+m) - (\lambda+m)\frac{p}{\lambda}(e^{\lambda\tau} - e^{\lambda\sigma})\right] e^{-\lambda t}\phi_n(t-\sigma) \leqq$$

$$\leqq (p+q)(e^{\lambda\sigma} - e^{\lambda\tau})\, e^{-\lambda t}\phi_n(t-\sigma) < 0$$

which proves that

$$(\lambda + m) \in \Lambda(z_{n+1})$$

and completes the proof in this case.

Case 2. $p(\tau - \sigma) > 1$. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $y(t)$. Setting

$$w(t) = -z(t) = -x(t) + p\int_{t-\tau}^{t-\sigma} x(s)\, ds\,,$$

by Lemma 3(b), we conclude that $w(t)$ is an eventually positive and increasing solution of Eq. (1) such that

$$w'(t) - (p+q)\, x(t-\sigma) = 0\,, \quad t \geqq t_0\,.$$

Set

$$w_0(t) = w(t)$$

and

(19) $$w_n(t) = -w_{n-1}(t) + p\int_{t-\tau}^{t-\sigma} w_{n-1}(s)\, ds\,, \quad n = 1, 2, \ldots.$$

Then, for each $n = 1, 2, \ldots$, the function $w_n(t)$ is also an eventually positive and increasing solution of Eq. (1) and such that

(20) $$w'_n(t) - (p+q)\, w_{n-1}(t-\sigma) = 0\,, \quad t \geqq t_0\,.$$

For each $n = 1, 2, \ldots$, we define the set

$$M(w_n) = \{\mu > 0 \colon w'_n(t) - \mu\, w_n(t) \geqq 0\}\,.$$

We also set

$$\mu_0 = \frac{p+q}{p(\tau-\sigma)} \quad \text{and} \quad m_0 = \frac{m\mu_0}{p}\, e^{\mu_0\sigma}\,.$$

The proof will be accomplished by proving that $M(w_n)$ has the following contradictory properties:

(i) For each $n = 1, 2, \ldots$

$$\mu_0 \in M(w_n) \quad \text{and} \quad -q \notin M(w_n),$$

that is, the set $M(w_n)$ is nonempty and bounded from above by a number independent of n.

(ii) For every $\mu \in M(w_n)$, with $\mu > \mu_0$,

$$(\mu + m_0) \in M(w_{n+1}), \quad n = 1, 2, \ldots.$$

First, we will prove (i). From (19), we see that for each $n = 1, 2, \ldots$

$$w_n(t) \leqq p(\tau - \sigma)\, w_{n-1}(t - \sigma)$$

and, using this in (20), we find that

$$w'_n(n) - \frac{p + q}{p(\tau - \sigma)}\, w_n(t) \geqq 0.$$

Hence

$$\mu_0 = \frac{p + q}{p(\tau - \sigma)} \in M(w_n), \quad n = 1, 2, \ldots.$$

Now, from (20) and the fact that $w_n(t)$ is a solution of Eq. (1), we have

$$(p + q)\, w_{n-1}(t - \sigma) = w'_n(t) = -p w_n(t - \tau) - q w_n(t - \sigma)$$

which implies that $q < 0$ and

$$(p + q)\, w_{n-1}(t - \sigma) < -q w_n(t - \sigma) < -q w_n(t).$$

Using this in (20), we find that

$$w'_n(t) - (-q)\, w_n(t) < 0$$

which proves that

$$-q \notin M(w_n), \quad n = 1, 2, \ldots.$$

Finally, we will prove (ii). Let $\mu \in M(w_n)$ with $\mu \geqq \mu_0$, and set

$$w_n(t) = e^{\mu t} \psi_n(t).$$

Then

$$\psi'_n(t) = e^{-\mu t}[w'_n(t) - \mu\, w_n(t)] \geqq 0$$

and, from (19), we find that

$$w_{n+1}(t) = -e^{\mu t}\psi_n(t) + p \int_{t-\tau}^{t-\sigma} e^{\mu s} \psi_n(s)\, ds \leqq \tag{21}$$

$$\leqq -e^{\mu t}\psi_n(t) + \frac{p}{\mu}(e^{-\mu\sigma} - e^{-\mu\tau})\, e^{\mu t}\psi_n(t - \sigma) \leqq$$

$$\leqq e^{\mu t}\psi_n(t - \sigma)\left[-1 + \frac{p}{\mu}(e^{-\mu\sigma} - e^{-\mu\tau})\right].$$

Also, from (20), we have

$$w'_{n+1}(t) = (p + q)\, e^{-\mu\sigma} e^{\mu t} \psi_n(t - \sigma)\,. \tag{22}$$

Hence, from (21) and (22), we obtain

$$w'_{n+1}(t) - (\mu + m_0)\, w_{n+1}(t) \geqq$$

$$\geqq e^{\mu t} \psi_n(t - \sigma) \left[(p + q)\, e^{-\mu\sigma} + (\mu + m_0) - (\mu + m_0) \frac{p}{\mu} (e^{-\mu\sigma} - e^{-\mu\tau}) \right] \geqq$$

$$\geqq e^{\mu t} \psi_n(t - \sigma) \left[(\mu + p e^{-\mu\tau} + q e^{-\mu\sigma}) + m_0 \left(1 - \frac{p}{\mu} e^{-\mu\sigma} \right) \right] \geqq$$

$$\geqq e^{\mu t} \psi_n(t - \sigma) \left[m + m_0 \left(1 - \frac{p}{\mu_0 e^{\mu_0 \sigma}} \right) \right] = e^{\mu t} \psi_n(t - \sigma)\, m_0 \geqq 0$$

which implies that

$$(\mu + m_0) \in M(w_{n+1})\,, \quad n = 1, 2, \ldots\,.$$

The proof of the theorem is complete.

References

[1] *G. Ladas:* Sharp conditions for oscillations caused by delays, Applicable Analysis *9* (1979), 93–98.

[2] *G. Ladas, Y. G. Sficas* and *I. P. Stavroulakis:* Necessary and sufficient conditions for oscillations, Amer. Math. Monthly *90* (1983), 637–640.

[3] *M. I. Tramov:* Conditions for oscillation of solutions of first order differential equations with a delayed argument, Izv. Vysš. Učebn. Zaved., Matematika *19*, 3 (1975), 92–96 (Russian).

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