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INTRODUCTION

In classical potential theory the notion of energy played a considerable role [5]. It is no exaggeration to say that the central result responsible for this was a theorem of Cartan asserting the completeness of the set of excessive functions of finite energy.

It is this result we shall generalise to very general Markov processes.

GENERALITIES

We assume that we have a Hunt process X with a locally compact second countable state space. We assume that there is a σ-finite Radon excessive reference measure ξ also denoted dx.

Notation will generally be as in Blumenthal-Getoor [1].

All excessive functions will be assumed to be finite almost everywhere.

An excessive function s is called a *class* (D) *potential* if

$$P_{T_n} s \downarrow 0 \quad \text{almost everywhere}$$

whenever the sequence of stopping times T_n increases to infinity almost surely. To every class (D) potential corresponds a natural additive functional A defined off a polar set. For more complete details on this point see [7].

Also to every class (D) potential corresponds a measure — not necessarily σ-finite — called its *Revuz measure* [10]. The total mass of the Revuz measure of a class (D) potential s will be called its *mass functional* and denoted $L(s)$. The mass functional L is monotone and continuous from below i.e. $s \leqq r$ implies $L(s) \leqq L(r)$ and $s_n \uparrow s$ implies $L(s_n) \uparrow L(s)$. It is useful to note that

$$L(s) = \sup \{(s, g) \mid g > 0 \text{ and } \hat{U}g \leqq 1\} .$$

Now we are ready to introduce energy.

Let s be a class (D) potential with natural additive functional A. If $p = E^{\cdot}(A_\infty^2)$ is finite almost everywhere, it is necessarily a class (D) potential. We shall refer to p as the *energy function* of s. If $L(p) < \infty$ we shall say s has finite energy and we put

$$\|s\|_e^2 = L(p) .$$

Now it is clear how to define the mutual energy of two class (D) potentials r and s of finite energy. If the additive functionals A and B correspond to r and s the mutual energy $\langle r, s\rangle_e$ is defined by

$$\langle r, s\rangle_e = L(E^\cdot(A_\infty \cdot B_\infty)) .$$

Polarisation allows one to define energy of a difference, in the usual way. Thus defined the energy becomes a Hilbertian seminorm on the linear space generated by class (D) potentials of finite energy.

It is the purpose of this paper to prove under some conditions that the set of class (D) potentials is complete in energy norm.

We collect here some simple facts. These will be used without further mention. Proofs may be found in [7].

Let s be a class (D) potential of finite energy with additive functional A. Let $g > 0$ and $\hat{U}g \leqq 1$. Then

$$(s^2, g) \leqq (E^\cdot(A_\infty^2), g) \leqq L(E^\cdot(A_\infty^2)) = \|s\|_e^2 .$$

In particular s is square integrable relative to g.

If s, t are class (D) potentials of finite energy with Revuz measures μ and ν then

$$(s, \nu) + (\mu, t) \leqq \langle s, t\rangle_e .$$

Equality holds if at least one of s or t is regular i.e. with continuous additive functional. In particular s belongs to $L^1(\nu)$.

Proposition 0.1. *Let $f \geqq 0$. Suppose $(f, Ug) < \infty$ for all $g \geqq 0$, such that Ug has finite energy.*

Then Uf has finite energy. In particular if Uf has finite energy $U\hat{P}_t f$ also has finite energy for each $t \geqq 0$.

Proof. We claim that there is a $M \geqq 0$ such that

$$(f, Ug) \leqq M \quad \text{if} \quad \|Ug\|_e \leqq 1 .$$

If not we can find g_n such that

$$\|Ug_n\|_e \leqq 1 \quad \text{and} \quad (f, Ug_n) \geqq 2^{2n} ,$$

but then $Ug = U(\Sigma 2^{-n} g_n)$ has energy less or equal to 1 but $(f, Ug) = \infty$. A contradiction.

Let $g > 0$ s.t. Ug has finite energy. For all $n \geqq 1$ $U(f \wedge ng)$ has finite energy. Thus

$$(f \wedge ng, U(f \wedge ng)) \leqq (f, U(f \wedge ng)) \leqq M\|U(f \wedge ng)\|_e .$$

This gives

$$\|U(f \wedge ng)\|_e \leqq 2M$$

for all $n \geqq 1$. Letting $n \to \infty$ we get

$$\|Uf\|_e \leqq 2M .$$

Let us apply this to $\hat{P}_t f$, where Uf has finite energy. We have in this case

$$(\hat{P}_t f, Ug) = (f, P_t Ug) \leqq (f, Ug) \leqq \|Uf\|_e \, \|Ug\|_e .$$

We conclude

$$\|U\hat{P}_t f\|_e^2 = 2(U\hat{P}_t f, \hat{P}_t f) \leqq 2\|Uf\|_e^2 .$$

Proposition 0.2. *Let $g > 0$ and $g \in L^1(\xi)$, (s_n) a sequence of excessive functions, uniformly integrable as a subset of $L^1(g)$. Suppose* $\lim s_n = s$ *weakly in $L^1(g)$, with s excessive. Then*

$$\liminf s_n \geqq s \quad \textit{everywhere} .$$

Proof. For all $\alpha > 0$, $n \geqq 1$ and $N \geqq 1$

$$U_\alpha s_n(x) = \int s_n(y) u_\alpha(x, y) \, \mathrm{d}y \geqq \int s_n(y) u_\alpha(x, y) \wedge N g(y) \, \mathrm{d}y \underset{n \to \infty}{\to} \int s(y) \, u_\alpha(x, y) \wedge N \, g(y) \, \mathrm{d}y .$$

But since $g > 0$ the right hand side increases to $U_\alpha \, s(x)$ for $N \to \infty$, and $\alpha U_\alpha s_n \uparrow s_n$ and $\alpha U_\alpha s \uparrow s$ everywhere by excessivity. We conclude

$$\liminf_n s_n \geqq s \quad \text{everywhere} .$$

1. SOME GENERAL RESULTS

Proposition 1.1. *Let s be class (D) of finite energy. For $\alpha > 0$, let s^α be the unique α-excessive function such that*

$$s = s^\alpha + \alpha U s^\alpha = s^\alpha + \alpha U^\alpha s .$$

Then s^α is square integrable and

$$2\alpha \|s^\alpha\|_2^2 \leqq \|s\|_e^2 .$$

Proof. Sufficient to prove this for s of form Uf. Since $Uf \geqq \alpha U U^\alpha f$, $UU^\alpha f$ also has finite energy. Hence

$$0 \leqq (U(f - \alpha U^\alpha f), f - \alpha U^\alpha f)$$

i.e.

$$\alpha \|U^\alpha f\|_2^2 \leqq (U^\alpha f, f) \leqq (Uf, f) = \tfrac{1}{2} \|Uf\|_e^2 .$$

Definition. Let m be a σ-finite measure on a set Ω.

A bounded subset $F \subseteq L^1(m)$ will be called *uniformly integrable* if:

For each $\varepsilon > 0$ we can find $\delta > 0$ and a set C of finite m-measure such that for all $f \in F$

$$\int_{\Omega \setminus C} |f| \, \mathrm{d}m \leqq \varepsilon$$

and whenever $H \subset \Omega$ with $m(H) < \delta$ we have

$$\int_H |f| \, \mathrm{d}m \leqq \varepsilon .$$

This is just uniform integrability adapted to σ-finite measures. It is wellknown that, if (f_n) is uniformly integrable in the above sense and converges to f m-almost everywhere then (f_n) tends to f in $L^1(m)$.

Note that in the following lemma a little care is needed because the measures are not necessarily finite.

Proposition 1.2. *Let r be class (D) of finite energy with Revuz measure ν. The set*

$$\{s \mid s \text{ excessive } \|s\|_e \leqq 1\}$$

is a uniformly integrable subset of $L^1(\nu)$.

Proof. Let us find a set G whose complement has finite ν-measure such that

$$\int_G s \, d\nu \quad \text{for} \quad \|s\|_e \leqq 1$$

is uniformly small.

Now ν is σ-finite. Let F denote sets with $\nu(F) < \infty$ and $G = F^c$ and A the natural additive functional of r. Denote by the same letter a set and its indicator function.

As the sets F increase to full ν-measure the excessive functions $r_F = U_A F$ increase to r and hence r_G decrease to zero.

Since r_G is strongly dominated by r, $\|r_G\|_e$ is small for large F.

The Revuz measure of r_G is $\nu|_G$, so if $\nu(F)$ is large yeat finite by energy inequality

$$\int_G s \, d\nu \leqq \|r_G\|_e \quad \text{for} \quad \|s\|_e \leqq 1 \, .$$

Thus uniformly small.

The other item in the definition of uniform integrability is treated similarly.

In [8] it was claimed without proof that every limit of a sequence of class (D) potentials bounded in energy is itself class (D). The proof was not available as Professor P. A. Meyer pointed out to the authors. In [7] a general condition was then given to ensure this. This condition is as follows:

There is a strictly positive function g such that for every $\varepsilon > 0$, the set $(\hat{U}g \geqq \varepsilon)$ has finite measure.

This condition will be assumed throughout this section.

The following theorem is essentially theorem 0·5 in [7]. We present another proof, assuming strong duality. This seems to give a little more. However, this assumption is not necessary.

Recall that for each excessive function s and $\alpha > 0$, there is a unique α-excessive function s^α such that

$$s = s^\alpha + \alpha U^\alpha s$$

and if s is further purely excessive

$$s = s^\alpha + \alpha U s^\alpha \, . \tag{1.1}$$

Theorem 1.3. *Let the sequence (s_n) of class (D) potentials converge a.s. to an exces-*

sive function s. Suppose the sequence is bounded in energy. Assume that we have a strong Markov dual in addition to the above assumption.

Then s is class (D) and for each $\alpha > 0$

$$\lim s_n^\alpha = s^\alpha\,, \quad \lim Us_n^\alpha = Us^\alpha$$

on the set $(s < \infty) \cap (\lim_n s_n = s)$.

Proof. Suppose $\|s_n\|_e \leqq 1$ and let $\alpha = 1$. There is no loss of generality in assuming that the function g above also satisfies $0 < g < 1$, $g \in L^1(\xi)$ and $\hat{U}g \leqq 1$.

(s_n) and hence (s_n^1) is uniformly integrable as a subset of $L^1(g)$. Along a subsequence N we may assume (s_n^1) converges a.e. to a 1-excessive function t. By uniform integrability

$$\lim_{n\in N} s_n^1 = t \quad \text{in} \quad L^1(g)\,. \tag{1.2}$$

We assert (s_n^1) is also uniformly integrable as a subset of $L^1(\hat{U}g)$. Indeed

$$(s_n^1, \hat{U}g)^2 \leqq (Us_n^1, g)^2 \leqq (s_n, g)^2 \leqq (s_n^2, g)\,(g, 1) \leqq \|s_n\|_e^2\,(g, 1)\,.$$

If F is a set whose $\hat{U}g$-measure is small, denoting the indicator function of F by F

$$(s_n^1 F, \hat{U}g)^2 \leqq \|s_n^1\|_2^2\,(F, \hat{U}g)$$

is also uniformly small.

For each $\varepsilon > 0$, the set $(\hat{U}g \geqq \varepsilon)$ has finite ξ-measure by assumption, hence it has finite $\hat{U}g$-measure. Fixing ε let $C = (\hat{U}g < \varepsilon)$. Let C also denote its indicator function. Then

$$(s_n^1, C\hat{U}g) \leqq (s_n^1, \hat{P}_C\hat{U}g)\,. \tag{1.3}$$

Now $\hat{P}_C\hat{U}g \leqq \varepsilon$. The mass functional of the coexcessive function $\hat{P}_C\hat{U}g$ is atmost $(g, 1)$. Meyer's energy formula – applied to the dual – leads to

$$\|\hat{P}_C\hat{U}g\|_e^2 \leqq 2\varepsilon(g, 1)\,. \tag{1.4}$$

Now $(\hat{U}s_n^1, s_n^1) = (s_n^1, Us_n^1)$ and $Us_n^1 \leqq s_n$. It follows that

$$\|\hat{U}s_n^1\|_e \leqq 2\|s_n\|_e \leqq 2\,.$$

Apply the energy inequality to the right side of (1.3) and use (1.4)

$$(s_n^1, C\hat{U}g) \leqq 4\sqrt{\varepsilon(g, 1)}\,.$$

Thus (s_n^1) is uniformly integrable as a subset of $L^1(\hat{U}g)$. Now along the subsequence $N(s_n^1)$ converges a.e. to t. Hence

$$\lim_{n\in N} s_n^1 = t \quad \text{in} \quad L^1(\hat{U}g)\,.$$

In particular

$$\lim_{n\in N} (Us_n^1, g) = \lim_{n\in N} (s_n^1, \hat{U}g) = (t, \hat{U}g) = (Ut, g)\,.$$

However, by Fatou lemma

$$\liminf_{n\in N} Us_n^1 \geqq Ut\,. \tag{1.5}$$

Using (1.5) we can now assert that Us_n^1 converges in $L^1(g)$ to Ut. We must thus have

$$s = t + Ut\,. \tag{1.6}$$

The last identity easily leads to the proof that s is purely excessive. Indeed for each $r > 0$

$$P_r s = P_r t + P_r U t\,.$$

As $r \uparrow \infty$, $P_r U\, t(x) \downarrow 0$ at each point at which $U\, t(x) < \infty$.

Put

$$h = \lim_{r \uparrow \infty} P_r s\,.$$

Then

$$\lim_{r \uparrow \infty} P_r t = h \quad \text{a.e.}$$

but by Fatou lemma

$$0 = \liminf_{r \to \infty} P_r U t = \liminf_{r \to \infty} U P_r t \geqq U h\,.$$

Thus $Uh = 0$ a.e. and hence $h = 0$.

This proves that s is purely excessive. That s must indeed be class (D) is now proved as in [7].

Take x in $(s < \infty) \cap (\lim s_n = s)$. From proposition 0.2 and (1.5) we have

$$s(x) = \lim s_n(x) \geqq \limsup s_n^1(x) + \liminf U\, s_n^1(x) \geqq$$
$$\geqq \liminf s_n^1(x) + U\, t(x) \geqq t(x) + U\, t(x) = s(x)$$

showing that $\lim\limits_n s_n^1(x) = t(x) = s^1(x)$. *q.e.d.*

Corollary 1.4. *Let $g > 0$ be such that Ug has finite energy. Then the function $t \to (P_t s, g)$ tends to zero uniformly as $t \to \infty$ on the set*

$$\{s \mid s \text{ excessive } \|s\|_e \leqq 1\}\,.$$

Proof. Suppose not. Then there exists $\varepsilon > 0$ and s_n and $t_n \to \infty$ such that

$$\|s_n\|_e \leqq 1 \quad (P_{t_n} s_n, g) \geqq \varepsilon\,. \tag{1.7}$$

By choosing a subsequence if necessary let us assume that (s_n) converges a.s. to an excessive function s. We know that s is necessarily class (D). For each $t > 0$ by propositions 1.2 and 0.1 (s_n) is uniformly integrable as a subset of $L^1(\hat{P}_t g)$. Therefore

$$\lim_n (P_t s_n, g) = \lim_n (s_n \hat{P}_t g) = (s, \hat{P}_t g) = (P_t s, g)\,. \tag{1.8}$$

(1.7) and (1.8) together give

$$(P_t s, g) \geqq \varepsilon \quad \text{for all} \quad t\,.$$

This is a contradiction because s is class (D). q.e.d.

Corollary 1.5. *Let $g \geqq 0$ integrable and $\hat{U} g \leqq 1$. Then as the compact sets K increase to the state space*

$$(P_D s, g) \downarrow 0$$

uniformly for s excessive with $\|s\|_e = 1$. $D = K^C$.

Proof. Using corollary 1.4 choose t so that

(1.9) $$(P_t s, g) \leqq \varepsilon \quad \text{for all } s \text{ with } \quad \|s\|_e \leqq 1 .$$

Writing T for T_D

(1.10) $$P_D s \leqq P_t s + E^{\cdot}[s(X_T), T \leqq t] .$$

By Cauchy-Schwarz the last term in (1.10) is dominated by

(1.11) $$(E^{\cdot}[A_\infty^2])^{1/2} (P^{\cdot}(T \leqq t))^{1/2} .$$

Integrating both sides of (1.10) relative to g using (1.9) and (1.11) and again Cauchy-Schwarz

(1.12) $$(P_D s, g) \leqq \varepsilon + (E^{\cdot}[A_\infty^2], g)^{1/2} (P^{\cdot}(T \leqq t), g)^{1/2} .$$

The first term on the right of (1.2) is at most 1. $P^{\cdot}(T \leqq t) \downarrow 0$ as the compact sets K increase to the state space because the process remains in compact sets in compact time intervals. Recalling that g is integrable the proof is finished. *q.e.d.*

Proposition 1.6. *For a Borel set A the hitting potential $P_A 1$ is class (D) and has finite energy provided its mass functional is finite.*

For a finely open set O: $s = P_O 1$ has finite energy iff $L(s) < \infty$ and we have

(1.13) $$\|s\|_e^2 = L(s) + (\mu, s)$$

where μ is its Revuz measure.

Proof. Put $r = P_A 1$ and suppose $L(r) < \infty$. Let us show that r is class (D).

If $Uf_n \uparrow r$

$$\|Uf_n\|_e^2 = 2(Uf_n, f_n) \leqq 2(f_n, 1) \leqq 2\, L(r) .$$

Theorem 1.3 garantees that r is class (D). Now Meyer's energy formula tells us that r has finite energy.

Now suppose O is finely open and $s = P_O 1$. By a theorem of Hunt we can choose g_n vanishing off O such that $s_n = Ug_n \uparrow s$. We know that if $L(s) < \infty$, s has finite energy. Suppose s has finite energy. By Theorem 1.4 [7] s_n tends weakly to s in energy. In particular

$$\lim_n \langle s_n, s \rangle_e = \|s\|_e^2 .$$

But since g_n vanishes off O and $s \equiv 1$ on O

(1.14) $$\langle s_n, s \rangle_e = (g_n, s) + (\mu, s_n) = (g_n, 1) + (\mu, s_n) .$$

Letting n tend to infinity in (1.14) we get (1.13).

Proposition 1.7. *Let s be class (D) with $\|s\|_e = 1$. For $\lambda > 0$ let $O = (s > \lambda)$. Then*

(1.15) $$\lambda^2 L(P_O 1) \leqq 8 , \quad \lambda L(P_O s) \leqq 2 , \quad \lambda v(1) + (v, P_O s) \leqq \|P_O s\|_e^2 ,$$

where v is the Revuz measure of $P_O s$.

Proof. Because $\lambda P_O 1 \leqq s$, the first inequality is immediate from (1.13).

Again if $s_n = Ug_n \uparrow P_O s$ with g_n vanishing off O

$$\langle s_n, P_O s\rangle_e = (g_n, P_O s) + (v, s_n) \geqq \lambda(g_n, 1) + (v, s_n)\,.$$

We let $n \to \infty$ and use Theorem 1.3 [7] to get the last inequality in (1.15).

Since $\|P_O s\|_e^2 \leqq 2\|s\|_e^2 \leqq 2$ the second inequality is immediate from the third.

q.e.d.

Remark. Let us keep the notation of the above proposition. Now $s \leqq \lambda + P_O s$. By a famous theorem of Mokobodzki [6] we can write

$$s = s_\lambda + t_\lambda$$

where $s_\lambda \leqq \lambda$ and $t_\lambda \leqq P_O s$ and both excessive. In particular the above proposition tells us that $L(t_\lambda) \leqq 2/\lambda$. In view of this it is tempting to inquire whether given $\varepsilon > 0$ we can find $\lambda > 0$ such that it is possible to decompose every excessive s with $\|s\|_e \leqq 1$ in the form

$$s = a_\lambda + b_\lambda$$

with $a_\lambda \leqq \lambda$ and $\|b_\lambda\|_e \leqq \varepsilon$ both excessive. This however is not possible as the following example shows.

Consider the d-dimensional Brownian Motion. For $\alpha > 0$ let

$$s_\alpha(x) = \alpha^{(d-2)/2}(|x| \vee \alpha)^{-d+2}\,.$$

s_α is the potential of the uniform measure of mass $\alpha^{(d-2)/2}$ on the sphere of radius α, $\|s_\alpha\|_e^2 = 2$. The Riesz measure of s_α is concentrated on the set $(s_\alpha = \alpha^{(-d+2)/2})$. The sought for decomposition is thus not possible.

Proposition 1.8. *Hypothesis* (B) *of Hunt holds iff for each compact set K the Revuz measure of $s = P_K 1$ is concentrated on K.*

Proof. Suppose Hyp. (B) holds. We can write

$$s = \Sigma s_n = \sum_n U_{A_n} 1$$

where each s_n has finite Revuz measure.

For any open set $D \supset K$ $P_D s = s$ therefore

$$P_D s_n = s_n \quad \text{for each} \quad n\,.$$

Fix one of these s_n and put $r = s_n$. Let D open $D \supset K$. By a theorem of Hunt there exists (f_n) vanishing off D such that $Uf_n \uparrow r$.

$(f_n(x)\,dx)$ converges weakly to the Revuz measure of r, see [9].

In particular the Revuz measure of r is concentrated on $\bar{D}$ and indeed D being arbitrary on K. The same is thus true for s.

Conversely suppose the Revuz measure of s is concentrated on K. If $s_n = U_{A_n} 1$ are as above we have for each n

$$L(U_{A_n} 1_{K^c}) = 0$$

and hence

$$s_n = U_{A_n} 1_K\,, \quad s = U_A 1_K\,.$$

Since A is natural by the last identity

$$s = P_D s \quad \text{for each open } D\,, \quad D \supseteq K\,.$$

Hypothesis (B) therefore holds [4].

2. MAIN RESULTS

The following assumption will be in force throughout this section

$$(2.1)\quad \begin{cases} \text{All excessive functions are increasing limits of continuous excessive functions.} \\ \text{Hyp. (B) is valid.} \\ \text{There is a density } u(\cdot,\cdot) \text{ of the kernel } U \text{ such that for each } x \text{ the map} \\ \qquad\qquad y \to u(x, y) \\ \text{is l.s.c.} \\ \text{There exists a function } \varphi > 0 \text{ such that } 0 < \hat{U}\varphi \leqq 1 \text{ and continuous.} \end{cases}$$

We now show that we can modify u so that in addition to the above properties, it is excessive in the first variable for each fixed second variable.

Proposition 2.1. *There is a density $v(\cdot, \cdot)$ of U such that in addition to* (2.1) *we also have*

$$x \to v(x, y) \quad \textit{excessive for all} \quad y\,.$$

Proof. Step 1. Fix y_0. Choose $f_n \geqq 0$ with compact support such that $f_n(y)\,\mathrm{d}y$ converges weakly to the Dirac measure at y_0. By l.s.c.

$$(2.2)\qquad \liminf Uf_n \geqq u(\cdot, y_0)\,.$$

By assumption $\hat{U}\varphi$ is continuous and $f_n(y)\,\mathrm{d}y$ converges to the point mass at y_0:

$$(2.3)\qquad \lim_n\,(Uf_n, \varphi) = \lim_n\,(f_n, \hat{U}\varphi) = \hat{U}\,\varphi(y_0)\,.$$

Now using 2.2 we conclude

$$\lim Uf_n = u(\cdot, y_0) \quad \text{in} \quad L^1(\varphi)\,.$$

So from (2.2)

$$(2.4)\qquad \liminf U f_n(x) = u(x, y_0) \quad \text{a.e.} \quad x\,.$$

Let $v(\cdot, y_0)$ be excessive regularisation of the supermedian function $\liminf\limits_n Uf_n$. (2.4) gives that $v(x, y_0) = u(x, y_0)$ a.e. x. Naturally this can be done for each y in a measurable way. We have shown the existence of $v(\cdot, \cdot)$ such that

$$x \to v(x, y) \quad \text{excessive for all} \quad y$$
$$v(\cdot, y) = u(\cdot, y) \quad \text{a.e.} \quad x \quad \text{for all} \quad y\,,$$

This last property also tells us that

$$\hat{U}\varphi = \int v(x, \cdot)\,\varphi(x)\,\mathrm{d}x$$

which is continuous.

Step 2. v is a density for U. By Fubini and step 1 for a.e. x

$$u(x, y) = v(x, y) \quad \text{a.e.} \quad y\,.$$

In particular for $f \geqq 0$: $Uf = \int v(\cdot, y) f(y)\, dy$ a.e. and hence everywhere by excessivity.

Step 3. $y \to v(x, y)$ is l.s.c. for all x. Indeed $v(\cdot, y)$ being excessive

$$\alpha U^\alpha(v(\cdot, y)) \uparrow v(\cdot, y)\,. \tag{2.5}$$

In (2.5) we can by step 1 replace v by u. Hence for each α the expression in (2.5) is l.s.c. for each x.

The proof is concluded.

From now on we drop the v notation and stick to u. Thus we will assume that the given kernel has the additional property of being excessive in the first variable.

There are strictly positive potentials: by the maximum principle f must vanish a.e. on $(Uf = 0)$. Our assumptions guarantee that the Revuz measure of every class (D) potential is Radon. In particular by proposition 1.6 the hitting potential of every relatively compact set has finite energy.

Proposition 2.2. *Let K be a compact polar set. There is a potential s of finite energy such that*

$$K \subseteq (s = \infty)\,.$$

Proof. For each relatively compact open set D the Revuz measure μ_D of $P_D 1$ sits on $\bar{D}$. As $D \supset K$ $\mu_D(1) \downarrow 0$ because if $\mu_D \to \mu$ weakly, K being polar we have

$$0 = \liminf P_D 1 = \liminf U\mu_D \geqq U\mu$$

and hence $\mu = 0$.

For suitable D_n, $s = \sum_n P_{D_n} 1$ is then the required function.

Proposition 2.3. *The set*

$$N = (y|\ \exists D \text{ open } D \ni y\ u(\cdot, y) \neq P_D u(\cdot, y))$$

is polar.

Proof. Fix an open D and a function $g > 0$ with $\hat{U}g \leqq 1$. Put

$$A = \left(y \mid y \in D \int (u - P_D u)(x, y)\, g(x)\, dx > 0\right).$$

We claim A is polar. To show this let K be a compact subset of A. Hyp. (B) being valid, the Revuz measure ν of $s = P_K 1$ is concentrated on K by proposition 1.8. Further

$$s = U\nu \tag{2.6}$$

by [9] and proposition 2.1.

Apply P_D to both sides of (2.6) and integrate relative to $g(x)\, dx$. Since $P_D s = s$ and $K \subseteq A$ this leads to the conclusion that $\nu = 0$, i.e. A is polar.

Since $u \geqq P_D u$ and both are excessive in the first variable we must have $u(\cdot, y) = P_D u(\cdot, y)$ except perhaps for a polar subset of D.

Using a countable base of open sets the proof is concluded.

Proposition 2.4. *Let $s = Um$ be class (D). If m does not charge polar sets it must be the Revuz measure of s.*

Proof. The proof is that of theorem 5 in [2]. We supply a proof of completeness.

Suppose first that m is concentrated on a compact set K. Then $P_D s = s$ for each open D, $D \supseteq K$. There is a sequence (Uf_n):

$$Uf_n \uparrow s \quad f_n \text{ vanishing off } D .$$

The Revuz measure μ of s is then concentrated on $\bar{D}$. This being true for all open D containing K μ must be concentrated on K.

We know that $s = U\mu$, Theorem 3 [9].

For any compact set K define

$$s_K = \int_K u(\cdot, y)\, m(\mathrm{d}y) .$$

Since s dominates s_K in the strong order, μ dominates the Revuz measure of s_K, which from above must be concentrated on K. Thus, if m charges a compact set so does μ. Consider the Radon-Nikodym derivative

$$f = \frac{\mathrm{d}m}{\mathrm{d}(m + \mu)} \quad \text{and } L \text{ compact} \subseteq (f > \tfrac{1}{2})$$

and let μ_1 be the Revuz measure of s_L. Then μ_1 is concentrated on L, $\mu_1 \leqq \mu$ and

$$s_L = U\mu_1 .$$

Assuming $(m + \mu)(L) > 0$ would lead to

$$\int_L u(\cdot, y)\, \mu(\mathrm{d}y) \geqq U\mu_1 = s_L = \int_L u(\cdot, y)\, m(\mathrm{d}y) > \frac{1}{2}\int_L u(\cdot, y)\, (\mu + m)(\mathrm{d}y) .$$

Consequently by subtraction

$$\int_L u(\cdot, y)\, \mu(\mathrm{d}y) > \int_L u(\cdot, y)\, m(\mathrm{d}y) > \int_L u(\cdot, y)\, \mu(\mathrm{d}y) .$$

This contradiction forces that $f \leqq \frac{1}{2}$. On the other hand $U\mu = Um$. Thus we must have $\mu = m$. q.e.d.

Theorem 2.5. *Let $s_n = U\mu_n$ be excessive of class (D). Assume (s_n) is bounded in energy and $\lim_n s_n = s$ a.e. Then μ_n converges vaguely to the Revuz measure μ of s.*

Proof. Step 1. By assumption there is $\varphi > 0$ such that $0 < \hat{U}\varphi \leqq 1$ is continuous. By making φ smaller we may assume φ integrable.

Then $U\varphi$ has finite energy. In particular

$$((\mu_n, \hat{U}\varphi)) = ((s_n, \varphi)) \quad \text{is bounded} .$$

In other words the sequence $(\mu_n(\mathrm{d}y)\, \hat{U}\, \varphi(y))$ of measures is uniformly bounded.

Let K be compact, D its complement and denote by D the indicator of D. Then

$$UD\mu_n \leqq P_D s_n .$$

Corollary 1.5 then tells us that the above sequence of measures is indeed tight.

Let m denote a vague limit of (μ_n) along a sequence N. The above considerations imply that $(\hat{U}\, \varphi(y)\, \mu_n(\mathrm{d}y))$ converge weakly to $\hat{U}\, \varphi(y)\, m(\mathrm{d}y)$ along the sequence N.

Step 2. (s_n) converges to s a.e. and is uniformly integrable relative to $\varphi(x)\, \mathrm{d}x$ by proposition 1.2. By step 1

$$(Um, \varphi) = (m, \hat{U}\varphi) = \lim_{n \in N} (\mu_n, \hat{U}\varphi) = (s, \varphi) .$$

For any bounded measurable ϱ, $\hat{U}(\varphi\varrho)$ is also bounded and continuous. Arguing as above we find

$$(Um, \varphi\varrho) = (s, \varphi\varrho)$$

or that $s = Um$ a.e. and hence everywhere.

We claim m cannot charge polar sets. If K is compact polar there is a class (D) potential r of finite energy such that $K \subseteq (r = \infty)$ by proposition 2.2.

By l.s.c. of r

$$(m, r) \leqq \liminf\, (\mu_n, r) \leqq \|s_n\|_e\, \|r\|_e < \infty .$$

Therefore m cannot charge K. By proposition 2.4 m is the Revuz measure of s.

q.e.d.

Proposition 2.6. *The following are equivalent.*

1) *All excessive functions are increasing limits of continuous excessive functions.*

2) *There is a positive function b such that Ub is continuous.*

Proof. We need only prove that 1) implies 2). Let $f > 0$. Choose a compact K of large measure such that Uf restricted to K is continuous. Denote by K and D the indicator functions of K and the complement of K. Now

$$Uf = UfK + UfD .$$

We deduce that the restriction of UfK to K is continuous. Let p_n be continuous excessive and increase to UfK. Given $\varepsilon > 0$, by Dini there is an n such that $p_n + \varepsilon > UfK$ on K.

We have

$$p_n + \varepsilon \geqq P_K(p_n + \varepsilon) \geqq P_K(UfK) = UfK \geqq p_n ,$$

that is UfK is the uniform limit on the whole space of continuous excessive functions. Using the maximum principle we can now produce a strictly positive b such that Ub is continuous and bounded.

Theorem 2.7. *The space of class (D) potentials of finite energy is complete in energy norm.*

Proof. Let (s_n) be a sequence of class (D) potentials which converges weakly in energy. The sequence is therefore bounded in energy norm.

Along a subsequence N, s_n converges a.e. to a class (D) potential s of finite energy. It is shown in [7] that the set of potentials Uf of finite energy is dense in energy in the space of all excessive functions of finite energy. By proposition 2.6, there is $b > 0$ s.t. Ub is continuous. We may assume $b \in L^1$ and Ub is bounded.

If μ_n is the Revuz measure of s_n

$$\langle s_n, Ub\rangle_e = (\mu_n, Ub) + (b, s_n)\,, \tag{2.7}$$

(s_n) is uniformly integrable as a subset of $L^1(b)$ by proposition 1.2. Therefore

$$\lim_{n\in N} (s_n, b) = (s, b)\,. \tag{2.8}$$

Let K denote a compact set and $D = K^C$. We will also denote by the same letters their indicator functions.

Choose K such that $P_D Ub$ has small energy — see corollary 1.5. Now

$$(\mu_n, Ub) = (\mu_n, KUb) + (\mu_n, DUb) \leqq (\mu_n, KUb) + (\mu_n, P_D Ub)\,. \tag{2.9}$$

KUb is u.s.c.

By theorem 2.5 (μ_n) converges vaguely to μ

$$\limsup_{n\in N} (\mu_n, KUb) \leqq (\mu, KUb)\,. \tag{2.10}$$

The last term in (2.9) is small so from (2.10)

$$\limsup_{n\in N} (\mu_n, Ub) \leqq (\mu, Ub)\,.$$

Since Ub is l.s.c. we get finally

$$\lim_{n\in N} (\mu_n, Ub) = (\mu, Ub)\,. \tag{2.11}$$

(2.8) together with (2.11) can be written

$$\lim_{n\in N} \langle s_n, Ub\rangle = \langle s, Ub\rangle_e\,.$$

Now every excessive function is an increasing limit of continuous excessive functions of the form Ub — this is the content of proposition 2.6. By Theorem 1.4 of [7] it follows that the set

$$\{Ub\colon 0 < b \in L^1,\ Ub \text{ continuous bounded}\}$$

is weakly and hence strongly dense in energy in the space of all class (D) potentials of finite energy. In view of this, (2.11) can be reformulated to say that (s_n) converges along N, weakly in energy to s. Since the whole sequence is assumed to be weakly convergent in energy, the proof is complete.

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