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Czechoslovak Mathematical Journal, Vol. 37 (1987), No. 2, 231–247

Persistent URL: http://dml.cz/dmlcz/102151

Czechoslovak Mathematical Journal, 37 (112) 1987, Praha

SEQUENTIALLY DETERMINED CONVERGENCE SPACES

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(Received September 11, 1985)

1. INTRODUCTION AND PRELIMINARIES

It occurs repeatedly in doing analysis and functional analysis in convergence spaces that countability conditions of one kind or another are present. (See e.g. [A], [Be 1], [Be 2], [KM]). In such cases, sequential arguments are often both easier and more natural than their filter counterparts. Even series arguments are occasionally called for.

Unfortunately, in general, first countable convergence spaces are not determined by their convergent sequences. Distinct first countable convergence spaces may have precisely the same convergent sequences with the result that continuity and sequential continuity are very different notions.

The problem which arises then is the following: if λ is a convergence structure and $\Lambda(\lambda)$ is the set of all first countable convergence structures having the same convergent sequences as λ, to choose in $\Lambda(\lambda)$ a "special" convergence structure. As we know from [BeBuH], $\Lambda(\lambda)$ is a complete lattice with a largest element $\varphi(\lambda)$ and a smallest element $\gamma(\lambda)$, and so these are two natural candidates. Both have their advantages. The convergence spaces obtained in the first way were introduced by Frič and later studied by Frič and Kent, who called them sequential. Sequential convergence spaces have some very nice properties and are, moreover, very well suited to study for example the sequential continuous convergence on $\mathscr{C}(X)$ – see [FMR]. On the other hand, $\varphi(\lambda)$ is rarely topological even when λ is – for example even $\varphi(\nu)$ is not topological for the natural topology ν on the real line. Also φ does not preserve algebraic compatibility.

In this paper we study the convergence spaces which arise by choosing $\gamma(\lambda)$; we call them sequentially determined.

In Section 2 we show that large classes of convergence spaces are sequentially determined, in particular all first countable pretopological spaces, the c-duals of separable Fréchet spaces and all web-spaces. We study basic properties, such as permanence properties, and show that the notions of continuity and sequential continuity coincide for sequentially determined convergence spaces.

In Section 3, we show that γ preserves algebraic structure. We study the convergence group completion of an abelian Hausdorff convergence group due to Frič and Kent [FKe 2] and show the following: if the convergence group is sequentially determined, so is its completion and if its sequential convergence is maximal, so is that of its completion. A consequence of these results is a close relationship between this completion and the sequential convergence group completion of Novák [N]: One can always construct Novák's completion using that of Frič and Kent and under special conditions one can go the other way as well.

Although we are mainly interested in convergence spaces we will also deal with the more general notion of a filter convergence space introduced in [BeBuH]. This will enable us to better connect convergence structures defined by filters and those defined by sequences.

In the notation we follow [B]. Furthermore, if X is a set, then $P(X)$, $F(X)$ and $S(X)$ denote its power set, the set of all fiters and the set of all sequences on X, respectively. If $\xi \in S(X)$, then $\langle \xi \rangle$ is the Fréchet filter generated by ξ and sometimes we write also $\xi = (\xi(n))_{n \in N}$ or simply $\xi = (\xi(n))$.

1.1. Definition. The pair $(X, \mathscr{L})$ is called a *sequential convergence space* and $\mathscr{L}$ a *sequential convergence structure* on X if X is a set and $\mathscr{L}$ a mapping from X into the power set of $S(X)$ satisfying for all $x \in X$ the following conditions:

(S1) If $\xi \in S(X)$ and $\xi(n) = x$ for all $n \in N$, then $\xi \in \mathscr{L}(x)$.

(S2) If $\xi \in \mathscr{L}(x)$ then $\xi' \in \mathscr{L}(x)$ for each subsequence ξ' of ξ.

(S3) If $\xi \in S(X)$ and $(\xi(n + 1))_{n \in N} \in \mathscr{L}(x)$, then $\xi \in \mathscr{L}(x)$.

Both $(X, \mathscr{L})$ and $\mathscr{L}$ are called *maximal* or *Urysohn* if moreover the following is satisfied

(S4) If $\xi \in S(X)$ and if every subsequence ξ' of ξ contains a subsequence $\xi'' \in \mathscr{L}(x)$ then $\xi \in \mathscr{L}(x)$.

Finally, a mapping $f: (X, \mathscr{L}) \to (Y, \mathscr{M})$ between sequential convergence spaces is called *continuous* if $f \circ \xi \in \mathscr{M}(f(x))$ for all $x \in X$ and all $\xi \in \mathscr{L}(x)$.

1.2. Example. If (X, λ) is a convergence space then

$$\mathscr{L}(\lambda): X \to P(S(X)) \quad \text{defined by} \quad \xi \in \mathscr{L}(\lambda)(x) \quad \text{if} \quad \langle \xi \rangle \in \lambda(x) \quad \text{for all} \quad x \in X$$

is a sequential convergence structure on X.

1.3. Definition. For every sequential convergence space $(X, \mathscr{L})$ we define

$$\gamma(\mathscr{L}): X \to P(F(X))$$

by stating that a filter $\mathscr{F} \in F(X)$ belongs to $\gamma(\mathscr{L})(x)$ for some $x \in X$ if there is a filter $\mathscr{G} \subset \mathscr{F}$ with a countable basis satisfying the condition

$$S_x(\mathscr{L}) \quad \text{If} \quad \xi \in S(X) \quad \text{and} \quad \langle \xi \rangle \supset \mathscr{G} \quad \text{then} \quad \xi \in \mathscr{L}(x).$$

We set $\gamma(\lambda) = \gamma(\mathscr{L}(\lambda))$ for a convergence structure λ on X. Evidently a filter $\mathscr{G}$ with a countable basis belongs to $\gamma(\mathscr{L})(x)$ for some $x \in X$ if and only if it satisfies the condition $S_x(\mathscr{L})$. Furthermore $\gamma(\mathscr{L})(x)$ contains $\dot{x}$, the trivial ultrafilter generated by x and also $\mathscr{F} \in \gamma(\mathscr{L})(x)$ implies $\mathscr{G} \in \gamma(\mathscr{L})(x)$ for all filters $\mathscr{G} \supset \mathscr{F}$. But in general $\gamma(\mathscr{L})(x)$ is not closed under finite intersections and so $\gamma(\mathscr{L})$ is not a convergence structure on X. Therefore we define:

1.4. Definition. The pair (X, λ) is called a *filter convergence space* and λ a *filter convergence structure* on X if X is a set and λ is a mapping from X into the power set of $F(X)$ such that the following are satisfied for all $x \in X$:
(F1) $\dot{x} \in \lambda(x)$
(F2) If $\mathscr{F} \in \lambda(x)$ then $\mathscr{G} \in \lambda(x)$ for all $\mathscr{G} \in F(X)$ with $\mathscr{G} \supset \mathscr{F}$.

Clearly a filter convergence space (X, λ) is a convergence space if and only if $\mathscr{F} \cap \mathscr{G} \in \lambda(x)$ for all $x \in X$ and all $\mathscr{F}, \mathscr{G}$ in $\lambda(x)$. We transfer all definitions which are usually given for convergence spaces, like adherence, separation axioms or continuity, in the obvious way to filter convergence spaces. Also the definition of $\mathscr{L}(\lambda)$ in 1.2 makes sense for filter convergence spaces and yields a sequential convergence structure again.

Recall that a filter convergence space (X, λ) and the structure λ are called *first countable* if for every $x \in X$ and every $\mathscr{F} \in \lambda(x)$ there is a coarser filter $\mathscr{G} \subset \mathscr{F}$ such that $\mathscr{G} \in \lambda(x)$ and $\mathscr{G}$ has a countable basis.

1.5. Lemma. *If $\mathscr{L}$ is a sequential convergence structure on X then $\gamma(\mathscr{L})$ is a first countable filter convergence structure on X. Moreover, if λ is a convergence structure on X, then $\gamma(\lambda)$ is also a convergence structure.*

Proof. We only prove the last part. So assume that λ is a convergence structure on X and that $\mathscr{F}^\sim, \mathscr{G}^\sim \in \gamma(\lambda)(x)$ for some $x \in X$. Then there are filters $\mathscr{F} \subset \mathscr{F}^\sim$ and $\mathscr{G} \subset \mathscr{G}^\sim$ with countable bases

$$\{F_n: n \in \mathbf{N}\} \quad \text{and} \quad \{G_n: n \in \mathbf{N}\},$$

respectively, satisfying $S_x(\mathscr{L}(\lambda))$. Without loss of generality one may assume that $F_{n+1} \subset F_n$ and $G_{n+1} \subset G_n$ for all $n \in N$. Take any sequence $\xi \in S(X)$ with $\langle\xi\rangle \supset \supset \mathscr{F} \cap \mathscr{G}$. Define

$$I = \{i \in N\text{: there is a } k \in \mathbf{N} \text{ such that } \xi(i) \in F_k \setminus G_k\},$$

and $J = N \setminus I$. Then for all $n \in \mathbf{N}$

$$\{\xi(i): i \in I\} \cap (F_n \cup G_n) \subset F_n$$

and

$$\{\xi(i): i \in J\} \cap (F_n \cup G_n) \subset G_n .$$

If $\langle\xi\rangle \not\supset \mathscr{F}$ and $\langle\xi\rangle \not\supset \mathscr{G}$ then I and J are infinite and $(\xi(i))_{i \in I}$ and $(\xi(i))_{i \in J}$ define subsequences ξ' and ξ'' of ξ with

$$\langle\xi'\rangle \supset \mathscr{F}, \quad \langle\xi''\rangle \supset \mathscr{G} \quad \text{and} \quad \langle\xi\rangle = \langle\xi'\rangle \cap \langle\xi''\rangle .$$

The following lemma contains basic facts needed in the sequel. These can be proved by routine arguments.

1.6. Lemma. *Let X be a set, $\mathscr{L}$ a sequential convergence structure on X and λ a first countable filter convergence structure on X. Then the following hold:*

(i) $\mathrm{id}\colon (X, \mathscr{L}(\gamma(\mathscr{L}))) \to (X, \mathscr{L})$ *is continuous.*
(ii) $\mathrm{id}\colon (X, \lambda) \to (X, \gamma(\lambda))$ *is continuous.*
(iii) $\mathscr{L}(\gamma(\mathscr{L}(\lambda))) = \mathscr{L}(\lambda)$ *and* $\gamma(\mathscr{L}(\gamma(\mathscr{L}))) = \gamma(\mathscr{L})$.
(iv) $\mathscr{L}(\lambda)$ *is Hausdorff if and only if λ is Hausdorff.*
(v) *If $\mathscr{L}$ is Hausdorff then $\gamma(\mathscr{L})$ is Hausdorff.*
(vi) *λ and $\mathscr{L}(\lambda)$ share the same adherence.*

It is easy to give an internal description of sequential convergence spaces $(X, \mathscr{L})$ satisfying the condition $\mathscr{L} = \mathscr{L}(\gamma(\mathscr{L}))$. They were called (FL)-*spaces* by Frič and Kent and Proposition 2 in [BeBuH] contains a list of equivalent conditions. We note that $(X, \mathscr{L})$ is an (FL)-space if $\mathscr{L}$ is a maximal sequential convergence structure. Filter convergence spaces satisfying $\lambda = \gamma(\lambda)$ are our main objects of interest in this paper.

It was shown in [BeBuH] that $\mathscr{L}$ induces a concrete functor from the category of all filter convergence spaces into that of all sequential convergence spaces and this functor has a left adjoint. Likewise, γ induces a concrete functor from the category of all sequential convergence spaces into that of all first countable filter convergence spaces and this functor also has a left adjoint. A consequence of this is the following proposition which can also be verified by direct calculation:

1.7. Proposition.

(i) *$\mathscr{L}$ preserves embeddings and products, i.e. if $f\colon (X, \lambda) \to (Y, \mu)$ is an embedding between filter convergence spaces then $f\colon (X, \mathscr{L}(\lambda)) \to (Y, \mathscr{L}(\mu))$ is an embedding and if $\{(X_i, \lambda_i)\colon i \in I\}$ is a family of filter convergence spaces then* $(\Pi X_i, \mathscr{L}(\Pi \lambda_i)) = (\Pi X_i, \Pi\, \mathscr{L}(\lambda_i))$.
(ii) *γ preserves embeddings and countable products.*

2. SEQUENTIALLY DETERMINED CONVERGENCE SPACES

2.1. Definition. A filter convergence space (X, λ) is called *sequentially determined* if $\lambda = \gamma(\lambda)$.

Our first task will be to exhibit large classes of sequentially determined filter convergence spaces. We start with an easy but useful characterization:

2.2. Proposition. *For every filter convergence space (X, λ) the following are equivalent:*

(i) *(X, λ) is sequentially determined.*
(ii) *(X, λ) is first countable and a filter $\mathscr{F}$ with a countable basis belongs to $\lambda(x)$ for some $x \in X$ whenever it satisfies the condition* $S_x(\lambda) := S_x(\mathscr{L}(\lambda))$.
(iii) *There is a sequential convergence structure $\mathscr{L}$ on X with $\lambda = \gamma(\mathscr{L})$.*

A (filter) convergence space is called *pretopological* if for all $x \in X$

$$\mathcal{U}(x) = \bigcap\{\mathcal{F}: \mathcal{F} \in \lambda(x)\} \in \lambda(x) .$$

2.3. Proposition. *A pretopological convergence space (X, λ) is sequentially determined if and only if it is first countable.*

Proof. The non-trivial direction follows from the fact that for every filter $\mathcal{F} \in F(X)$ with a countable basis we have

$$\mathcal{F} = \bigcap\{\langle\xi\rangle: \xi \in S(X),\ \langle\xi\rangle \supset \mathcal{F}\} .$$

2.4. Definition. A filter convergence space (X, λ) is called *strongly first countable* if for each $x \in X$ there is a countable local basis of (X, λ) at x, i.e. there is a collection $\mathcal{B}_x$ of countably many subsets of X with the property that, for any filter $\mathcal{F} \in \lambda(x)$, there is a coarser filter $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G} \in \lambda(x)$ and $\mathcal{G}$ has a basis consisting of elements of $\mathcal{B}_x$.

Strong first countability is another generalization of the notion of first countability of a topological space and was thoroughly studied in [BeBu].

2.5. Proposition. *Every strongly first countable filter convergence space (X, λ) is sequentially determined.*

Proof. Evidently (X, λ) is first countable. Choose any $x \in X$ and any filter $\mathcal{F} \in F(X)$ with a countable basis satisfying $S_x(\lambda)$. Let $\mathcal{B}_x^\sim$ be a local basis of (X, λ) at x and assume $\mathcal{F} \notin \lambda(x)$. Then

$$\mathcal{B}_x = \{B \in \mathcal{B}_x^\sim : B \notin \mathcal{F}\} = \{B_n : n \in N\}$$

is also a local basis of (X, λ) at x and $\mathcal{B}_x \cap \mathcal{F} = \emptyset$. Choose a basis $\{F_n : n \in N\}$ of $\mathcal{F}$ with $F_{n+1} \subset F_n$ for all $n \in N$. Then $F_m \not\subset B_n$ for all m, n and so there are elements $x_{m,n} \in F_m \setminus B_n$ for all m, n. List the collection $\{x_{m,n} : m \geqq n\}$ to obtain a sequence $\xi = (x_{m(i),n(i)})$. We show that $\langle\xi\rangle \supset \mathcal{F}$.

For all $m \in N$ the set $\{(\mu, \nu): \nu \leqq \mu \leqq m\}$ is finite and so there is some $k_m \in N$ such that $m(i) > m$ for all $i \geqq k_m$. Hence

$$\xi(i) \in F_m \quad \text{for all} \quad i \geqq k_m .$$

Thus $\langle\xi\rangle \supset \mathcal{F}$ and so $\langle\xi\rangle \in \lambda(x)$. This implies the existence of some $r_0 \in N$ with $B_{r_0} \in \langle\xi\rangle$ and so there is some $i_0 \in N$ with

$$\{\xi(i): i \geqq i_0\} \subset B_{r_0} .$$

However, $x_{m,r_0} \notin B_{r_0}$ for all $m \geqq r_0$ and so there are infinitely many $i \in N$ with $\xi(i) \notin B_{r_0}$.

This contradiction completes the proof.

In [Be 1] the class of web-spaces was introduced in order to generalize and sharpen the classical closed graph theorem.

2.6. Corollary. *Every second countable filter convergence space is sequentially determined and also every web-space.*

Proof. Both classes are strongly first countable [BeBu].

Given a topological vector space E we denote by $\mathscr{L}_c E$ its c-dual, that is the space of all continuous linear functionals on E endowed with the continuous convergence structure [B]. In the following proposition we characterize the sequential determinedness of the c-dual of a Fréchet space:

2.7. Proposition. *For a metrizable locally convex topological vector space E the following are equivalent:*

(i) *E is separable.*

(ii) *$\mathscr{L}_c E$ is strongly first countable.*

(iii) *$\mathscr{L}_c E$ is sequentially determined.*

(iv) *$\mathscr{L}_c E$ is first countable.*

Proof. (ii) $\Rightarrow$ (iii) follows from 2.5 and (iii) $\Rightarrow$ (iv) is clear.

(i) $\Rightarrow$ (ii): Since E is second countable also $\mathscr{L}_c E$ is second countable (see [B], Theorem 79) and the result follows from 2.6.

(iv) $\Rightarrow$ (i): Let $\{U_n: n \in N\}$ be a zero neighbourhood basis of E and denote by σ_n the restriction of the weak*-topology $\sigma(\mathscr{L}E, E)$ to U_n^0, the polar of U_n. Then all (U_n^0, σ_n) are subspaces of $\mathscr{L}_c E$ and therefore also first countable. Consequently there are sequences

$\{A_{n,m}: n, m \in N\}$ of finite subsets of E such that for all $n \in N$

$$\{U_n^0 \cap A_{n,m}^0: m \in N\}$$

is a zero neighbourhood basis in (U_n^0, σ_n). Then

$$A = \bigcup \{A_{n,m}: n, m \in N\}$$

is countable. We claim that the Q-vector space generated by A is dense in E and in order to prove this it is sufficient to show that $A^0 = \{0\}$. Take any $\varphi \in A^0$. Then there is an $n \in N$ such that $\varphi \in U_n^0$ and therefore $\varphi \in U_n^0 \cap A_{n,m}^0$ for all $m \in N$. Since σ_n is Hausdorff we get $\varphi = 0$.

The following theorem summarizes the permanence properties of sequentially determined spaces and therefore yields more classes of sequentially determined spaces.

2.8. Theorem. *Subspaces, sums and countable products of sequentially determined filter convergence spaces are sequentially determined.*

Proof. Subspaces and countable products follow from 1.7, sums by an easy calculation.

As the following example shows, inductive limits and hence also quotients of sequentially determined (filter) convergence spaces may fail to be sequentially determined.

2.9. Example. Let E be an infinite dimensional Fréchet space and define a convergence structure λ on E by stating that a filter $\mathscr{F}$ in $F(E)$ belongs to $\lambda(x)$ for some

$x \in E$ if and only if $\mathscr{F}$ converges in E to x and $\mathscr{F}$ contains a subspace of E of countable dimension.

Then (E, λ) is a first countable convergence (vector) space and a (strict) inductive limit of the subspaces of E of countable dimension. Since E and (E, λ) have the same convergent sequences, we get $E = (E, \gamma(\lambda))$ and since (E, λ) is evidently not topological it is not sequentially determined.

We note that, with regard to the above example, the identity mapping

$$\text{id}: E \to (E, \lambda)$$

is sequentially continuous but clearly not continuous since otherwise E and (E, λ) would be isomorphic. This situation cannot occur for sequentially determined spaces as the following theorem shows.

2.10. Theorem. *Let (X, λ) and (Y, μ) be filter convergence spaces, (X, λ) first countable and (Y, μ) sequentially determined. Then $f: (X, \lambda) \to (Y, \mu)$ is sequentially continuous if and only if f is continuous.*

Proof. The sequential continuity of $f: (X, \lambda) \to (Y, \mu)$ is precisely the continuity of $f: (X, \mathscr{L}(\lambda)) \to (Y, \mathscr{L}(\mu))$. Since $\mathscr{L}$ is functorial, continuity implies sequential continuity. On the other hand, if f is sequentially continuous,

$$f: (X, \gamma(\mathscr{L}(\lambda))) \to (Y, \gamma(\mathscr{L}(\mu)))$$

is continuous since γ is functorial and the continuity of f now follows from 1.6(ii).

2.11. Corollary.

(i) *A mapping between sequentially determined filter convergence spaces is continuous if and only if it is sequentially continuous.*

(ii) *Two sequentially determined filter convergence structures on a set with the same sequential convergence coincide.*

(iii) *If λ is a first countable filter convergence structure on a set X, then $\gamma(\lambda)$ is the only sequentially detemined filter convergence structure on X with the same sequential convergence as λ.*

At this point let us compare $\gamma(\mathscr{L})$ with Frič's modification ([F 1]): Given a sequential convergence space $(X, \mathscr{L})$, the filter convergence structure $\varphi(\mathscr{L})$ on X is defined by

$$\mathscr{F} \in \varphi(\mathscr{L})(x) \quad \text{if there is a} \quad \xi \in \mathscr{L}(x) \quad \text{with} \quad \mathscr{F} \supset \langle \xi \rangle$$

for all $x \in X$, $\mathscr{F} \in F(X)$; Frič and Kent called a *filter convergence space* (X, λ) sequential if $\lambda = \varphi(\mathscr{L})$ for some sequential convergence structure $\mathscr{L}$ on X. As the following shows, $\varphi(\mathscr{L})$ is very seldom topological:

2.12. Proposition. *If $(X, \mathscr{L})$ is a Hausdorff* (FL)*-sequential convergence space such that $(X, \varphi(\mathscr{L}))$ is topological, then $\varphi(\mathscr{L}) = \gamma(\mathscr{L})$ and each point in $(X, \varphi(\mathscr{L}))$ has a basis consisting of countable, open, compact sets.*

Proof. Since $(X, \varphi(\mathscr{L}))$ is topological it is sequentially determined by 2.3. Since

$(X, \mathscr{L})$ is an (FL)-space, we have

$$\mathscr{L}(\varphi(\mathscr{L})) = \mathscr{L} = \mathscr{L}(\gamma(\mathscr{L}))$$

by Proposition 2 in [BeBuH] and then $\varphi(\mathscr{L}) = \gamma(\mathscr{L})$ by 2.11(iii). Take now any $x \in X$ and any neighbourhood U of x. Since $\varphi(\mathscr{L})$ is topological, there is a sequence $\xi \in \mathscr{L}(x)$ such that $\langle\xi\rangle$ is the neighbourhood filter of x. Therefore there is a $k \in \boldsymbol{N}$ such that

$$\{\xi(i): i \geqq k\} \subset U .$$

Furthermore, there is an open neighbourhood V of x and an $n \in \boldsymbol{N}$ with

$$W := \{\xi(i): i \geqq n\} \subset V \subset \{\xi(i): i \geqq k\}$$

and so W is open since $V \setminus \{\xi(i): i \geqq n\}$ is finite. Evidently W is countable and compact.

2.13. Corollary. *If a non-trivial Hausdorff topological space is either connected or uncountable Lindelöf then it is not sequential.*

Since large classes of filter convergence spaces consist of sequentially determined spaces one cannot expect that these spaces have as attractive properties as their sequential counterparts. On the other hand, one can hope that good sequential properties will imply good properties of the filter convergence space itself. One such example is given in 2.15 below.

2.14. Definition. A filter convergence space (X, λ) is called a *Choquet space* if a filter $\mathscr{F}$ on X belongs to $\lambda(x)$ for some $x \in X$ whenever every ultrafilter $\mathscr{G} \supset \mathscr{F}$ belongs to $\lambda(x)$.

Similarly, (X, λ) is called *countably Choquet* if this property holds for every $\mathscr{F}$ with a countable basis.

Choquet spaces are also called pseudotopological [FKe 1] and solid [S].

2.15. Theorem. *If $\mathscr{L}$ is a maximal sequential convergence structure on a set X, then $(X, \gamma(\mathscr{L}))$ is countably Choquet.*

Proof. Let $\mathscr{F}$ be a filter with a countable basis such that $\mathscr{F} \notin \gamma(\mathscr{L})(x)$ for some $x \in X$. We show that there is an ultrafilter $\mathscr{G} \supset \mathscr{F}$ such that $\mathscr{G} \notin \gamma(\mathscr{L})(x)$.

There is a sequence $\xi \in S(X)$ such that $\langle\xi\rangle \supset \mathscr{F}$ and $\xi \notin \mathscr{L}(x)$. By the maximality of $\mathscr{L}$ there is a subsequence η of ξ no subsequence of which is in $\mathscr{L}(x)$; take any ultrafilter $\mathscr{G} \supset \langle\eta\rangle$, we show that $\mathscr{G} \notin \gamma(\mathscr{L})(x)$. Assume on the contrary that $\mathscr{G} \in \gamma(\mathscr{L})(x)$. Then there is a filter $\mathscr{H} \in \gamma(\mathscr{L})(x)$ with a countable basis $\{H_n: n \in \boldsymbol{N}\}$ such that $\mathscr{H} \subset \mathscr{G}$. Consequently,

$$\{\eta(i): i \geqq k\} \cap H_n \neq \emptyset \quad \text{for all} \quad k, n \quad \text{in} \quad \boldsymbol{N}.$$

and so by induction one can choose a subsequence ζ of η with $\zeta(n) \in H_n$ for all $n \in \boldsymbol{N}$. But then $\langle\zeta\rangle \supset \mathscr{H}$ and so $\langle\zeta\rangle \in \gamma(\mathscr{L})(x)$, implying $\zeta \in \mathscr{L}(\gamma(\mathscr{L}))(x) = \mathscr{L}(x)$ since $\mathscr{L}$ is maximal, contradicting the choice of η.

2.16. Corollary. *If (X, λ) is a sequentially determined filter convergence space such that $\mathscr{L}(\lambda)$ is maximal, then (X, λ) is countably Choquet.*

As the following examples show, the above results cannot be essentially strengthened:

2.17. Examples.

(i) Let $\mathscr{A} = \{A \subset \boldsymbol{R}: \sum_{a \in A} |a| < \infty\}$ and define a convergence structure λ on $\boldsymbol{R}$ as follows:

$\mathscr{F} \in \lambda(x)$ for $x \neq 0$ if $\mathscr{F} = \dot{x}$

$\mathscr{F} \in \lambda(0)$ if $\mathscr{F}$ converges to 0 in the natural topology and if $\mathscr{F} \cap \mathscr{A} \neq \emptyset$.

It it easy to see that any subsequence of $\xi = (1/n)$ contains a subsequence in $\mathscr{L}(\lambda)(0)$ but $\xi \notin \mathscr{L}(\lambda)(0)$. Thus $(\boldsymbol{R}, \lambda)$ is Choquet and sequentially determined but $\mathscr{L}(\lambda)$ is not maximal.

(ii) Define a sequential convergence structure $\mathscr{L}$ on $X = [0, 1]$ as follows:

$$\xi \in \mathscr{L}(x) \quad \text{for} \quad x \neq 0 \quad \text{if} \quad \{i: \xi(i) \neq x\} \quad \text{is finite}$$
$$\xi \in \mathscr{L}(0) \quad \text{if} \quad \xi^{-1}(\{x\}) \quad \text{is finite for all} \quad x \neq 0\,.$$

We claim that $(X, \gamma(\mathscr{L}))$ is not a Choquet space. A filter $\mathscr{F} \in F(X)$ belongs to $\gamma(\mathscr{L})(0)$ if and only if there is a countable family $\{F_n: n \in \boldsymbol{N}\} \subset \mathscr{F}$ with $\bigcap\{F_n: n \in \boldsymbol{N}\} \subset \{0\}$. Therefore

$$\mathscr{F}_0 = \{F \subset X: X \setminus F \text{ is finite}\} \notin \gamma(\mathscr{L})(0)\,.$$

But if $\mathscr{G} \supset \mathscr{F}_0$ is an ultrafilter then $\mathscr{G}$ converges in the natural topology to a point $x \in X$ and so there is a countable family $\{U_n: n \in \boldsymbol{N}\}$ of neighbourhoods of x in the natural topology with $\bigcap\{U_n: n \in \boldsymbol{N}\} = \{x\}$. Since $\mathscr{G} \supset \mathscr{F}_0$ we have $\mathscr{G} \neq \dot{x}$ and so $U_0 = X \setminus \{x\} \in \mathscr{G}$, implying that

$$\{U_n: n \in \boldsymbol{N} \cup \{0\}\} \subset \mathscr{G} \quad \text{and} \quad \bigcap\{U_n: n \in \boldsymbol{N} \cup \{0\}\} = \emptyset\,.$$

Consequently $\mathscr{G} \in \gamma(\mathscr{L})(0)$ for all ultrafilters $\mathscr{G} \supset \mathscr{F}_0$. Evidently $\mathscr{L}$ is maximal and so $(X, \gamma(\mathscr{L}))$ is countably Choquet and sequentially determined, but, as was just shown, it is not a Choquet space.

We cannot characterize sequentially determined Choquet or countably Choquet spaces by means of their sequential convergence. For strongly first countable filter convergence spaces, however, we know more: such a space is countably Choquet if and only if its sequential convergence is maximal (see [BeBu]).

3. SEQUENTIALLY DETERMINED CONVERGENCE GROUPS AND THEIR COMPLETIONS

As usual, a (filter) convergence group or a sequential convergence group is a group carrying a (filter) convergence structure or a sequential convergence structure, respectively, making the composition and the inverse operations continuous. Since we are mainly interested in abelian groups, the group operation will be assumed to be addition. We start with a result which, in spite of its easy proof, is of fundamental importance.

3.1. Proposition. *If $(G, \mathscr{L})$ is a sequential convergence group, then $(G, \gamma(\mathscr{L}))$ is a filter convergence group.*

Proof. Since the addition is $\mathscr{L} \times \mathscr{L} - \mathscr{L}$-continuous, it is also $\gamma(\mathscr{L} \times \mathscr{L}) - \gamma(\mathscr{L})$-continuous. But $\gamma(\mathscr{L} \times \mathscr{L}) = \gamma(\mathscr{L}) \times \gamma(\mathscr{L})$ by 1.7 and so the addition is a continuous operation on $(G, \gamma(\mathscr{L}))$. Evidently the inverse operation is $\gamma(\mathscr{L}) - \gamma(\mathscr{L})$-continuous.

3.2. Corollary. *If (G, λ) is a filter convergence group, then $(G, \gamma(\lambda))$ is also a filter convergence group.*

Proof. Clearly $(G, \mathscr{L}(\lambda))$ is a sequential convergence group.

Similarly one can show that if $(A, \mathscr{L})$ is a sequential convergence vector space or a sequential convergence algebra, then $(A, \gamma(\mathscr{L}))$ is a filter convergence vector space or a filter convergence algebra, respectively. Thus γ preserves algebraic compatibility. This is not true for φ as the following shows:

3.3. Proposition. *Let $(G, \mathscr{L})$ be a Hausdorff sequential convergence group. Then $(G, \varphi(\mathscr{L}))$ is a filter convergence group if and only if $\mathscr{L}$ is discrete. In particular, if (G, λ) is a non-discrete first countable topological group, then $(G, \varphi(\mathscr{L}(\lambda)))$ is not a filter convergence group.*

Proof. We show the "only if" part. If $\mathscr{L}$ is not discrete, there is a sequence $\xi = (x_n) \in \mathscr{L}(0)$ with

$$x_m \neq x_n \neq 0 \quad \text{for all} \quad m \neq n\,.$$

We claim that $\langle\xi\rangle + \langle\xi\rangle \notin \varphi(\mathscr{L})(0)$. Assume on the contrary that there is a sequence $\eta = (y_n) \in \mathscr{L}(0)$ with $\langle\xi\rangle + \langle\xi\rangle \supset \langle\eta\rangle$. Then there is a $k \in N$ such that

$$\{y_i \colon i \in N\} \supset \{x_i + x_j \colon i, j \geqq k\}\,.$$

Consequently,

$$x_n + x_k \in \{y_i \colon i \in N\} \quad \text{for all} \quad n \geqq k\,,$$

and so there are $i_n \in N$ such that $x_n + x_k = y_{i_n}$ for all $n \geqq k$. Since $x_m + x_k \neq x_n + x_k$ for all $m \neq n$, we have $i_m \neq i_n$ for all $m \neq n$ and so there is a subsequence $(y_{i_{n_r}})$ of η such that

$$x_{n_r} + x_k = y_{i_{n_r}} \quad \text{for all} \quad r \in N\,.$$

It follows that the constant sequence with the value x_k converges to 0. This contradiction completes our proof.

As 3.3 shows, one has to consider in all non-trival cases the (filter) convergence group modification $(G, q, +)$ of $(G, \varphi(\mathscr{L}))$. This was done in [F 1] but then the situation often gets rather complicated, as for example the lengthy proof of 3.2 there shows.

This modification differs from $\gamma(\mathscr{L})$ even for maximal convergences as the following example shows: Denote by $\mathscr{L}$ the natural convergence on the reals. Then

$\gamma(\mathscr{L})$ is the natural topology on the reals. On the other hand, every q-convergent filter contains a countable set.

Let (G, λ) be a filter convergence group with the property that $\mathscr{F} \cap 0^{\bullet} \in \lambda(0)$ whenever $\mathscr{F} \in \lambda(0)$. Then for all $\mathscr{F}, \mathscr{G} \in \lambda(0)$

$$\mathscr{F} \cap \mathscr{G} \supset (\mathscr{F} \cap 0^{\bullet}) + (\mathscr{G} \cap 0^{\bullet}) \in \lambda(0)$$

and so (G, λ) is indeed a convergence group. Therefore, the final filter convergence group structure on a group G with respect to a family of group homomorphisms from convergence groups into G is again a convergence structure. Consequently, quotients, direct sums (coproducts) and (inductive) limits of convergence groups in the category of filter convergence groups are convergence spaces and therefore coincide with the corresponding constructions in the category of convergence groups. In addition to the permanence properties proved in 2.8 we get the following for groups:

3.4. Proposition. *If $(G_i, \lambda_i)_{i \in I}$ is a family of sequentially determined filter convergence groups, then $\bigoplus_{i \in I} (G_i, \lambda_i)$ is also sequentially determined.*

Proof. Clearly $(G, \lambda) = \bigoplus_{i \in I} (G_i, \lambda_i)$ is first countable. For any $i \in I$ we denote by $\pi_i: (G, \lambda) \to (G_i, \lambda_i)$ the projection and for any finite set $J \subset I$ we set $(G_J, \lambda_J) = \prod_{i \in J} (G_i, \lambda_i)$. Finally, we denote by $e_J: (G_J, \lambda_J) \to (G, \lambda)$ the embedding. Then λ is the final filter convergence structure with respect to the family (e_J). Take any $z \in G$ and any filter $\mathscr{F} \in F(G)$ with a countable basis satisfying the condition $S_x(\lambda)$. We first show the existence of a finite set $J \subset I$ with $e_J(G_J) \in \mathscr{F}$: If this is not the case, then we construct a sequence (z_n) in G and a sequence (i_n) of pairwise distinct elements in I with the property that $z_n \in F_n$ and $\pi_{i_n}(z_n) \neq 0$ for all $n \in N$. We do this by induction as follows: Clearly $F_1 \neq \{0\}$ and so there are $z_1 \in F_1$ and $i_1 \in I$ such that $\pi_{i_1}(z_1) \neq 0$. If $z_1, \ldots, z_n$ and $i_1, \ldots, i_n$ have been chosen we set $J = \{i_1, \ldots, i_n\}$. Since $F_{n+1} \not\subset e_J(G_J)$ there are $z_{n+1} \in F_{n+1}$ and $i_{n+1} \notin J$ such that $\pi_{i_{n+1}}(z_{n+1}) \neq 0$.

Since $\langle (z_n) \rangle \supset \mathscr{F}$ we have $\langle (z_n) \rangle \in \lambda(z)$ and so there is a finite set $J \subset I$ with $e_J(G_J) \in \mathscr{F}$. Therefore there is an $n_0 \in N$ with

$$\pi_i(z_n) = 0 \quad \text{for all} \quad i \in I \setminus J \quad \text{and all} \quad n \geqq n_0 .$$

But $\{i_n: n \geqq n_0\}$ is infinite and so $i_k \notin J$ for some $k \geqq n_0$, implying that

$$\pi_{i_k}(z_k) \neq 0 \quad \text{and} \quad i_k \in I \setminus J \quad \text{and} \quad k \geqq n_0 .$$

This contradiction shows that there is indeed a finite set $J \subset I$ with $e_J(G_J) \in \mathscr{F}$. Without loss of generality, we may also assume that $z \in e_J(G_J)$; let $u \in G_J$ be such that $e_J(u) = z$. Now $e_J: (G_J, \gamma(\lambda_J)) \to (G, \gamma(\lambda))$ is an embedding by 1.7 and therefore $e_J^{-1}(\mathscr{F}) \in \gamma(\lambda_J)(u)$. But $\gamma(\lambda_J) = \lambda_J$ by 2.8 and so $\mathscr{F} = e_J(e_J^{-1}(\mathscr{F})) \in \lambda(z)$.

A direct sum of uncountably many non-trivial sequentially determined convergence groups is perhaps the simplest example of a sequentially determined space which is not strongly first countable.

Up to now, the theories for sequentially determined convergence spaces and for their filter convergence counterparts are more or less parallel. The situation becomes different, however, in dealing with completions. It is well-known that every abelian, Hausdorff convergence group has a completion. An analogous result can be shown for abelian, Hausdorff filter convergence groups. But the filter convergence group completion of a non-complete convergence group is not a convergence space and therefore differs from its convergence group completion. As was already pointed out, our main interest lies in the study of convergence spaces and therefore in what follows we will study the convergence group completion.

3.5. Definition. The triple $(\hat{G}, \hat{\lambda}, e_G)$ is called a *completion* of the abelian, Hausdorff convergence group (G, λ) if $(\hat{G}, \hat{\lambda})$ is a complete abelian, Hausdorff convergence group and e_G is an embedding from (G, λ) onto a dense subgroup of $(\hat{G}, \hat{\lambda})$ satisfying the following property:

(EP) Given a complete abelian, Hausdorff convergence group (H, μ) and a continuous group homomorphism $f: (G, \lambda) \to (H, \mu)$ there is a unique continuous group homomorphism

$$\hat{f}: (\hat{G}, \hat{\lambda}) \to (H, \mu) \quad \text{such that} \quad \hat{f} \circ e_G = f.$$

The following theorem summarizes the main results of [FKe 2]:

3.6. Theorem. *Every abelian Hausdorff convergence group has a completion.*

Evidently, by (EP), a completion of a convergence group (G, λ) is unique up to isomorphism. We will refer to it as the completion and denote it by $(\hat{G}, \hat{\lambda}, e_G)$. We also need the following results the first of which was proved in [FKe 2].

3.7. Lemma. *If $\mathscr{H}$ is a filter which converges in the completion of the abelian Hausdorff convergence group (G, λ), then there are elements $z_1, \dots, z_n \in \hat{G}$ such that $(z_1 + e_G(G)) \cup \dots \cup (z_n + e_G(G)) \in \mathscr{H}$.*

3.8. Lemma. *Let $\mathscr{H}$ be a Cauchy filter in a first countable filter convergence group (G, λ). Then there exists a Cauchy filter $\mathscr{H}_0 \subset \mathscr{H}$ with a countable basis.*

Proof. There is a filter $\mathscr{D} \in \lambda(0)$ with a countable basis $\{D_n: n \in N\}$ such that $\mathscr{D} \subset \mathscr{H} - \mathscr{H}$. Choose for all $n \in N$ a set $H_n \in \mathscr{H}$ such that $D_n \supset H_n - H_n$ and $H_{n+1} \subset H_n$ for all $n \in N$. Then $\{H_n: n \in N\}$ is the basis of a Cauchy filter $\mathscr{H}_0$ in (G, λ) such that $\mathscr{H}_0 \subset \mathscr{H}$.

The following two theorems illustrate properties preserved by the completion.

3.9. Theorem. *Let $(\hat{G}, \hat{\lambda}, e_G)$ be the completion of the abelian Hausdorff convergence group (G, λ). Then the following hold:*

(i) *If λ is first countable, then $\hat{\lambda}$ is first countable.*

(ii) *If (G, λ) is sequentially determined, then $(\hat{G}, \hat{\lambda})$ is also sequentially determined.*

Proof.

(i) Assume that λ is first countable and define a convergence structure μ on $\hat{G}$

by stating for all $z \in \hat{G}$ and all $\mathscr{H} \in F(\hat{G})$:

$\mathscr{H} \in \mu(z)$ if there is a filter $\mathscr{H}_0 \in \hat{\lambda}(z)$ with a countable basis such that $\mathscr{H} \supset \mathscr{H}_0$. Evidently $(\hat{G}, \mu)$ is a convergence group and $e_G: (G, \lambda) \to (\hat{G}, \mu)$ an embedding. We show that $(\hat{G}, \mu)$ is complete.

Given a Cauchy filter $\mathscr{H}$ on $(\hat{G}, \mu)$ there is by 3.8 a Cauchy filter $\mathscr{H}_0 \subset \mathscr{H}$ with a countable basis. Since $\mathscr{H}_0$ converges in $(\hat{G}, \hat{\lambda})$, it also converges in $(\hat{G}, \mu)$.

Since $e_G: (G, \lambda) \to (\hat{G}, \mu)$ is continuous, by (EP) also id: $(\hat{G}, \hat{\lambda}) \to (\hat{G}, \mu)$ is continuous and so $\hat{\lambda} = \mu$.

(ii) Assume that λ is sequentially determined and that $\mathscr{H} \in \gamma(\hat{\lambda})(z)$ has a countable basis $\{H_n: n \in N\}$. We first show that there are elements $w_1, \ldots, w_k \in \hat{G}$ with $(w_1 + e_G(G)) \cup \ldots \cup (w_k + e_G(G)) \in \mathscr{H}$. If this is not the case, one can inductively find $z_n \in H_n$ such that

$$z_n \notin (z_1 + e_G(G)) \cup \ldots \cup (z_{n-1} + e_G(G)) \quad \text{for all} \quad n \geqq 2\,.$$

But then $\langle (z_n) \rangle \supset \mathscr{H}$ and so $\langle (z_n) \rangle \in \hat{\lambda}(z)$. On the other hand we have that $z_n - z_k \notin$ $\notin e_G(G)$ for all $n \neq k$, contradicting 3.7. So we can choose $w_1, \ldots, w_k$ in $\hat{G}$ with

$$(w_1 + e_G(G) \cup \ldots \cup (w_k + e_G(G)) \in \mathscr{H}\,.$$

Without loss of generality we may assume that $\mathscr{H}$ has a trace on every $w_i + e_G(G)$ and so for every $i \in \{1, \ldots, k\}$

$$\{H_n \cap (w_i + e_G(G)): n \in N\}$$

is the basis of a filter $\mathscr{H}_i$ on $\hat{G}$. We have

$$\mathscr{H} = \mathscr{H}_1 \cap \ldots \cap \mathscr{H}_k\,.$$

Since $\gamma(\hat{\lambda})$ is Hausdorff by 1.6 we are done if we prove that $\mathscr{H}_i$ is a Cauchy filter in $(\hat{G}, \hat{\lambda})$ for every $i \in \{1, \ldots, k\}$.

Since $(\hat{G}, \gamma(\hat{\lambda}))$ is a convergence group by 3.2,

$$(\mathscr{H}_i - w_i) - (\mathscr{H}_i - w_i) = \mathscr{H}_i - \mathscr{H}_i \in \gamma(\hat{\lambda})(0)\,.$$

If now

$$\mathscr{F} = \{F \subset G: e_G(F) \in \mathscr{H}_i - w_i\}\,,$$

then $e_G(\mathscr{F}) = \mathscr{H}_i - w_i$ and so

$$e_G(\mathscr{F} - \mathscr{F}) = e_G(\mathscr{F}) - e_G(\mathscr{F}) = (\mathscr{H}_i - w_i) - (\mathscr{H}_i - w_i) \in \gamma(\hat{\lambda})(0)\,.$$

But $e_G: (G, \gamma(\lambda)) \to (\hat{G}, \gamma(\hat{\lambda}))$ is an embedding by 1.7 and therefore $\mathscr{F}$ is a Cauchy filter in (G, λ) and so $\mathscr{H}_i - w_i = e_G(\mathscr{F})$ is a Cauchy filter in $(\hat{G}, \hat{\lambda})$.

3.10. Theorem. *If (G, λ) is a sequentially determined, abelian Hausdorff convergence group and $(\hat{G}, \hat{\lambda}, e_G)$ its completion, then $\mathscr{L}(\hat{\lambda})$ is maximal if and only if $\mathscr{L}(\lambda)$ is maximal.*

Proof. The "only if" part is clear so assume that $\mathscr{L}(\lambda)$ is maximal. Take an element $z \in \hat{G}$ and a sequence $\zeta \in S(\hat{G})$ with the property that every subsequence contains a subsequence in $\mathscr{L}(\hat{\lambda})(z)$. With the help of 3.7 one can show in the usual way that there are $w_1, \ldots, w_k$ in $\hat{G}$ with $(w_1 + e_G(G)) \cup \ldots \cup (w_k + e_G(G)) \in \langle\zeta\rangle$.

Since $\hat{\lambda}$ is a convergence structure we may assume that $k = 1$, i.e.

$$w + e_G(G) \in \langle\zeta\rangle \quad \text{for some} \quad w \in \hat{G}\,.$$

Since $(\hat{G}, \hat{\lambda})$ is Hausdorff, it is enough to show that $\langle\zeta\rangle$ is a Cauchy filter. By 3.9(ii) we know that $(\hat{G}, \hat{\lambda})$ is sequentially determined and so consider any $\eta \in S(\hat{G})$ with $\langle\eta\rangle \supset \langle\zeta\rangle - \langle\zeta\rangle$. Since

$$e_G(G) = (w + e_G(G)) - (w + e_G(G)) \in \langle\zeta\rangle - \langle\zeta\rangle \subset \langle\eta\rangle\,,$$

we can assume that $\eta(n) \in e_G(G)$ for all $n \in \boldsymbol{N}$ and since $\mathscr{L}(\lambda)$ is maximal and e_G an embedding we are done if we show that η contains a subsequence which converges to 0.

Now there is an $n_1 \in \boldsymbol{N}$ with

$$\{\zeta(i) - \zeta(j)\colon i, j \in \boldsymbol{N}\} \supset \{\eta(n)\colon n > n_1\}$$

and so one can choose i_1 and j_1 with

$$\eta(n_1) = \zeta(i_1) - \zeta(j_1)\,.$$

Since

$$\{\zeta(i) - \zeta(j)\colon i > i_1,\ j > j_1\} \in \langle\eta\rangle\,,$$

there are $n_2 > n_1$, $i_2 > i_1$ and $j_2 > j_1$ with

$$\eta(n_2) = \zeta(i_2) - \zeta(j_2)\,.$$

Inductively one gets strictly monotone sequences (n_k), (i_k), and (j_k) with

$$\eta(n_k) = \zeta(i_k) - \zeta(j_k) \quad \text{for all} \quad k \in \boldsymbol{N}\,.$$

By assumption there is a strictly monotone sequence (k_r) such that both $(\zeta(i_{k_r}))$ and $(\zeta(j_{k_r}))$ converge to z and so $(\eta(n_{k_r}))$ converges to 0.

The above theorem allows us to set this completion theory in relation to the completion theory for abelian maximal sequential convergence groups developed in [N].

3.11. Proposition. *If $(G, \mathscr{L})$ is an abelian sequential convergence group and ξ a sequence in G, then $\langle\xi\rangle$ is a Cauchy filter in $(G, \gamma(\mathscr{L}))$ if and only if $\xi \circ \alpha - \xi \circ \beta \in \mathscr{L}(0)$ for all finite-to-one mappings $\alpha, \beta\colon \boldsymbol{N} \to \boldsymbol{N}$.*

Proof. Assume that $\langle\xi\rangle$ is a Cauchy filter in $(G, \gamma(\mathscr{L}))$ and that $\alpha, \beta\colon \boldsymbol{N} \to \boldsymbol{N}$ are finite-to-one mappings. Then

$$\langle\xi \circ \alpha\rangle = \langle\xi \circ \beta\rangle = \langle\xi\rangle$$

and therefore

$$\langle\xi \circ \alpha - \xi \circ \beta\rangle \supset \langle\xi \circ \alpha\rangle - \langle\xi \circ \beta\rangle = \langle\xi\rangle - \langle\xi\rangle \in \gamma(\mathscr{L})(0)$$

and so $\xi \circ \alpha - \xi \circ \beta \in \mathscr{L}(0)$.

Conversely, if ξ has the required property and η is a sequence in G with $\langle\eta\rangle \supset$

$\supset \langle \xi \rangle - \langle \xi \rangle$, then for all $n \in \boldsymbol{N}$ there is a $k_n \in \boldsymbol{N}$ with

$$\{\xi(i) - \xi(j): i, j \geqq n\} \supset \{\eta(k): k \geqq k_n\}\,.$$

Consequently, for all $k \geqq k_1$ there are $\alpha(k)$ and $\beta(k)$ with

$$\eta(k) = \xi(\alpha(k)) - \xi(\beta(k))$$

and

$$\alpha(k),\ \beta(k) \geqq n \quad \text{if} \quad k \geqq k_n\,.$$

But then α and β are finite-to-one mappings and we have therefore $\eta \in \mathscr{L}(0)$, implying that $\langle \xi \rangle - \langle \xi \rangle \in \gamma(\mathscr{L})(0)$.

3.12. Corollary. *If $(G, \mathscr{L})$ is a maximal abelian sequential convergence group and ξ a sequence in G, then $\langle \xi \rangle$ is a Cauchy filter in $(G, \gamma(\mathscr{L}))$ if and only if $\xi' - \xi'' \in \mathscr{L}(0)$ for all subsequences ξ' and ξ'' of ξ.*

Proof. We have to show that $\xi \circ \alpha - \xi \circ \beta \in \mathscr{L}(0)$ for all finite-to-one mappings α, β if $\xi' - \xi'' \in \mathscr{L}(0)$ for all subsequences ξ' and ξ'' of ξ. Now any subsequence of $\xi \circ \alpha - \xi \circ \beta$ contains a subsequence of the form $\xi' - \xi''$ where $\xi'$and ξ'' are subsequences of ξ and so $\xi \circ \alpha - \xi \circ \beta \in \mathscr{L}(0)$ by the maximality of $\mathscr{L}$.

J. Novák called a sequence ξ in a maximal, abelian Hausdorff sequential convergence group $(G, \mathscr{L})$ a *Cauchy sequence* if $\xi' - \xi'' \in \mathscr{L}(0)$ for all subsequences ξ' and ξ'' of ξ and called *$(G, \mathscr{L})$-complete* if every Cauchy sequence in $(G, \mathscr{L})$ converges. He then showed that $(G, \mathscr{L})$ can be densely embedded into a complete, abelian Hausdorff sequential convergence group. (See [N].) We are now in a position to prove this result in the framework of the theory of convergence group completions.

3.13. Lemma. *A maximal, abelian Hausdorff sequential convergence group $(G, \mathscr{L})$ is complete if and only if $(G, \gamma(\mathscr{L}))$ is complete.*

Proof. The "if" part follows directly from 3.12. Now, assume that $\mathscr{H}$ is a Cauchy filter in $(G, \gamma(\mathscr{L}))$. By 3.8 there is a Cauchy filter $\mathscr{H}_0 \subset \mathscr{H}$ with a countable basis. Choose a sequence $\xi \in S(G)$ with $\langle \xi \rangle \supset \mathscr{H}_0$. Then ξ is a Cauchy sequence in the complete space $(G, \mathscr{L})$. Since $\xi \in \mathscr{L}(x)$ for some $x \in G$ from $\langle \xi \rangle \in \gamma(\mathscr{L})(x)$ and

$$\mathscr{H} \supset \mathscr{H}_0 \supset \mathscr{H}_0 - \mathscr{H}_0 + \langle \xi \rangle$$

we get $\mathscr{H} \in \gamma(\mathscr{L})(x)$. This proves the "only if" part.

The sequential convergence structure $\mathscr{L}(\widehat{\gamma(\mathscr{L})})$ will be of great importance to us in what follows. Therefore we introduce a simpler and more suggestive notation for it. For any sequential group convergence structure $\mathscr{L}$ on a group G we set

$$\hat{\mathscr{L}} = \mathscr{L}(\widehat{\gamma(\mathscr{L})})\,.$$

3.14. Theorem. *Let $(G, \mathscr{L})$ be a maximal abelian Hausdorff sequential convergence group. Then $(\hat{G}, \hat{\mathscr{L}})$ is a complete maximal abelian Hausdorff sequential convergence group and e_G is an embedding onto a dense subgroup.*

Proof. Since $\mathscr{L}$ is maximal we have $\mathscr{L} = \mathscr{L}(\gamma(\mathscr{L}))$ and so $\mathscr{L} = \hat{\mathscr{L}}(\widehat{\gamma(\mathscr{L})})$ is maximal according to 3.10. But then $(\hat{G}, \hat{\mathscr{L}}) = (\hat{G}, \mathscr{L}(\widehat{\gamma(\mathscr{L})}))$ is complete by 3.13 and 3.10. Since $\gamma(\mathscr{L})$ is first countable, $\widehat{\gamma(\mathscr{L})}$ has the same property by 3.9(i) and so $e_G(G)$ is dense in $(\hat{G}, \hat{\mathscr{L}})$ by 1.6(vi). Finally, e_G is an embedding by 1.7.

3.15. Theorem. *Let $(G, \mathscr{L})$ and $(H, \mathscr{M})$ be maximal abelian Hausdorff sequential convergence groups and $(H, \mathscr{M})$ complete. Then to every continuous group homomorphism $f: (G, \mathscr{L}) \to (H, \mathscr{M})$ there is a unique group homomorphism $\hat{f}: (\hat{G}, \hat{\mathscr{L}}) \to$ $\to (H, \mathscr{M})$ such that $\hat{f} \circ e_G = f$.*

Proof. The uniqueness is clear. Since $(H, \gamma(\mathscr{M}))$ is complete by 3.13, we get from 3.6 the existence of a continuous group homorphism $\hat{f}: (G, \widehat{\gamma(\mathscr{L})}) \to (H, \gamma(\mathscr{M}))$ with $\hat{f} \circ e_G = f$. But then $\hat{f}$ is also $\mathscr{L}(\widehat{\gamma(\mathscr{L})}) - \mathscr{L}(\gamma(\mathscr{M}))$ – continuous and since $\mathscr{M}$ is maximal we have $\mathscr{M} = \mathscr{L}(\gamma(\mathscr{M}))$.

3.16. Corollary. *Let $(G, \mathscr{L})$ be a maximal abelian Hausdorff sequential convergence group. Then $(\hat{G}, \hat{\mathscr{L}})$ is isomophic to Novák's completion $(\tilde{G}, \tilde{\mathscr{L}})$.*

Proof. This is a direct consequence from 3.15 and the fact that the analogous statement of 3.15 also holds for $(\tilde{G}, \tilde{\mathscr{L}})$, as was shown in [F 2] (see also [R]).

The above results show that Novák's completion can always be constructed from the convergence group completion. The next two results show that if (G, λ) has a maximal sequential convergence, then the convergence group completion can be constructed from that of Novák. That this condition cannot be relaxed is clear since $\hat{\mathscr{L}}$ is always maximal.

3.17. Theorem. *If $(G, \mathscr{L})$ is a maximal abelian sequential convergence group then $(\hat{G}, \widehat{\gamma(\mathscr{L})}) = (\hat{G}, \gamma(\hat{\mathscr{L}}))$.*

Proof. $\gamma(\hat{\mathscr{L}}) = \gamma(\mathscr{L}(\widehat{\gamma(\mathscr{L})})) = \widehat{\gamma(\mathscr{L})}$ by 3.8(ii).

3.18. Corollary. *If (G, λ) is an abelian sequentially determined Hausdorff convergence group such that $\mathscr{L}(\lambda)$ is maximal, then*

$$(\hat{G}, \hat{\lambda}) = (\hat{G}, \gamma(\widehat{\mathscr{L}(\lambda)})) .$$

Proof. $\gamma(\widehat{\mathscr{L}(\lambda)}) = \widehat{\gamma(\mathscr{L}(\lambda))} = \hat{\lambda}$ by 3.17.

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