Rosa Anna Marinosci

A unified approach to some theorems on homogeneous Riemannian and affine spaces

Czechoslovak Mathematical Journal, Vol. 37 (1987), No. 1, 151–156

Persistent URL: http://dml.cz/dmlcz/102143

Czechoslovak Mathematical Journal, 37 (112) 1987, Praha

A UNIFIED APPROACH TO SOME THEOREMS ON HOMOGENEOUS RIEMANNIAN AND AFFINE SPACES*)

ROSA ANNA MARINOSCI, Lecce

(Received September 16, 1985)

0. Introduction. In this note we present a unified version of some theorems on homogeneous Riemannian and affine spaces. W. Ambrose and I. M. Singer proved ([1]) that a connected, complete, simply connected Riemannian manifold (M, g) is homogeneous (i.e., its full group of isometries $I(M)$ acts transitively on M) if and only if there exists a skew-symmetric tensor field S on (M, g) such that $\nabla_X R = S_X(R)$ and $\nabla_X S = S_X(S)$ for any vector field X on M (∇ is the Levi-Civita connection of (M, g) and R its curvature tensor). K. Sekigawa gave a characterization of homogeneous almost-Hermitian manifolds in a similar way ([7]). The affine case was investigated by B. Kostant in 1960; he proved in [3] that a connected and simply connected manifold M with an affine connection ∇ is a reductive homogeneous space with respect to a connected Lie group G of ∇-affine transformations of M if and only if there exists a complete connection $\tilde{\nabla}$ on M such that $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}S = 0$ where $\tilde{R}$ and $\tilde{T}$ are the curvature and torsion tensors of $\tilde{\nabla}$ respectively and S is the difference tensor $\nabla - \tilde{\nabla}$.

Here we give a very short proof of each of these theorems using essentially some concepts and theorems of the theory of generalized symmetric spaces ([5]).

In order to give a self-contained presentation of the results we recall, in section 1 below, some notions on Riemannian and affine manifolds and in particular on affine reductive homogeneous spaces. We shall follow essentially the book "Generalized symmetric spaces" by O. Kowalski ([5], [6]). The reader may see also [2] for more details on Propositions A 1, A 2, A 3, A 4, A 6, A 7.

1. Proposition A 1. *Let M and M' be connected and simply connected, complete analytic Riemannian manifolds. Then every isometry between connected open subsets of M and M' can be uniquely extended to an isometry between M and M'.*

Proposition A 2. *Let M be a differentiable manifold with an affine connection ∇ such that $\nabla T = 0$ and $\nabla R = 0$. With respect to any atlas consisting of normal coordinate systems, M is an analytic manifold and the connection ∇ is also analytic.*

---

*) This research was partially supported by funds of M.P.I..

Proposition A 3. *Let M and M' be differentiable manifolds with the affine connections ∇ and ∇' respectively. Assume*: $\nabla T = 0$, $\nabla R = 0$ *and* $\nabla' T' = 0$, $\nabla' R' = 0$. *If F is a linear isomorphism of $T_{x_0}(M)$ onto $T_{y_0}(M')$ mapping the tensors T_{x_0} and R_{x_0} at x_0 into the tensors T'_{y_0} and R'_{y_0} at y_0 respectively, then there is an affine isomorphism f of a normal neighborhood U_{x_0} onto a normal neighborhood V_{y_0} such that $f(x_0) = y_0$ and $(f)_{*x_0} = F$.*

Proposition A 4. *In Proposition* A 3 *let M and M' be connected, simply connected and complete. Then there exists a unique affine isomorphism f of M onto M' such that $f(x_0) = y_0$ and the differential of f at x_0 coincides with F.*

Proposition A 5. *Let ∇ and $\tilde{\nabla}$ be affine connections on a differentiable manifold M such that the tensor field S defined by $S_X Y = \nabla_X Y - \tilde{\nabla}_X Y$ satisfies $\tilde{\nabla} S = 0$. Then for all X and Y vector fields on M we have*

(1) $$\tilde{T}(X, Y) = T(X, Y) - S_X Y + S_Y X$$

(2) $$\tilde{R}(X, Y) = R(X, Y) - [S_X, S_Y] - S_{\tilde{T}(X,Y)}$$

Proof. The first equation follows immediately from the definition of T and $\tilde{T}$. The relation $\tilde{\nabla} S = 0$ implies:

$$\tilde{\nabla}_X S_Y - S_{\tilde{\nabla}_X Y} - S_Y \tilde{\nabla}_X = 0\,,$$

or equivalently

$$[\tilde{\nabla}_X, S_Y] = S_{\tilde{\nabla}_X Y}\,.$$

Hence we get:

(3) $$\begin{aligned}[\nabla_X, \nabla_Y] &= [\tilde{\nabla}_X + S_X, \tilde{\nabla}_Y + S_Y] = \\ &= [\tilde{\nabla}_X, \tilde{\nabla}_Y] + [\tilde{\nabla}_X, S_Y] + [S_X, \tilde{\nabla}_Y] + [S_X, S_Y] = \\ &= [\tilde{\nabla}_X, \tilde{\nabla}_Y] + S_{\tilde{\nabla}_X Y} - S_{\tilde{\nabla}_Y X} + [S_X, S_Y] = \\ &= [\tilde{\nabla}_X, \tilde{\nabla}_Y] + S_{[X,X]} + S_{\tilde{T}(X,Y)} + [S_X, S_Y]\,.\end{aligned}$$

Since

(4) $$\nabla_{[X,Y]} = \tilde{\nabla}_{[X,Y]} + S_{[X,Y]}$$

we obtain (2) subtracting (4) from (3).

Now let G be a connected Lie group and let H be one of its closed subgroups; let $\mathbf{g}$ and $\mathbf{h}$ be the Lie algebras of G and H respectively; we say that the homogeneous space G/H is reductive if there exists a subspace $\mathbf{m}$ of $\mathbf{g}$ such that $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ and $\mathrm{ad}(H)\,\mathbf{m} \subseteq \mathbf{m}$. Let X_a be a tangent vector of the tangent space $T_a(G)$ at $a \in G$; let p be a point of G/H; we define the tangent vector X_p^* as follows:

$$X_p^* = (\mathrm{d}/\mathrm{d}t)_{t=0}\, \tau_{\exp_a tX}(p)\,,$$

where τ is the natural action of G on G/H and $\exp_a = L_a \circ \exp \circ (L_{a^{-1}})_*$; more geometrically X_p^* is the tangent vector at p to the orbit $t \to \tau_p(\exp_a tX_a)$. If G/H is a reductive homogeneous space then the following is true:

Proposition A 6. *There exists a unique G-invariant affine connection ∇ on G/H*

such that: $(\nabla_{X^*}Y)_{p_0} = [X^*, Y]_{p_0}$ *for each* $X \in \mathbf{m}$ *and for each vector field* Y *on* G/H (*where* $p_0 = \pi(e)$ *and* π *is the canonical projection of* G *on* G/H).

The above connection is called the *canonical connection* of the reductive homogeneous space G/H.

Some geometrical properties of the canonical connection are the following:

a) For each $X \in \mathbf{m}$ the parallel displacement of the tangent vector at p_0 along the curve $t \to \tau_{p_0}(\exp tX)$ $(0 \leqq t \leqq s)$ coincides with the differential $(\tau_{\exp sX})_{*p_0}$.

b) For each $X \in \mathbf{m}$ the curve $t \to \tau_{p_0}(\exp tX)$ is a geodesic with respect to ∇; conversely each geodesic of ∇ starting from p_0 is of the form: $t \to \tau_{p_0}(\exp tX)$ for some X of $\mathbf{m}$.

c) The connection ∇ is complete.

Proposition A 7. ([2]). *Any G-invariant tensor field S on* G/H *is parallel with respect to the canonical connection* ∇.

As a consequence of this proposition we have that, in particular, the curvature tensor R and the torsion tensor T of ∇ are parallel tensor field.

Now let (M, ∇) be a connected manifold with an affine connection. An affine transformation $f: M \to M$ is called a *transvection* of (M, ∇) if for each point $p \in M$ there is a piece-wise differentiable curve starting at p and ending at $f(p)$ such that the tangent map $(f_*)_p$ coincides with the parallel displacement along this curve. It is obvious that the set $\mathrm{Tr}(M)$ of all transvections of (M, ∇) is a normal subgroup of the group $A(M)$ of all affine transformations of M. The following proposition gives an intrinsic characterization of all manifolds with affine connection ∇ which come from reductive homogeneous spaces.

Proposition A 8. *Let M be a connected manifold with an affine connection* ∇; *the following conditions are equivalent*:

(i) *The transvection group* $\mathrm{Tr}(M)$ *acts transitively on each holonomy bundle* $P(u)$, *where* $u \in L(M)$ *is a tangent frame.*

(ii) *M can be expressed as the reductive homogeneous space* G/H *with respect to a decomposition* $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, *where G is effective on M and* ∇ *is the canonical connection of* G/H.

Moreover if (ii) *is satisfied, then* $\mathrm{Tr}(M)$ *is a connected Lie group, namely, it is a normal subgroup of G and its Lie algebra is isomorphic to the ideal* $1 = \mathbf{m} + [\mathbf{m}, \mathbf{m}]$ of $\mathbf{g}$ (see [5] p. 37).

The following definition (see [5], p. 41) is a consequence of the previous theorem:

Definition 1.1. A connected manifold (M, ∇) with an affine connection is called an *affine reductive space* if the group $\mathrm{Tr}(M)$ acts transitively on each holonomy bundle $P(u) \subset L(M)$.

Proposition A 9. *On an affine reductive space* (M, ∇) *a tensor field is parallel if it is invariant with respect to the transvection group* $\mathrm{Tr}(M)$.

Proof. It is a consequence of Proposition A 7, Proposition A 8 and Definition 1.1.

Proposition A 10. *Let* $(M, \tilde{\nabla})$ *be a connected and simply connected manifold with a complete affine connection such that* $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$. *Then* $(M, \tilde{\nabla})$ *is an affine reductive space* (see [5], p. 44).

2. We present now a unified version of theorems by Ambrose and Singer ([1]), by Sekigawa ([7]), and by Kostant ([3]) which follows naturally from the results given in section 1. We shall start with the following basic lemma.

Basic lemma 2.1.

A) *Let* (M, g) *be a homogeneous Riemannian manifold; then there exists a metric connection* $\tilde{\nabla}$ *such that:*

a$_1$) $\tilde{\nabla}\tilde{R} = 0$ *and* a$_2$) $\tilde{\nabla}S = 0$, *where* $S = \nabla - \tilde{\nabla}$, ∇ *is the Levi-Civita connection of* (M, g) *and* $\tilde{R}$ *is the curvature tensor of* $\tilde{\nabla}$.

B) *Let* (M, g) *be a connected, simply connected and complete Riemannian manifold and suppose that there exists a metric connection* $\tilde{\nabla}$ *satisfying* a$_1$) *and* a$_2$); *then* (M, g) *is homogeneous and* $(M, \tilde{\nabla})$ *is an affine reductive space.*

Proof of part A). Let $M = G/H$ be a Riemannian homogeneous manifold; as well-known M is also reductive. Let $\tilde{\nabla}$ be its canonical connection, $\tilde{\nabla}$ is a G-invariant connection; but also the Levi-Civita connection ∇ is G-invariant, hence the difference tensor $S = \nabla - \tilde{\nabla}$ is G-invariant. Now we apply $\tilde{\nabla}$ to the G-invariant tensors g, $\tilde{R}$, S and by Proposition A 7 we obtain $\tilde{\nabla}g = 0$, $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}S = 0$, so $\tilde{\nabla}$ is metric and satisfies $-$ a$_1$) and a$_2$).

Proof of part B)[1]). We must prove that for any two points $x, y \in M$ there exists an isometry f of M such that $f(x) = y$. Because the torsion tensor T of the Levi-Civita connection ∇ is zero, the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ has, by formula (1) of Proposition A 5, the following expression:

$$\tilde{T}(X, Y) = S_Y X - S_X Y,$$

so that by condition a$_2$), we get $\tilde{\nabla}\tilde{T} = 0$.

Now $\tilde{\nabla}\tilde{R} = 0$ and $\tilde{\nabla}\tilde{T} = 0$ imply (by Proposition A 2) that $(M, \tilde{\nabla})$ is analytic; since $\tilde{\nabla}$ is metric, also (M, g) is analytic.

For any $x, y \in M$ we consider the $\tilde{\nabla}$-parallel displacement $h_\gamma^{x,y}$ along any piece-wise differentiable curve γ joining x to y. Because $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$ and $\tilde{\nabla}g = 0$, $h_\gamma^{x,y}$ maps the tensors $\tilde{R}_x$, $\tilde{T}_x$, g_x onto $\tilde{R}_y$, $\tilde{T}_y$, g_y respectively and hence, by Proposition A 3, there exists a local affine diffeomorphism f such that $(f_*)_x = h_\gamma^{x,y}$. Because $(f_*)(g_x) = g_y$ and $\tilde{\nabla}g = 0$ f is also a local isometry. Because (M, g) is connected, simply connected, complete and analytic, then by Proposition A1, f may be extended to a global isometry of (M, g). The fact that $(M, \tilde{\nabla})$ is an affine reductive space follows

[1]) This proof was suggested to us by a very short proof of the Ambrose-Singer's theorem in [6].

from Proposition A 10 because the connection $\tilde{\nabla}$ is metric and hence also complete (see [5] p. 25–26).

Theorem 2.2 (By W. Ambrose and I. M. Singer)

A′) *Let (M, g) be a homogeneous Riemannian manifold then there exists a skew-symmetric tensor field S of type $(1, 2)$ on M such that for any vector field X on M:* a′$_1$) $\nabla_X R = S_X(R)$ *and* a′$_2$) $\nabla_X S = S_X(S)$ *where ∇ and R denote the Levi-Civita connection and the curvature tensor field of (M, g) respectively.*

B′) *Let (M, g) be a connected, simply connected, complete Riemannian manifold of class $\mathscr{C}^\infty$ and suppose that there is a skew-symmetric tensor field S of type* $(1, 2)$ *satisfying the above conditions* a′$_1$) *and* a′$_2$). *Then (M, g) is homogeneous.*

Proof. It is sufficient to prove the equivalence of the following conditions:

(i) On a Riemannian manifold (M, g) there exists a metric connection $\tilde{\nabla}$ satisfying a$_1$) and a$_2$) of the Basic lemma 2.1.

(ii) On a Riemannian manifold (M, g) there exists a skew-symmetic tensor field S of type $(1, 2)$ satisfying the above conditions a′$_1$) a′$_2$).

Proof of (i) ⇒ (ii): put $S = \nabla - \tilde{\nabla}$, where ∇ is the Levi-Civita connection and R its curvature tensor. Because ∇ and $\tilde{\nabla}$ are metric, we have $S(g) = 0$ i.e. S is skew-symmetric. The conditions a$_1$), a$_2$) and formula (1) and (2) of Proposition A 5 give $\tilde{\nabla}_X R = 0$ (for any vector field X on M), hence $\nabla_X R = \tilde{\nabla}_X R + S_X(R) = S_X(R)$; a′$_2$) follows immediately from $\tilde{\nabla} + S = \nabla$ and a$_2$).

Proof of (ii) ⇒ (i): Put $\tilde{\nabla} = \nabla - S$. Then $\tilde{\nabla} R = 0$, $\tilde{\nabla} S = 0$ and $\tilde{\nabla} g = 0$ according to a′$_1$) a′$_2$). Now we use (1) and (2) of Proposition A 5 to get $\tilde{\nabla}\tilde{R} = 0$.

Lemma 2.3.

A) *Let (M, g, J) be a homogeneous almost Hermitian manifold; then there exists a metric connection $\tilde{\nabla}$ such that:*
a$_1$) $\tilde{\nabla}\tilde{R} = 0$; a$_2$) $\tilde{\nabla} S = 0$; a$_3$) $\tilde{\nabla} J = 0$, *where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection and $\tilde{R}$ is the curvature of $\tilde{\nabla}$.*

B) *Let (M, g, J) be a connected, simply connected and complete almost Hermitian manifold and suppose that there is a metric connection $\tilde{\nabla}$ satisfying* a$_1$) a$_2$) a$_3$); *then (M, g, J) is a homogeneous almost Hermitian manifold.*

Proof of part A). It is a consequence of part A of the Basic lemma 2.1 and Proposition A 7.

Proof of part B). Because a$_1$) and a$_k$) are satisfied we obtain from part B of the Basic lemma 2.1 that (M, g) is homogeneous and $(M, \tilde{\nabla})$ is an affine reductive space. Because $\tilde{\nabla} J = \tilde{\nabla} g = 0$, the almost complex structure J and the metric g are invariant with respect to the transvection group $G = \mathrm{Tr}(M)$ of $(M, \tilde{\nabla})$ (see Proposition A 9). Then G is a transvection group of holomorphic isometries of (M, g, J) and this completes the proof.

From Lemma 2.3 we obtain immediately the following theorem:

Theorem 2.4 (By K. Sekigawa).

A′) *Let* (M, g, J) *be a homogeneous almost Hermitian manifold. Then there exists a skew-symmetric tensor field* S *of type* $(1,2)$ *on* M *satisfying* $a_1')$ $a_2')$ *of* Theorem 2.2 *and, furthermore,* $a_3')$ $\nabla_X J = S_X(J)$.

B′) *Let* (M, g, J) *be a connected, simply connected and complete almost Hermitian manifold and suppose that there exists a skew-symmetric tensor field* S *of type* $(1, 2)$ *on* M *satisfying the above conditions* $a_1')$ $a_2')$ *and* $a_3')$. *Then* (M, g, J) *is a homogeneous almost Hermitian manifold.*

Theorem 2.5 (By B. Kostant).

A) *Let* $M = G/H$ *be a reductive homogeneous space with a G-invariant connection* ∇. *Then there exists a complete G-invariant connection* $\tilde{\nabla}$ *such that* $a_1)$ $\tilde{\nabla}\tilde{T} = 0$, $\tilde{\nabla}\tilde{R} = 0$ *and* $a_2)$ $\tilde{\nabla}S = 0$ *where* $S = \nabla - \tilde{\nabla}$ *and* $\tilde{T}, \tilde{R}$ *are the torsion and curvature tensor of* $\tilde{\nabla}$ *respectively.*

B) *Let* (M, ∇) *be a connected, simply connected affine manifold with an affine connection and suppose that there is a complete connection* $\tilde{\nabla}$ *satisfying the above conditions* $a_1)$ *and* $a_2)$. *Then* M *is a reductive homogeneous space* G/H *and* ∇, $\tilde{\nabla}$ *are G-invariant connections.*

Proof of part A). If $M = G/H$ is a reductive homogeneous space, then its canonical connection $\tilde{\nabla}$ is complete and satisfies $a_1)$. Since ∇ and $\tilde{\nabla}$ are G-invariant, the tensor $S = \nabla - \tilde{\nabla}$ is also G-invariant, so by Proposition A 7 we obtain $a_2)$.

Proof of part B). We apply Proposition A 10 and obtain that $(M, \tilde{\nabla})$ is an affine reductive homogeneous space. Let G be the transvection group of $(M, \tilde{\nabla})$; then we get a reductive homogeneous space $(M = G/H, \tilde{\nabla})$ where $\tilde{\nabla}$ is the canonical connection.

The above condition $a_2)$ and Proposition A 9 imply that S is G-invariant. It follows that $\nabla = \tilde{\nabla} + S$ is also G-invariant and this completes the proof.

References

[1] *Ambrose W. - Singer I. M.:* On homogeneous Riemannian manifolds, Duke Math. J. *25* (1958) p. 647—669.

[2] *Kobayashi S. - Nomizu K.:* Foundations of differential geometry, Vol. I and II (1963 and 1969) New York Interscience Publishers.

[3] *Kostant B.:* A characterization of invariant affine connections, Nagoya Math. J. *16* (1960) p. 35—50.

[4] *Kowalski O.:* Affine reductive spaces, Beitrage zur Algebra und Geometrie *8* (1979) p. 99—105.

[5] *Kowalski O.:* Generalized symmetric spaces, Lecture notes in mathematics *805*, Springer-Verlag 1980.

[6] *Kowalski O.:* Generalized symmetric spaces, revised Russian version, MIR, Moscow 1984.

[7] *Sekigawa K.:* Notes on homogeneous almost Hermitian manifolds, Hokkaido Math. J. Vol. 7 (1978) p. 206—213.

[8] *Tricerri F. - Vanecke L.:* Homogeneous Riemannian structures, London Math. Soc. Lecture Notes Series 1983 — Cambridge University Press.

Author's address: Dipartimento di Matematica, Universitá di Lecce, Via Arnesano, 73100 Lecce, Italy.