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CLOSURE OPERATORS ON RADICAL CLASSES OF LATTICE-ORDERED GROUPS

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1. INTRODUCTION AND BACKGROUND

We will assume that the reader is familiar with the common terms, definitions, theorems, and notation of lattice-ordered groups (the reader unfamiliar with these can find them either in [2] or [6]) and go on to the following less familiar terms or more recent developments.

An *l-variety* is an equationally-defined collection of l-groups; from a well-known result due to Birkhoff [3, p. 153], a collection $\mathscr{V}$ of l-groups is an l-variety if and only if $\mathscr{V}$ is closed with respect to l-subgroups, l-homomorphic images, and cardinal products of its elements. The l-varieties used in this paper are $\mathscr{E}$, the l-variety of one-element l-groups; $\mathscr{L}$, the collection of all l-groups; $\mathscr{N}$, the l-variety of normal-valued l-groups; and $\mathscr{A}$, the l-variety of abelian l-groups. A *torsion class* [18] is a collection of l-groups closed with respect to convex l-subgroups, joins of convex l-subgroups, and l-homomorphic images. A *quasitorsion class* [16] is a collection closed with respect to convex l-subgroups, joins of convex l-subgroups, and complete l-homomorphic images. A *radical class* [14] is a collection of l-groups closed with respect to l-isomorphic images, convex l-subgroups, and joins of convex l-subgroups. It is known [12] that every l-variety is a torsion class and clearly every torsion class is a quasitorsion class and every quasitorsion class is a radical class.

For every radical class $\mathscr{R}$ and l-group G, there exists a convex l-subgroup $\mathscr{R}(G)$ of G which is the join of all convex l-subgroups of G that are in $\mathscr{R}$. By definition, $\mathscr{R}(G) \in \mathscr{R}$ and clearly $\mathscr{R}(G)$ is an l-ideal of G [14]. $\mathscr{R}(G)$ is called the *$\mathscr{R}$-kernel* of G.

For any radical class $\mathscr{R}$, the mapping $G \to \mathscr{R}(G)$ on the collection of l-groups has the following two properties:

i) For any $C \in \mathscr{C}(G)$, $\mathscr{R}(C) = C \cap \mathscr{R}(G)$,

ii) if $\phi\colon G \to H$ is an l-bijection, $\phi[\mathscr{R}(G)] = \mathscr{R}(H)$.

Conversely, any mapping f on the class of l-groups satisfying the above properties is called a *radical operator* or a *radical mapping*; such radical operators always define a unique radical class $\mathscr{R}$ such that $\mathscr{R}(G) = f(G)$ for every l-group G [14].

If $\mathscr{R}$ and $\mathscr{S}$ are radical classes, define $\mathscr{R}\cdot\mathscr{S}$ to be the class of l-groups G such that $\mathscr{S}(G/\mathscr{R}(G)) = G/\mathscr{R}(G)$. $\mathscr{R}\cdot\mathscr{S}$ is then a radical class [15], called the *product* of $\mathscr{R}$ with $\mathscr{S}$. It is known that the product of l-varieties remains an l-variety [17] and that the product of torsion classes remains a torsion class [18].

Inclusion gives a natural lattice ordering on the collection of radical classes of l-groups. This lattice is then complete and Brouwerian [14]. The subcollections of torsion classes and quasitorsion classes are complete sublattices of this lattice ([18] and [16], respectively.)

Every l-group G can be represented as a set of order-preserving permutations of a totally-ordered set Ω [11]. When such a representation is important, we will use the notation (G, Ω) and refer to (G, Ω) as an *l-permutation group*.

If G is an l-group and (H, Ω) is an l-permutation group, the *wreath product* of G with (H, Ω) [13], denoted $G \mathscr{W}r\,(H, \Omega)$, is the l-group defined on the set ${}^{\Omega}G \times H$ with binary operation $(f_1, h_1)(f_2, h_2) = (g, h_1h_2)$, where $g\colon \Omega \to G\colon \lambda \to f_1(\lambda)f_2(\lambda h_1)$ and with (f, h) being positive if for all $\lambda \in \Omega$, $\lambda h \geqq \lambda$ and for all $\lambda \in \Omega$ with $\lambda h = \lambda$, $f(\lambda) \geqq e$ in G. The *restricted wreath product* of G with (H, Ω), denoted $G\, wr\,(H, \Omega)$, is the l-subgroup of $G \mathscr{W}r\,(H, \Omega)$ consisting of those (f, h) such that $f(\lambda) = e$ for all but a finite number of $\lambda \in \Omega$.

The group operation of an l-group will be denoted multiplicatively. $\boldsymbol{Z}$ will denote the group of integers and $\boldsymbol{R}$ the group of real numbers under the usual additions and orders. For any l-group G, $G\, wr\,(n)\, \boldsymbol{Z}$ denotes the l-subgroup of $G \mathscr{W}r\,(\boldsymbol{Z}, \boldsymbol{Z})$ of all elements (f, h) such that if $k \equiv m \pmod n$, $f(k) = f(m)$.

2. THE PRIME PRODUCT OF TWO LATTICE-ORDERED GROUPS

In [1] was developed a general theory for obtaining extensions of one l-group A by another l-group B. Let $\mathscr{U}(B)$ denote the lattice of principal convex l-subgroups of B, $\mathscr{S}(A)$ the lattice of cardinal summands of A, and $\mathscr{O}(A)$ the l-automorphisms of A. Let π be a lattice homomorphism of $\mathscr{U}(B)$ into $\mathscr{S}(A)$ and σ a group homomorphism of B into $\mathscr{O}(A)$ such that:

i) $\pi((e)) = (e)$

ii) $\sigma(b)\,[\pi(B(b))]'$ is the identity for any $b \in B$, and

iii) for any $b_1, b_2 \in B$, $\sigma(b_1)\,[\pi(B(b_2))] = \pi(B(b_1^{-1}b_2b_1))$.

Then, defining $(a_1, b_1)(a_2, b_2) = (a_1\sigma(b_1)(a_2), b_1b_2)$ and defining $(a, b) \geqq (e, e)$ if $b \geqq e$ and $b'(a) \geqq e$ (where $a = \hat{b}(a)\, b'(a)$ and $\hat{b}(a) \in \pi(B(b))$ and $b'(a) \in [\pi(B(b))]'$), $A \times B$ is then an l-group, called in this paper the *upper product of A by B determined by π and σ*, and will be denoted $A \times_{\pi,\sigma} B$.

Among the standard extensions of one l-group by another that are upper products for suitable π and σ are cardinal sums, wreath products, and if B is an o-group, lex extensions. We add one more now.

Let P be a prime l-ideal of B. Then for any b_1 and b_2 in B, $B(b_1) \cap B(b_2) \subseteq P$ implies that either b_1 or b_2 is an element of P. Thus, if we let $\sigma\colon B \to \mathscr{O}(A)$ be the trivial homomorphism and define

$$\pi(B(b)) = \begin{cases} A, & b \notin P, \\ (e), & b \in P \end{cases}$$

π and σ satisfy conditions (i), (ii), and (iii). We will call this upper product the *P-product* of A and B and will denote this by $A \times_P B$. Note that if $P = B$, we have the cardinal sum of A and B and if B is an o-group and $P = (e)$, $A \times_P B = A \times^{\leftarrow} B$. Note that $(a, b) \geqq (e, e)$ in $A \times_P B$ if $b \in B^+ \setminus P$ or $b \in P^+$ and $a \in A^+$.

According to the theory developed in [1], $A \times P$ is a prime subgroup of $A \times_P B$. For every $(a, e) \in A \times_P B$, a value of (a, e) is of the form $M \times P$, where M is a value of a in A. Thus $A \times P$ contains every value of any element of the form (a, e) and so, if P is a regular subgroup of B then $A \times_P P$ is an essential value in $A \times_P B$ and so is closed. This fact will be used often in the next section in the building of examples.

3. THE ORDER-CLOSURE OPERATOR

In [10], Conrad developed the theory of K-radical classes. A K-radical class $\mathscr{R}$ is a radical class such that for any l-group $G \in \mathscr{R}$, $\mathscr{K}(G)$ is an element of a predetermined class T of lattices. Some of the more prominent K-radical classes are the completely-distributive l-groups ($\mathscr{C}d$), the class of l-groups whose root system of regular subgroups has a minimal plenary subset ($\mathscr{E}ss$), the class of l-groups with bases ($\mathscr{B}as$), the special-valued l-groups ($\mathscr{S}pec$) which are those l-groups in which every positive element is a join of disjoint special elements, and the archimedean l-groups ($\mathscr{A}rch$). For any K-radical class $\mathscr{R}$, it is known [10] that $\mathscr{R}(G)$ is a closed l-ideal. In this section, we deal with closed-kernel radical classes, showing that for each radical class $\mathscr{R}$, there exists a minimal closed-kernel radical class $\mathscr{R}^c$ containing $\mathscr{R}$ and we discuss this closure property.

Theorem 3.1. *Let $\mathscr{R}$ be a radical class and G an l-group. Define $\mathscr{R}^c(G) = \mathrm{cl}(\mathscr{R}(G))$, the order closure of $\mathscr{R}(G)$ in G. Then $G \to \mathscr{R}^c(G)$ is a radical mapping and $\mathscr{R}^c$ is the least closed-kernel radical class containing $\mathscr{R}$.*

Proof. Let $C \in \mathscr{C}(G)$. $\mathrm{cl}^C(\mathscr{R}(C)) = \mathrm{cl}^C(C \cap \mathscr{R}(G)) = \mathrm{cl}^G(C \cap \mathscr{R}(G)) \cap C \subseteq$ $\subseteq \mathrm{cl}^G(\mathscr{R}(G)) \cap C$. If $e \leqq x \in \mathrm{cl}^G(\mathscr{R}(G)) \cap C$, $x = \bigvee g_\alpha$ for $\{g_\alpha\} \subseteq \mathscr{R}(G)^+$ and since C is convex, each g_α is an element of C. Thus $x \in \mathrm{cl}^C(\mathscr{R}(C))$ and so $\mathscr{R}^c(C) = C \cap \mathscr{R}^c(G)$.

Now let $\phi: G \to H$ be a bijective l-homomorphism. Then $\phi[\mathscr{R}(G)] = \mathscr{R}(H)$ and so $\phi[\mathscr{R}^c(G)] = \mathscr{R}^c(H)$. So we have a radical mapping.

Now if $\mathscr{S}$ is a closed-kernel radical class containing $\mathscr{R}$, clearly $\mathscr{R}^c(G) = \mathrm{cl}(\mathscr{R}(G)) \subseteq \mathrm{cl}(\mathscr{S}(G)) = \mathscr{S}(G)$ for every l-group G and so $\mathscr{R}^c \subseteq \mathscr{S}$. $\square$

The following proposition, which is easy to prove, shows that the map $\mathscr{R} \to \mathscr{R}^c$ is a closure operator on the lattice of radical classes.

Proposition 3.2. *Let $\mathscr{R}$ and $\mathscr{S}$ be radical classes.*

a) $\mathscr{R}^{cc} = \mathscr{R}^c$

b) *If* $\mathscr{R} \subseteq \mathscr{S}$, $\mathscr{R}^c \subseteq \mathscr{S}^c$

c) $(\mathscr{R} \cap \mathscr{S})^c = \mathscr{R}^c \cap \mathscr{S}^c$.

In contrast to the later closure operators we will consider, for any radical class $\mathscr{R}$, $\mathscr{R}^c$ is usually a radical class that closely resembles $\mathscr{R}$. In fact, quite a nice result holds if we take the closed-kernel closure of a quasitorsion class.

Theorem 3.3. *If $\mathscr{R}$ is a quasitorsion class, so is $\mathscr{R}^c$.*

Proof. Suppose $G = \mathrm{cl}(\mathscr{R}(G))$ and that K is a closed l-ideal of G. Then $\mathscr{R}(G) \cap K$ is a closed l-ideal of $\mathscr{R}(G)$ and

$$\frac{\mathscr{R}(G)\, K}{K} \cong \frac{\mathscr{R}(G)}{\mathscr{R}(G) \cap K}$$

which is in $\mathscr{R}$.

However, $\mathrm{cl}(\mathscr{R}(G/K)) = A/K$ for some closed convex l-subgroup A of G. Since

$$\frac{\mathscr{R}(G)\, K}{K} \subseteq \mathscr{R}\left(\frac{G}{K}\right),$$

$G = \mathrm{cl}(\mathscr{R}(G)) \subseteq \mathrm{cl}(\mathscr{R}(G)\, K) \subseteq A$, implying $\mathrm{cl}(\mathscr{R}(G/K) = G/K$. □

The above result is not true for torsion classes, however, as the following example shows.

Example 3.4. Let $\mathscr{F}\!\mathit{in}$ be the torsion class of finite-valued l-groups and $\mathscr{S}\!\mathit{pec}$ the quasitorsion class of special-valued l-groups. Then $\mathscr{F}\!\mathit{in} \subset \mathscr{S}\!\mathit{pec}$ and so $\mathscr{F}\!\mathit{in}^c \subseteq \subseteq \mathscr{S}\!\mathit{pec}^c = \mathscr{S}\!\mathit{pec}$.

Let $G = \prod_{i=1}^{\infty} \boldsymbol{Z}$. G is then in $\mathscr{F}\!\mathit{in}^c$ but $G/\sum_{i=1}^{\infty} \boldsymbol{Z}$ is not special-valued and so not in $\mathscr{F}\!\mathit{in}^c$. □

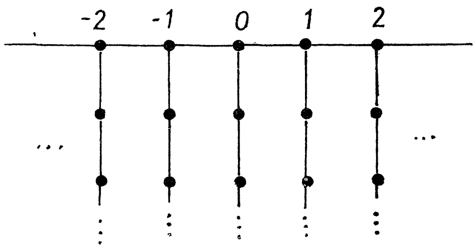
We should remark here that not every closed-kernel quasitorsion class is $\mathscr{T}^c$ for some torsion class $\mathscr{T}$. The largest torsion class contained in $\mathscr{A}\mathit{rch}$ is $\mathscr{H}\!\mathit{yp}$, the hyperarchimedean l-groups, but $\mathscr{H}\!\mathit{yp}(\boldsymbol{C}(\boldsymbol{R})) = (0)$ and so $\mathscr{H}\!\mathit{yp}^c(\boldsymbol{C}(\boldsymbol{R})) = (0)$, implying that $\mathscr{A}\mathit{rch} \neq \mathscr{H}\!\mathit{yp}^c$ and thus $\mathscr{A}\mathit{rch} \neq \mathscr{T}^c$ for any torsion class $\mathscr{T}$.

At this time, it is not known whether $\mathscr{S}\!\mathit{pec} = \mathscr{T}^c$ for some torsion class $\mathscr{T}$. In fact, as far as the author knows, the (unique) largest torsion class contained in $\mathscr{S}\!\mathit{pec}$ has not been determined. Along these lines, we can remark that for any positive integer n, $(\mathscr{F}\!\mathit{in}^c)^n \neq \mathscr{S}\!\mathit{pec}$, as for any positive integer n, $\boldsymbol{Z}\,\mathscr{W}\!\mathit{r}^{n+1}\,\boldsymbol{Z}$ is in $\mathscr{S}\!\mathit{pec}$ but not in $(\mathscr{F}\!\mathit{in}^c)^n$.

Theorem 3.5. *The closed-kernel radical classes form a complete lattice under inclusion, where $\mathscr{R} \wedge_K \mathscr{S} = \mathscr{R} \cap \mathscr{S}$ and $\mathscr{R} \vee_K \mathscr{S} = (\mathscr{R} \vee \mathscr{S})^c$.*

The following example shows that the lattice of closed-kernel radical classes is not a sublattice of the lattice of radical classes.

Example 3.6. Let

$\Delta =$ 

and for each integer n of the real line, let V_n be the Hahn group $V(\Gamma_n, \boldsymbol{R})$, where

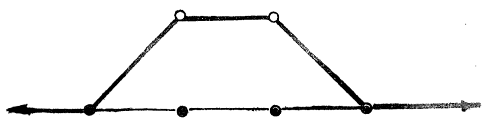
$\Gamma_n =$ 

For each integer n, we identify the point n of the real line with the top point of Γ_n to get Δ above.

Let G be the set of functions on Δ generated by $\sum_{-\infty}^{\infty} V_n$ and the continuous functions on the top real line. G is then an l-subgroup of $V(\Delta, \boldsymbol{R})$.

$\mathcal{A}rch(G) = \{g \in G\colon g(\delta) = 0$ for all $\delta \in \bigcup_{-\infty}^{\infty} \Gamma_n$ and such that g has bounded support on the top real line$\}$, while $\mathcal{S}pec(G) = \sum_{-\infty}^{\infty} V_n$. Since $\mathcal{A}rch(G) \cap \mathcal{S}pec(G) = (0)$, $(\mathcal{A}rch \vee \mathcal{S}pec)(G) = \mathcal{A}rch(G) \boxplus \mathcal{S}pec(G)$.

$$\text{Let}\quad a_n(\delta) = \begin{cases} 1\,, & n < \delta < n + 1 \text{ on the real line} \\ \delta - (n - 1)\,, & n - 1 \leqq \delta < n \text{ on the real line} \\ -\delta + (n + 2)\,, & n + 1 < \delta \leqq n + 2 \text{ on the real line} \\ 0\,, & \text{elsewhere}\,. \end{cases}$$



Graph of a_n

$$\text{and define}\quad s_m(\delta) = \begin{cases} 1\,, & \delta = m \text{ on the real line} \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then $a_n \in \mathcal{A}rch(G)$ for all integers n and $s_m \in \mathcal{S}pec(G)$ for all integers m. The function g which is identically 1 for all points on the real line and 0 elsewhere is easily seen to be the join in G of $\{\{a_n\} \cup \{s_m\}\}$ for all integers m and n, but $g \notin$ $\notin \mathscr{A}rch(G) \boxplus \mathscr{S}pec(G)$. $\square$

In this, the lattice of closed-kernel radical classes has much the same relationship to the lattice of radical classes that the lattice of closed convex l-subgroups has to the lattice of convex l-subgroups of an l-group. We review here some of the peculiarities of $\mathscr{K}(G)$ that complicate matters when dealing with the closed-kernel radical and quasitorsion classes.

$\mathscr{K}(G)$ is not a sublattice of $\mathscr{C}(G)$ [5]; in fact, the join of a tower of closed convex l-subgroups need not be closed. If K is a closed l-ideal of G and A/K is a closed convex l-subgroup of G/K, then A is closed in G but if K is a closed l-ideal of G and A is a closed convex l-subgroup of G containing K, A/K need not be closed in G/K. Example 3.9 will have such a closed l-ideal and closed convex l-subgroup.

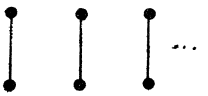
Lemma 3.7. $(\mathscr{R}^c \,.\, \mathscr{S}^c)^c = \mathscr{R}^c \,.\, \mathscr{S}^c$.

Proof. $\dfrac{(\mathscr{R}^c \,.\, \mathscr{S}^c)(G)}{\mathscr{R}^c(G)} = \mathrm{cl}(\mathscr{S}(G/\mathscr{R}^c(G)))$ and so $(\mathscr{R}^c \,.\, \mathscr{S}^c)(G) \in \mathscr{K}(G)$. Hence $(\mathscr{R}^c \,.\, \mathscr{S}^c)^c = \mathscr{R}^c \,.\, \mathscr{S}^c$. $\square$

Theorem 3.8. *The closed-kernel radical classes are a subsemigroup of the radical classes.*

Theorem 3.8 is not true for quasitorsion classes, however; in fact, the product of a closed-kernel quasitorsion class with itself need not be a quasitorsion class, as the following example shows.

Example 3.9. Let

$\Delta =$ 

and let H be the l-subgroup of $V(\Delta, \boldsymbol{Z})$ generated by

$$\begin{pmatrix} 1 \; 1 \; 1 & \\ & \cdots \\ 0 \; 0 \; 0 & \end{pmatrix}, \begin{pmatrix} 1 \; 2 \; 3 & \\ & \cdots \\ 0 \; 0 \; 0 & \end{pmatrix}, \begin{pmatrix} 1 \; 4 \; 9 \; 16 & \\ & \cdots \\ 0 \; 0 \; 0 \; \; 0 & \end{pmatrix},$$

and $\Sigma(\Delta, \boldsymbol{Z})$. Then $\Sigma(\Delta, \boldsymbol{Z})$ is a prime l-ideal of H. Let $G = \boldsymbol{R} \times_{\Sigma(\Delta, \boldsymbol{Z})} H$.

$\mathscr{A}rch(G) = \boldsymbol{R} \times \{h \in H\colon h$ has support only on the lower tier$\}$ and $G/\mathscr{A}rch(G)$ is archimedean also. So $G \in \mathscr{A}rch^2$.

But $K = \boldsymbol{R} \times_{\Sigma(\Delta, \boldsymbol{Z})} \Sigma(\Delta, \boldsymbol{Z})$ is a closed l-ideal of G and G/K is l-isomorphic to $\boldsymbol{Z} \times^{\leftarrow} \boldsymbol{Z} \times^{\leftarrow} \boldsymbol{Z}$, which is not in $\mathscr{A}rch^2$.

The map $\mathscr{R} \to \mathscr{R}^c$ is not a semigroup endomorphism. For let H be the l-subgroup of $\prod_{i=1}^{\infty} \boldsymbol{Z}$ generated by $(1, 2, 3, \ldots)$ and $(1, 1, 1, \ldots)$; let $G = H \mathscr{W}r\, \boldsymbol{Z}$. Then $\mathscr{F}in(G) =$ $= (\sum_{-\infty}^{\infty} \sum_{i=1}^{\infty} \boldsymbol{Z}) \times \{0\}$ and $G/\mathscr{F}in(G)$ is not finite-valued. So $\mathscr{F}in^2(G) \neq G$.

But $\mathscr{F}in^c(G) = \prod_{-\infty}^{\infty} H \times \{0\}$ and $(\mathscr{F}in^c)^2(G) = G$.

In [15], Jakubík showed that if $\mathscr{R} \subseteq \mathscr{S}$ and $\mathscr{T}$ are all radical classes, then $\mathscr{R}^{\bullet}\mathscr{T}$ need not be contained in $\mathscr{S}^{\bullet}\mathscr{T}$. In fact, an easy example shows that $\mathscr{T}$ need not be contained in $\mathscr{S}^{\bullet}\mathscr{T}$ even for quasitorsion classes, which of course gives Jakubík's result if we let $\mathscr{R} = \mathscr{E}$, the l-variety of one-element l-groups. For if H is the same l-group as in the preceding paragraph, $\mathscr{F}in(H) = \sum_{i=1}^{\infty} Z$. But $H/\mathscr{F}in(H)$ is l-isomorphic to $Z \times^{\leftarrow} Z$, which is not in $\mathscr{A}rch$. So $\mathscr{A}rch \nsubseteq \mathscr{F}in^{\bullet}\mathscr{A}rch$.

Jakubík's result is still true in general for closed-kernel quasitorsion classes.

Example 3.10. Let H be the same l-group as above. Then $\sum_{i=1}^{\infty} Z$ is a prime l-ideal of H. Let $G = R \times_{\sum_{i=1}^{\infty} Z} H$.

Let $\mathscr{R}ealsum$ denote the class of l-groups that are cardinal sums of the reals, R. $\mathscr{R}ealsum$ is a torsion class [18] and so $\mathscr{R}ealsum^c$ is a quasitorsion class. For G, $\mathscr{R}ealsum(G) = \mathscr{R}ealsum^c(G) = R \times_{\sum_{i=1}^{\infty} Z} (0)$.

That $G \in \mathscr{R}ealsum^{c\bullet}\mathscr{A}rch$ is plain and $\mathscr{R}ealsum^c \subseteq \mathscr{A}rch$ is also clear. But $\mathscr{A}rch(G) = R \times_{\sum_{i=1}^{\infty} Z} \sum_{i=1}^{\infty} Z$ and $G/\mathscr{A}rch(G)$ is l-isomorphic to $Z \times^{\leftarrow} Z \notin \mathscr{A}rch$. So $\mathscr{R}ealsum^{c\bullet}\mathscr{A}rch$ is not contained in $\mathscr{A}rch^2$.

Proposition 3.11. *If $\mathscr{S}$ and $\mathscr{T}$ are closed-kernel quasitorsion classes, then $\mathscr{T} \subseteq \mathscr{S}^{\bullet}\mathscr{T}$.*

Proof. Let $G \in \mathscr{T}$. $\mathscr{S}(G) \in \mathscr{K}(G)$ implies that $G/\mathscr{S}(G) \in \mathscr{T}$. □

We look now at the minimal K-radical class $\mathscr{R}^K$ containing a given radical class $\mathscr{R}$. This is easy to describe in terms of $\mathscr{R}$: Let T be the class of lattices that are isomorphic to $\mathscr{K}(G)$ for some $G \in \mathscr{R}$. $\mathscr{R}^K$ is the K-radical class determined by T.

Proposition 3.12. $\mathscr{F}in^K = \mathscr{S}pec$.

Proof. Let $G \in \mathscr{S}pec$ with $\Delta(G)$ being the minimal plenary subset of $\Gamma(G)$. Then $V(\Delta(G), R) \in \mathscr{S}pec$ and $\Sigma(\Delta(G), R) \in \mathscr{F}in$. So $\mathscr{K}(G)$ is isomorphic as a lattice to $\mathscr{K}(\Sigma(\Delta(G), R))$ and thus $G \in \mathscr{F}in^K$.

Clearly $\mathscr{F}in^K \subseteq \mathscr{S}pec$. □

Proposition 3.13. $\mathscr{H}yp^K = \mathscr{A}rch$.

Proof. Clearly $\mathscr{H}yp^K \subseteq \mathscr{A}rch$.

Let $G \in \mathscr{A}rch$. Then $\mathscr{K}(G) = \mathscr{P}(G)$ is a complete Boolean algebra. By the Stone Representation Theorem [20, p. 51], $\mathscr{K}(G)$ is isomorphic as a Boolean algebra to the clopen sets of a compact totally-disconnected Hausdorff space X.

Let H be the subgroup of $C(X, Z)$ generated by the characteristic functions of the clopen sets of X. H is then a Specker l-group [8] and $\mathscr{K}(H)$ is isomorphic to the clopen sets of X as a Boolean algebra. So $\mathscr{K}(G)$ is isomorphic to $\mathscr{K}(H)$ and thus $G \in \mathscr{H}yp^K$. $\square$

Not every closed-kernel radical class is a K-radical class. For let $\mathscr{R} = \mathscr{F}in \cap \mathscr{A}$. Then $\mathscr{R}^K = \mathscr{S}pec$ (by the same proof that $\mathscr{F}in^K = \mathscr{S}pec$) and $\mathscr{R}^c \neq \mathscr{S}pec$ as $Z\, wr(2)\, Z$ is not in $\mathscr{R}^c$ but is in $\mathscr{S}pec$.

4. THE POLAR CLOSURE OPERATOR

In the last section, we defined new radical classes from old by taking the order closure of the kernels. In this section, we define new radical classes analogously by taking the double polar of the kernels. This yields the class of radical classes whose kernels are always polars. Among these radical classes are the completely-distributive l-groups, the class of l-groups whose root system of regular subgroups has a unique minimal plenary subset, and the class of l-groups with bases.

Since the lattice of radical classes is complete, for any radical class $\mathscr{R}$, the join of all radical classes $\mathscr{S}$ such that $\mathscr{R} \cap \mathscr{S} = \mathscr{E}$ exists and will be denoted $\mathscr{R}^\perp$. Clearly $\mathscr{R} \cap \mathscr{R}^\perp = \mathscr{R} \cap (\bigvee \mathscr{S}) = \bigvee(\mathscr{R} \cap \mathscr{S}) = \mathscr{E}$.

Theorem 4.1. *For any radical class $\mathscr{R}$, $\mathscr{R}^\perp(G) = \mathscr{R}(G)'$.*

Proof. Let G be an l-group and $C \in \mathscr{C}(G)$. Denote the polar operation in G by $'$ and in C by $*$. Then $\mathscr{R}(C)^* \supseteq C \cap \mathscr{R}(C)' \supseteq C \cap \mathscr{R}(G)'$. Let $e \leqq g \in \mathscr{R}(C)$ and $e \leqq h \in C \cap \mathscr{R}(G)'$. Then

$$g \wedge h \in C \cap \mathscr{R}(G)' \cap \mathscr{R}(C) \subseteq \mathscr{R}(C)' \cap \mathscr{R}(C) = (e)\,.$$

So $\mathscr{R}(C)^* = C \cap \mathscr{R}(G)'$.

Clearly if $\phi: G \to H$ is a surjective l-isomorphism, $\phi[\mathscr{R}(G)] = \mathscr{R}(H)$, implying $\phi[\mathscr{R}(G)'] = \mathscr{R}(H)'$. Thus the map $G \to \mathscr{R}(G)'$ is a radical mapping defining a radical class $\mathscr{S}$.

That $\mathscr{S} \subseteq \mathscr{R}^\perp$ is obvious and for any l-group G, $\mathscr{R}^\perp(G) \cap \mathscr{R}(G) = (e)$ implies $\mathscr{R}^\perp G) \subseteq \mathscr{R}(G)' = \mathscr{S}(G)$. So $\mathscr{R}^\perp(G) = \mathscr{R}(G)'$.

Proposition 4.2. *If $\mathscr{R}$ is a radical class such that for any l-group G, $\mathscr{R}(G)$ is always a polar, $\mathscr{R} = \mathscr{R}^{\perp\perp}$.* $\square$

Theorem 4.3. *The lattice of radical classes is a pseudocomplemented lattice whose skeletal elements are precisely those radical classes with polar kernels.*

Thus the „polars" of the lattice of radical classes have polar kernels. This fact will give us a welcome respite from the pathologies of the order-closure operator that became evident in the examples of the last section.

Proposition 4.4. *For any radical class $\mathscr{R}$, $\mathscr{R}^{\perp\perp}$ is an idempotent radical class.*

Proof. If $\mathscr{R}^{\perp\perp}(G)' = (e)$, $G = \mathscr{R}^{\perp\perp}(G)'' = \mathscr{R}^{\perp\perp}(G)$.

If $\mathscr{R}^{\perp\perp}(G)' \neq (e)$ and $\mathscr{R}^{\perp\perp}(G) \neq (\mathscr{R}^{\perp\perp})^2(G)$, $A = (\mathscr{R}^{\perp\perp})^2(G) \cap \mathscr{R}^{\perp}(G) \neq (e)$. But

$$A \cong \frac{A \boxplus \mathscr{R}^{\perp\perp}(G)}{\mathscr{R}^{\perp\perp}(G)} \in \mathscr{R}^{\perp\perp}$$

and so $A \subseteq \mathscr{R}^{\perp\perp}(G)$, a contradiction. □

Corollary 4.5. *$\mathscr{C}d$, $\mathscr{E}ss$, and $\mathscr{B}as$ are idempotent radical classes.*

Proposition 4.6. *No proper quasitorsion class is a polar kernel radical class.*

Proof. Let $\mathscr{R}$ be a proper quasitorsion class. Let $G \in \mathscr{R}$ and let (H, Ω) be an l-permutation group which is not in $\mathscr{R}$. Let $A = G \, wr \, (H, \Omega)$. Then $\mathscr{R}(A) = \sum_{\omega \in \Omega} G$ and $A/\mathscr{R}(A) \simeq H$ which is not in $\mathscr{R}$. But $\mathscr{R}(A)'' = A$. □

This of course shows there exist idempotent radical classes that do not have polar kelners, for $\mathscr{N}$, the l-variety of normal-valued l-groups, is idempotent.

Clearly the map $\mathscr{R} \to \mathscr{R}^{\perp\perp}$ is a closure operator and $\mathscr{R}^{\perp\perp}(G) = \mathscr{R}(G)''$. This closure operator will be much nicer than the order-closure operator but does not have much resemblance to the original radical class.

Lemma 4.7. *For any radical classes $\mathscr{R}$ and $\mathscr{S}$,*

$$\mathscr{S}(\mathscr{R}^{\perp}(G)) = (\mathscr{R} \cdot \mathscr{S})(\mathscr{R}^{\perp}(G)).$$

Proof.

$$(\mathscr{R} \cdot \mathscr{S})(\mathscr{R}^{\perp}(G)) = \frac{(\mathscr{R} \cdot \mathscr{S})(\mathscr{R}^{\perp}(G))}{(e)} = \frac{(\mathscr{R} \cdot \mathscr{S})(\mathscr{R}^{\perp}(G))}{\mathscr{R}(\mathscr{R}^{\perp}(G))} = \mathscr{S}\left(\frac{\mathscr{R}^{\perp}(G)}{\mathscr{R}(\mathscr{R}^{\perp}(G))}\right) = \mathscr{S}(\mathscr{R}^{\perp}(G)).$$

□

Theorem 4.8. *The map $\mathscr{R} \to \mathscr{R}^{\perp\perp}$ is a semigroup endomorphism.*

Proof. First, both $(\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(G)$ and $(\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G)$ are polars of G.

Suppose that $(\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(G) \nsubseteq (\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G)$ for some l-group G. Then $(\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(G) \cap (\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})^{\perp}(G) \neq (e)$. Let $e < y$ be in this intersection. Then $y \in \mathscr{R}^{\perp}(G)$ and so $y \in (\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(\mathscr{R}^{\perp}(G)) = \mathscr{S}^{\perp\perp}(\mathscr{R}^{\perp}(G)) = (\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(\mathscr{R}^{\perp}(G)) \subseteq$ $\subseteq (\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G)$, a contradiction.

Conversely, suppose $(\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G) \nsubseteq (\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(G)$. Then $(\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G) \cap$ $\cap (\mathscr{R} \cdot \mathscr{S})^{\perp}(G) \neq (e)$. Let $e < x$ be in this intersection. Then $x \in (\mathscr{R}^{\perp\perp} \cdot \mathscr{S}^{\perp\perp})(G) \cap$ $\cap \mathscr{R}^{\perp}(G) = \mathscr{S}^{\perp\perp}(\mathscr{R})^{\perp}(G))$. But $\mathscr{S}(\mathscr{R}^{\perp}(G)) \subseteq (\mathscr{R} \cdot \mathscr{S})(G)$ implies that $\mathscr{S}^{\perp\perp}(\mathscr{R}^{\perp}(G)) \subseteq$ $\subseteq (\mathscr{R} \cdot \mathscr{S})^{\perp\perp}(G)$, also a contradiction. □

Corollary 4.9. *The polar radical classes are a subsemigroup of the radical classes.*

Corollary 4.10. *For any ordinal α and radical class $\mathscr{R}$, $(\mathscr{R}^{\alpha})^{\perp\perp} = \mathscr{R}^{\perp\perp}$.*

Proof. Suppose for any ordinal $\beta < \alpha$, $(\mathscr{R}^{\beta})^{\perp\perp} = \mathscr{R}^{\perp\perp}$.
Then if $\alpha = \gamma + 1$, $(\mathscr{R}^{\alpha})^{\perp\perp} = (\mathscr{R}^{\gamma} \cdot \mathscr{R})^{\perp\perp} = (\mathscr{R}^{\gamma})^{\perp\perp} \cdot \mathscr{R}^{\perp\perp} = \mathscr{R}^{\perp\perp} \cdot \mathscr{R}^{\perp\perp} = \mathscr{R}^{\perp\perp}$.
If α is a limit ordinal, $\mathscr{R}^{\alpha} = \bigvee_{\beta < \alpha} \mathscr{R}^{\beta}$ and so $\mathscr{R}^{\alpha} \subseteq \mathscr{R}^{\perp\perp}$ implies $\mathscr{R}^{\perp\perp} \subseteq (\mathscr{R}^{\alpha})^{\perp\perp} \subseteq \mathscr{R}^{\perp\perp}$. □

This now allows us to answer a question raised by the author in [4].

Corollary 4.11. *For any ordinal* α, $\mathcal{S}pec^{\alpha} \subseteq \mathcal{C}d$.

Theorem 4.12. *A polar kernel radical class is closed with respect to l-homomorphic images by normal polars.*

Proof. Let $\mathcal{S}$ be a polar kernel radical class, $G \in \mathcal{S}$, and P be a normal polar of G. Then $P' \in \mathcal{S}$ and $P' \cong (P' \boxplus P)/P$ implies that $(P' \boxplus P)/P \subseteq \mathcal{S}(G/P)$. But $\mathcal{S}(G/P) = Q/P$ for some polar Q of G and $P' \boxplus P \subseteq Q$ implies that $Q = G$. So $G/P \in \mathcal{S}$. □

Corollary 4.13. *If* $\mathcal{R} \subseteq \mathcal{S}$ *are polar kernel radical classes, then* $\mathcal{R}^{\bullet}\mathcal{S} = \mathcal{S}$.

Corollary 4.14. *If* $\mathcal{R} \subseteq \mathcal{S}$ *and* $\mathcal{T}$ *are polar kernel radical classes,* $\mathcal{R}^{\bullet}\mathcal{T} \subseteq \mathcal{S}^{\bullet}\mathcal{T}$.

Corollary 4.15. *If* $\mathcal{R} \subseteq \mathcal{S}$ *and* $\mathcal{T} \subseteq \mathcal{U}$ *are all polar kernel radical classes, then* $\mathcal{R}^{\bullet}\mathcal{T} \subseteq \mathcal{S}^{\bullet}\mathcal{U}$.

Theorem 4.16. *The polar kernel radical classes are an l-subsemigroup of the radical classes.*

One might be tempted to investigate radical classes whose kernels are always cardinal summands. These classes of l-groups, however, are not very interesting.

Proposition 4.17. $\mathcal{L}$ *and* $\mathcal{E}$ *are the only radical classes whose kernels are always cardinal summands.*

Proof. Suppose not. Thus there exists a proper radical class $\mathcal{R}$ such that $G = \mathcal{R}(G) \boxplus \mathcal{R}^{\perp}(G)$ for any l-group G. Let $(e) \neq A \in \mathcal{R}$ and $(e) \neq B \in \mathcal{R}^{\perp}$; let $G = (A \boxplus B) \times^{\leftarrow} Z$. $\mathcal{R}(G) = A$, which is not a cardinal summand of G. □

5. CLOSURES WITH RESPECT TO l-HOMOMORPHIC IMAGES AND TO l-SUBGROUPS

If $\{\mathcal{R}_{\lambda}\}$ is a collection of radical classes that are all closed with respect to l-subgroups or are all closed with respect to l-homomorphic images, then clearly $\bigcap \mathcal{R}_{\lambda}$ is likewise closed with respect to l-subgroups or l-homomorphic images, respectively. Thus, since the lattice of radical classes of l-groups is complete and since $\mathcal{L}$, the l-variety of all l-groups, is a radical class, we have the following theorem.

Theorem 5.1. *For any radical class* $\mathcal{R}$, *there exists unique minimal radical classes* $\mathcal{R}^{s}$ *and* $\mathcal{R}^{h}$, *closed with respect to l-subgroups and l-homomorphic images, respectively, that contain* $\mathcal{R}$. *Moreover, the collections of s-closed and h-closed radical classes form complete lattices under inclusion.*

Clearly the mapping $\mathcal{R} \to \mathcal{R}^{s}$ and $\mathcal{R} \to \mathcal{R}^{h}$ are closure operators on the lattice of radical classes. Since these operators are not based on the convex l-subgroup structure of the l-groups, the results in this section are not nearly as nice or complete as in the last two sections.

Proposition 5.2. *For any radical class $\mathscr{R}$, $\mathscr{R}^h$ is the least torsion class that contains $\mathscr{R}$.*

Proof. $\mathscr{R}^h$ is a radical class that is closed with respect to l-homomorphisms and so is a torsion class. Clearly if $\mathscr{R} \subseteq \mathscr{S}$ and $\mathscr{S}$ is a torsion class, $\mathscr{R}^h \subseteq \mathscr{S}$. $\square$

The following is an immediate consequence.

Corollary 5.3. *The h-closed radical classes form a lattice subsemigroup of the radical classes.*

We can also construct the kernels of $\mathscr{R}^s$ and $\mathscr{R}^h$ from that of $\mathscr{R}$ by using the following lemma.

Lemma 5.4. *Let $\mathscr{S}$ be a class of l-groups that is closed with respect to convex l-subgroups. Let G be an l-group and define $\mathscr{R}(G) = \bigvee\{C \in \mathscr{C}(G)$: C is l-isomorphic to an element of $\mathscr{S}\}$. Then $G \to \mathscr{R}(G)$ is a radical operator which generates the least radical class containing $\mathscr{S}$.*

Proof. Let $C \in \mathscr{C}(G)$. $\mathscr{R}(C) = \bigvee\{D \in \mathscr{C}(C)$: D is l-isomorphic to an element of $\mathscr{S}\} = \bigvee\{C \cap D$: $D \in \mathscr{C}(G)$ and D is l-isomorphic to an element of $\mathscr{S}\} = C \cap$ $\cap \bigwedge\{D \in \mathscr{C}(G)$: D is l-isomorphic to an element of $\mathscr{S}\} = C \cap \mathscr{R}(G)$.

If $\phi: G \to H$ is an l-bijection and if $C \in \mathscr{C}(G)$ with C being l-isomorphic to an element of $\mathscr{S}$, then $\phi(C)$ is also l-isomorphic to an element of $\mathscr{S}$ and so $\phi(\mathscr{R}(G)) =$ $= \mathscr{R}(H)$. So we have a radical operator and this clearly must generate the least radical class containing $\mathscr{S}$. $\square$

Proposition 5.5. *For any radical class $\mathscr{R}$ and l-group G, $\mathscr{R}^h(G) = \bigvee\{C \in \mathscr{C}(G)$: there exists $H \in \mathscr{R}$ and $L \in \mathscr{L}(H)$ such that $C \simeq H/L\}$.*

Proof. Let $\mathscr{S}$ be the collection of all l-homomorphic images of elements of $\mathscr{R}$ and let $\mathscr{T}$ denote the class of all l-groups G that equal $\bigvee\{C \in \mathscr{S} \cap \mathscr{C}(G)\}$.

To apply the above lemma, we must show that $\mathscr{S}$ is closed with respect to convex l-subgroups. But if $H \in \mathscr{S}$ and $C \in \mathscr{C}(H)$, then $H \simeq K/L$ for some $K \in \mathscr{R}$ and $L \in \mathscr{L}(K)$, implying that $C \simeq B/L$ for some $L \subseteq B \in \mathscr{C}(K)$. So $C \in \mathscr{S}$ and $\mathscr{T}$ is a radical class.

Now let $G \in \mathscr{T}$ and $M \in \mathscr{L}(G)$.

$G/M = \bigvee\{CM/M$: $C \in \mathscr{S} \cap \mathscr{C}(G)\}$. But each such CM/M is l-isomorphic to $C/(C \cap M)$ which is in $\mathscr{S}$ and so $G/M \in \mathscr{T}$. We have, then, that $\mathscr{R}^h \subseteq \mathscr{T}$ and clearly $\mathscr{T} \subseteq \mathscr{R}^h$. $\square$

The description of $\mathscr{R}^s$ is nearly as easy. Let $\mathscr{R}$ be a radical class and $\mathscr{R}_1$ be the radical class generated by all l-subgroups of elements of $\mathscr{R}$. Since the lattice of l-subgroups of an l-group is *not* Brouwerian, we can not claim that $\mathscr{R}_1 = \mathscr{R}^s$. So define $\mathscr{R}_2$ to be the radical class generated by all l-subgroups of elements of $\mathscr{R}_1$; define $\mathscr{R}_3, \mathscr{R}_4, \ldots$ analogously. It is easy to check, then, that $\mathscr{R}^s = \bigvee \mathscr{R}_i$.

Proposition 5.6. *For every nontrivial polar kernel radical class $\mathscr{R}$, $\mathscr{R}^s = \mathscr{R}^h = \mathscr{L}$.*

Proof. If $\mathscr{R} = \mathscr{L}$, we are done. Otherwise, let $(e) \neq G \in \mathscr{R}$ and let (H, Ω) be an l-permutation group not in $\mathscr{R}$. Then $G \mathscr{W}r(H, \Omega) \in \mathscr{R}$ and so $H \in \mathscr{R}^s$. Likewise,

$$H \simeq \frac{G \mathscr{W}r(H, \Omega)}{\Pi_H G}. \quad \square$$

The last proposition enables us to say that $\mathscr{C}d^s$, $\mathscr{E}ss^s$, and $\mathscr{B}as^s$ all equal $\mathscr{L}$. Also well-known [9] are that $\mathscr{A}rch^s = \mathscr{A}rch$, $\mathscr{H}yp^s = \mathscr{H}yp$, and $\mathscr{F}in^s = \mathscr{F}in$. In [19], [1], and [4] are various proofs that $\mathscr{S}pec^s = \mathscr{N}$.

Proposition 5.7. *For any two radical classes $\mathscr{R}$ and $\mathscr{S}$, $(\mathscr{R}^{h\bullet}\mathscr{S}^h)^h = \mathscr{R}^{h\bullet}\mathscr{S}^h$ and $(\mathscr{R}^\bullet\mathscr{S})^h \subseteq \mathscr{R}^{h\bullet}\mathscr{S}^h$.*

Proof. The first is true since the product of torsion classes is a torsion class. For the second, $\mathscr{R}^\bullet\mathscr{S}$ is clearly contained in $\mathscr{R}^{h\bullet}\mathscr{S}^h$ and so $(\mathscr{R}^\bullet\mathscr{S})^h \subseteq \mathscr{R}^{h\bullet}\mathscr{S}^h$. It is not known if $\mathscr{R}^{h\bullet}\mathscr{S}^h \subseteq (\mathscr{R}^\bullet\mathscr{S})^h$.

Thus the h-closure behaves relatively nicely through the product of two radical classes. The map $\mathscr{R} \to \mathscr{R}^h$ is easily seen to be a lattice homomorphism. $\square$

In marked contrast, Martinez [18] has shown that $\mathscr{R}^s \vee \mathscr{S}^s$ need not be closed with respect to l-subgroups and so the s-closed radical classes do not form a sublattice of the radical classes. We now show that $\mathscr{A}rch^\bullet\mathscr{A}rch$ is not closed with respect to l-subgroups, demonstrating that the s-closed radical classes are not closed under multiplication.

Example 5.8. Let H be the l-group of Example 3.9 and let A be the l-subgroup of H generated by

$$\begin{pmatrix} 1 & 1 & 1 & \\ & & & \cdots \\ 0 & 0 & 0 & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \\ & & & \cdots \\ 0 & 0 & 0 & \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 4 & 9 & 16 & \\ & & & & \cdots \\ 0 & 0 & 0 & 0 & \end{pmatrix},$$

and $0 \times_P \boldsymbol{R}$, where $P = \sum_{i=1}^{\infty} \boldsymbol{R}$. $\mathscr{A}rch(A) = \sum_{i=1}^{\infty} \boldsymbol{Z} \times_P \boldsymbol{R}$, but $A/\mathscr{A}rch(A)$ is l-isomorphic to $\boldsymbol{Z} \times^{\leftarrow} \boldsymbol{Z} \times^{\leftarrow} \boldsymbol{Z}$, which is not archimedean.

6. INTERCHANGING ORDERS OF CLOSURES

In this section, we show that usually the four closure operators outlined earlier do not commute with one another.

Example 6.1. *There exists a radical class $\mathscr{R}$ such that $\mathscr{R}^{ch} \neq \mathscr{R}^{hc}$.* Let $\mathscr{Z}sum$ denote the torsion class of cardinal sums of $\mathscr{Z}$. Then $\mathscr{Z}sum^h = \mathscr{Z}sum$ while $\boldsymbol{R} \notin \mathscr{Z}sum^c$. Thus $\boldsymbol{R} \notin \mathscr{Z}sum^{hc}$. But since $\mathscr{Z}sum^c$ includes all cardinal products of $\mathscr{Z}$, $\mathscr{Z}sum^{ch}$ [21] includes all abelian l-groups.

Proposition 6.2. *For any proper nontrivial torsion class $\mathscr{T}$, $\mathscr{T}^{ph} \neq \mathscr{T}^{hp}$.*

Proof. Let $\mathcal{T}$ be a proper nontrivial torsion class. Then $\mathcal{T}^p$ is a proper nontrivial polar radical class and so $\mathcal{T}^{ph} = \mathcal{L}$. Clearly $\mathcal{T}^{hp} = \mathcal{T}^p \neq \mathcal{L}$. □

Proposition 6.3. *For any radical class $\mathcal{R}$, $\mathcal{R}^{pc} = \mathcal{R}^{cp} = \mathcal{R}^p$.*

Proof. This is clear since polars are closed convex l-subgroups. □

Example 6.4. *There exists a radical class $\mathcal{R}$ such that $\mathcal{R}^{cs} \neq \mathcal{R}^{sc}$.* Let $\mathcal{R} = \mathcal{A} \cap \mathcal{F}in$. Since for every root system Δ, $\Sigma(\Delta, \boldsymbol{R}) \in \mathcal{R}$, $V(\Delta, \boldsymbol{R}) \in \mathcal{R}^c$ for every root system Δ and so $\mathcal{R}^{cs} = \mathcal{A}$.

But $\mathcal{R}^{sc} = \mathcal{R}^c \subseteq \mathcal{C}d$; thus $\boldsymbol{C}(\boldsymbol{R}) \notin \mathcal{R}^{sc}$.

Example 6.5. *There exists a radical class $\mathcal{R}$ such that $\mathcal{R}^{sp} \neq \mathcal{R}^{ps}$. Let $\mathcal{R} = \mathcal{N}$. Then $\mathcal{N}^{sp} = \mathcal{N}^p \neq \mathcal{L}$ while $\mathcal{N}^{ps} = \mathcal{L}$.*

The remaining possibility is whether or not $\mathcal{R}^{sh} = \mathcal{R}^{hs}$ for every radical class $\mathcal{R}$. Thus far no proof is known of this result and no counterexample is known. Indeed, in a surprising number of cases (though not all), $\mathcal{R}^{hs}$ and $\mathcal{R}^{sh}$ turn out to be l-varieties. This indicates that the s-closure and the h-closure might be strongly linked in some way. The following is a result in this direction and shows what can happen when one concatenates the various closure operators.

Proposition 6.6. *A radical class $\mathcal{R}$ is an l-variety if and only if $\mathcal{R} = \mathcal{R}^{shc}$.*

Proof. One direction is clear. For the converse, let $G \in \mathcal{R} = \mathcal{R}^{shc}$ and let A be an l-subgroup of G. Since $\mathcal{R}^s \subseteq \mathcal{R}^{shc} = \mathcal{R}$, $A \in \mathcal{R}$.

If $L \in \mathcal{L}(G)$, then $G/L \in \mathcal{R}^h \subseteq \mathcal{R}^{sh} \subseteq \mathcal{R}^{shc} = \mathcal{R}$.

Finally, if $\{G_\lambda\} \subseteq \mathcal{R}$, then $\Pi G_\lambda \in \mathcal{R}^c \subseteq \mathcal{R}^{shc} = \mathcal{R}$, and thus $\mathcal{R}$ is closed with respect to l-subgroups, l-homomorphic images, and cardinal products. So $\mathcal{R}$ is an l-variety. □

Concatenating the closure operators also allows us to obtain $\mathcal{A}rch$ from $\mathcal{H}yp$.

Proposition 6.7. $\mathcal{A}rch = \mathcal{H}yp^{cscs}$.

Proof. Let G be an archimedean l-group and let X be the Stone space of $\mathcal{P}(G)$. Then G can be viewed as an l-subgroup of $\mathcal{D}(X)$ [7]. However, for any topological space X, $\mathcal{D}(X)$ is the order closure of $\boldsymbol{C}(X)$.

Now since $\boldsymbol{R} \in \mathcal{H}yp$, every cardinal product of reals is in $\mathcal{H}yp^c$; therefore, for every topological space X, $\boldsymbol{C}(X)$ is in $\mathcal{H}yp^{cs}$. So $\mathcal{D}(X)$ is in $\mathcal{H}yp^{csc}$ and thus $G \in \mathcal{H}yp^{cscs}$.

Clearly $\mathcal{H}yp^{cscs} \subseteq \mathcal{A}rch$. □

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