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LIFTS OF DERIVATIONS TO THE FRAME BUNDLE

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INTRODUCTION

Let M be an n-dimensional C^∞ manifold and $\mathscr{F}M$ its frame bundle. A theory of complete and horizontal lifts to $\mathscr{F}M$ of tensor fields and connections on M are recently developped by several authors [1], [2], [4], [6] in such a way that – the results obtained may be closely compared with the corresponding results in the theory of lifts to the tangent bundle TM. In this paper, complete and horizontal lifts to $\mathscr{F}M$ of derivations of the tensorial algebra of the tensor fields on M are defined and their properties are studied. The particular case of covariant differentations is considered.

1. PRELIMINARIES

In this section, we shall fix our notations and recall, for later use, the definitions and some properties of the complete and horizontal lifts of tensor fields and connections to the frame bundle. Details can be found in Mok [6] and in [1], [2].

Manifolds, tensor fields and linear connections under consideration are all assumed to be differentiable and of class C^∞, and the manifolds to be connected.

1. Indices $i, j, k, \ldots; \alpha, \beta, \gamma, \ldots,$ have range in $\{1, \ldots, n\}$. We put $h_\alpha = \alpha n + h$. Summation over repeated indices is always implied.

2. Entries of matrices are written as A^i_j, A_{ij} or A^{ij}, and in all cases, i is the row index while j is the column index.

3. Let M be an n-dimensional manifold. Coordinate systems in M are denoted by (U, x^i), where U is the coordinate neighborhood and x^i are the coordinate functions. Components in (U, x^i) of geometric objects on U will be referred to simply as components in U, or just components. We denote the partial differentation $\partial/\partial x^i$ by ∂_i, and the Lie derivative by $\mathscr{L}_X$.

Let T_xM be the tangent space at a point $x \in M$, $(X_\alpha) = (X_1, \ldots, X_n)$ a linear frame at x and $\mathscr{F}M$ the frame bundle over M, that is the set of all frames at all points of M. Let $\pi\colon \mathscr{F}M \to M$ be the canonical projection of $\mathscr{F}M$ onto M; for the coordinate system (U, x^i) in M, we put $\mathcal{F}U = \pi^{-1}(U)$. A frame (X_α) at x can be expressed uniquely in the form $X_\alpha = X^i_\alpha(\partial/\partial x^i)_x$. The induced coordinate system in $\mathcal{F}M$ is $\{\mathcal{F}U, (x^i, X^i_\alpha)\}$. The matrix $[X^i_\alpha]$ is non-singular and its inverse will be written as $[X^\alpha_i]$.

4. Let ∇ be a linear connection on M with components Γ^h_{ji}. The curvature tensor and the torsion tensor of ∇ are denoted by R and T, respectively. The opposite connection $\hat{\nabla}$ of ∇ has components $\hat{\Gamma}^h_{ji} = \Gamma^h_{ij}$.

With a given linear connection ∇ on M, we can define two sets of global 1-forms on $\mathcal{F}M$, namely θ^γ and ω^ϱ_σ. Their expressions on $\mathcal{F}U$ are

$$\theta^\gamma = X^\gamma_h \, dx^h , \quad \omega^\varrho_\sigma = X^\varrho_h(\Gamma^h_{ji}X^i_\sigma \, dx^j + dX^h_\sigma)$$

and these $n + n^2$ global 1-forms are linearly independent everywhere. Actually, $\theta = (\theta^\gamma)$ is the canonical 1-form of $\mathcal{F}M$ and $\omega = (\omega^\varrho_\sigma)$ is the connection form of ∇. Let E_β, E^μ_λ be the $n + n^2$ global vector fields on $\mathcal{F}M$ dual to θ^γ, ω^ϱ_σ; they span the horizontal and the vertical distributions on $\mathcal{F}M$, respectively, and their expressions in $\mathcal{F}U$ are

$$E_\alpha = X^i_\alpha(\partial_i - \Gamma^j_{ik}X^k_\beta \, \partial_{j\beta}) , \quad E^\mu_\lambda = X^j_\lambda \, \partial_{j\mu} \tag{1.1}$$

where $\partial_{j\beta}$ stands for $\partial/\partial X^j_\beta$.

Note that $\lambda A = A^\alpha_\beta E^\beta_\alpha$ is the fundamental vector field on $\mathcal{F}M$ associated to $A = [A^\alpha_\beta] \in \mathrm{gl}(n, R)$, and $B\xi = \xi^\alpha E_\alpha$ is the basic vector field associated to $\xi = (\xi^1, \dots, \xi^n) \in R^n$.

From (1.1) we notice that the horizontal and the vertical distributions, when restricted to $\mathcal{F}U$, are spanned respectively by the local vector fields

$$D_j = \partial_j - \Gamma^h_{ji}X^i_\alpha \, \partial_{h\alpha} , \quad D_{k\beta} = \partial_{k\beta} . \tag{1.2}$$

The frame field $\{D_j, D_{k_\beta}\}$ will be said the *adapted frame* on $\mathcal{F}U$. The local 1-forms η^j, η^k_β on $\mathcal{F}U$ dual to D_j, D_{k_β} are given by

$$\eta^j = dx^j , \quad \eta^k_\beta = \Gamma^k_{ji}X^i_\beta \, dx^j + dX^k_\beta \tag{1.3}$$

and $\{\eta^j, \eta^k_\beta\}$ will be said the *adapted coframe* on $\mathcal{F}U$.

5. Let S be a tensor field on M of type $(1, s)$, $s \geqq 1$, with local components $S^h_{j_1 \dots j_s}$ in U; then, we associate to S a tensor field γS on $\mathcal{F}M$ of type $(1, s - 1)$ defined in $\mathcal{F}U$ by

$$\gamma S = S^h_{j_1 \dots j_s} X^{j_1}_\beta \, \partial_{h\beta} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_s} . \tag{1.4}$$

Let F be a tensor field on M of type (1,1) with local components F^h_j; let $F_\circ = [F^\beta_\alpha]$ be the $n \times n$ square matrix of functions globally defined on $\mathcal{F}M$ and given by $F^\beta_\alpha = F^h_j X^j_\alpha X^\beta_h$. For each $A = [A^\alpha_\beta] \in gl(n, R)$ we consider on $\mathcal{F}M$ a vertical vector field defined by

$$\lambda(F \circ A) = F^\beta_\alpha A^\alpha_\gamma E^\gamma_\beta . \tag{1.5}$$

From (1.4) and (1.5), we notice that if A is the unit matrix I, then $\lambda(F^\circ I) = \gamma F$; and if F is the identity tensor I on M, then $\lambda(I^\circ A) = \lambda A$.

6. Let S be a tensor field on M of type $(1, s)$, $s \geqq 0$, and let $S^h_{j_1 \dots j_s}$ be its local components in U; then, the complete lift S^C of S to $\mathscr{F}M$ is the tensor field of the same type given in $\mathscr{F}U$ by

$$S^C = S^h_{j_1 \dots j_s} \partial_h \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s} + \left(X^k_\alpha \partial_k S^h_{j_1 \dots j_s}\right) \partial_{h_\alpha} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s} +$$
$$+ \sum_{i=1}^{s} \delta^\beta_\alpha S^h_{j_1 \dots j_s} \partial_{h_\alpha} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}X^{j_i}_\beta \otimes \dots \otimes \mathrm{d}x^{j_s} .$$

So, if $s = 0$, that is, if X is a vector field on M with local components X^i, its complete lift X^C to $\mathscr{F}M$ is the vector field locally given by

$$X^C = X^i \partial_i + \left(X^k_\alpha \partial_k X^i\right) \partial_{i_\alpha} .$$

In [6], the following properties of these complete lifts are shown

$$(1.6) \qquad [X^C, Y^C] = [X, Y]^C, \quad [X^C, \lambda A] = 0$$
$$S^C(X^C_1, \dots, X^C_s) = \{S(X_1, \dots, X_s)\}^C$$

for any vector fields $X, Y, X_1, \dots, X_s$ and every tensor field S of type $(1, s)$ on M.

These definitions have been extended in [1] to covariant tensor fields on M as follows; let τ be an arbitrary 1-form on M with local components τ_i; then the functions

$$\gamma_\alpha \tau = X^i_\alpha \tau_i , \quad 1 \leqq \alpha \leqq n$$

are globally defined on $\mathscr{F}M$. Then, if f is a differentiable function on M, we define its complete lift f^C to $\mathscr{F}M$ by putting

$$f^C = \sum_{\alpha=1}^{n} f^{(\alpha)}$$

where $f^{(\alpha)} = \gamma_\alpha(\mathrm{d}f)$, $1 \leqq \alpha \leqq \eta$.

The complete lift τ^C of a 1-form τ on M to $\mathscr{F}M$ is defined as the unique 1-form on $\mathscr{F}M$ satisfying $\tau^C(X^C) = (\tau(X))^C$ for any vector field X on M. In general, if S is a tensor field on M of type $(0, s)$, $s \geqq 0$, its complete lift S^C to $\mathscr{F}M$ is defined as the unique tensor field of type $(0, s)$ on $\mathscr{F}M$ satisfying

$$(1.7) \qquad S^C(X^C_1, \dots, X^C_s) = \{S(X_1, \dots, X_s)\}^C$$

for any vector fields $X_1, \dots, X_s$ on M. If $S_{j_1 \dots j_s}$ are the components of S in U, then in $\mathscr{F}U$

$$S^C = \sum_{\alpha=1}^{n} \{(X^i_\alpha \partial_i S_{j_1 \dots j_s}) \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s} +$$
$$+ \sum_{k=1}^{s} S_{j_1 \dots j_s} \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}X^{j_k}_\alpha \otimes \dots \otimes \mathrm{d}x^{j_s}\} .$$

The vertical lift S^V to $\mathscr{F}M$ of a covariant tensor field S of type $(0, s)$, $s \geqq 0$, on M

is defined by setting $S^V = \pi^* S$. Thus, taking into account the previous definitions, one easily obtain the following formulas

$$(1.8) \qquad X^C f^{(\alpha)} = (Xf)^{(\alpha)}, \quad 1 \leqq \alpha \leqq n, \quad X^C f^C = (Xf)^C, \quad X^C f^V = (Xf)^V$$
$$\tau^V(X^C) = (\tau(X))^V, \quad \tau^V(\lambda A) = 0,$$

for any function f, any 1-form τ and any vector field X on M, and arbitrary $A \in \in \mathrm{gl}(n, R)$.

7. Let ∇ be a linear connection on M with local components Γ^h_{ji}. Let S be a tensor field on M of type (r, s) with $r = 0, 1$ and $s \geqq 0$, and suppose that the components of S in U are $S_{j_1 \dots j_s}$ for $r = 0$ or $S^h_{j_1 \dots j_s}$ for $r = 1$. Then a tensor field $\hat{\nabla}_\gamma S$ of the same type is defined on $\mathscr{F}M$ by putting in $\mathscr{F}U$

$$(1.9) \qquad \hat{\nabla}_\gamma S = \sum_{\alpha=1}^{n} (X^i_\alpha \hat{\nabla}_i S_{j_1 \dots j_s}) \, \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s} \quad \text{for} \quad r = 0$$
$$\hat{\nabla}_\gamma S = (X^i_\alpha \hat{\nabla}_i S^h_{j_1 \dots j_s}) \, \partial_{h_\alpha} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s} \quad \text{for} \quad r = 1,$$

where $\hat{\nabla}_i S_{j_1 \dots j_s}$ and $\hat{\nabla}_i S^h_{j_1 \dots j_s}$ denote the components of $\hat{\nabla} S$. We notice that $\hat{\nabla}_\gamma S = = \gamma(\hat{\nabla} S)$ for $r = 1$.

The horizontal lift S^H of a tensor field S on M of type $(0, s)$ or $(1, s)$, $s \geqq 0$, is defined by setting

$$(1.10) \qquad S^H = S^C - \hat{\nabla}_\gamma S.$$

Actually, X^H is the well known horizontal lift of a vector field X on M given in $\mathscr{F}U$ by

$$(1.11) \qquad X^H = X^j(\partial_j - \Gamma^h_{jk} X^k_\alpha \, \partial_{h_\alpha}).$$

If we take into account the various definitions of lifts, the following formulas are easily proved:

$$(1.12) \qquad f^H = 0, \quad X^H f^V = (Xf)^V, \quad X^H f^{(\alpha)} = (Xf)^{(\alpha)} - (\hat{\nabla}_\gamma X) f^{(\alpha)}, \quad 1 \leqq \alpha \leqq n$$
$$X^H f^C = (Xf)^C - (\hat{\nabla}_\gamma X) f^C$$

for every function f and any vector field X on M,

$$(1.13) \qquad [X^H, Y^H] = [X, Y]^H - \gamma R(X, Y), \quad [X^H, \lambda A] = 0,$$
$$[X^H, \gamma S] = \gamma(\nabla_X S)$$

for any vector fields X, Y and any tensor field S of type $(1, 1)$ on M and any $A \in \in \mathrm{gl}(n, R)$

$$(1.14) \qquad F^H X^H = (FX)^H, \quad F^H(\lambda A) = \lambda(F^\circ A)$$
$$F^H(\gamma S) = \gamma(FS), \quad F^H(\lambda S^\circ A) = \lambda((FS)^\circ A)$$

for every vector field X and any tensor fields F, S of type $(1, 1)$ on M and any $A \in \mathrm{gl}(n, R)$.

8. The complete lift to $\mathscr{F}M$ of a linear connection on M has been introduced and studied by Mok in [6] as follows.

Let ∇ be a linear connection on M; then the complete lift ∇^C of ∇ to $\mathscr{F}M$ is the linear connection on $\mathscr{F}M$ uniquely determined by the condition

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C \tag{1.15}$$

for any vector fields X, Y on M.

In [2], the following formulas are obtained:

$$\begin{gathered}\nabla^C_{\lambda A}\lambda B = \lambda(AB)\,, \quad \nabla^C_{\lambda A} X^C = \lambda((\nabla X)^\circ A)\,,\\ \nabla^C_{X^H}\lambda A = 0\,, \quad \nabla^C_{X^C}\lambda A = \lambda((\hat{\nabla} X)^\circ A)\,, \quad \nabla^C_{\lambda A} X^H = \lambda(T(-,X)^\circ A)\end{gathered} \tag{1.16}$$

for any vector field X on M, arbitrary A, $B \in \mathrm{gl}(n, R)$ and being $T(-, X)$ the tensor field of type $(1, 1)$ on M defined by $T(-, X)(Y) = T(X, Y)$, for any vector field Y on M, where T is the torsion tensor of ∇.

Moreover, in [2] Cordero and de León introduce the horizontal lift ∇^H of ∇ to $\mathscr{F}M$ as the linear connection on $\mathscr{F}M$ defined by the conditions

$$\begin{aligned}\nabla^H_{\lambda A}\lambda B &= \lambda(AB)\,, & \nabla^H_{X^H}\lambda B &= 0\\ \nabla^H_{\lambda A} X^H &= \lambda(T(-,X)^\circ A)\,, & \nabla^H_{X^H} Y^H &= 0\end{aligned} \tag{1.17}$$

for any vector fields X, Y on M and any A, $B \in \mathrm{gl}(n, R)$.

2. COMPLETE LIFTS OF DERIVATIONS

Let $\mathscr{T}(M) = \Sigma \mathscr{T}^r_s(M)$ be the tensorial algebra of the tensor fields on M. By a derivation of $\mathscr{T}(M)$, we shall mean a mapping $D\colon \mathscr{T}(M) \to \mathscr{T}(M)$ which satisfies the following conditions:

(a) $D\colon \mathscr{T}^r_s(M) \to \mathscr{T}^r_s(M)$
(b) $D(S + T) = DS + DT$, $S, T \in \mathscr{T}^r_s(M)$
(c) $D(S \otimes T) = D(S) \otimes T + S \otimes D(T)$, $S, T \in \mathscr{T}(M)$
(d) D commutes with every contraction of a tensor field.

The set $\mathscr{D}(M)$ of all derivations of $\mathscr{T}(M)$ forms a Lie algebra over R (of infinite dimensions) with respect to the natural addition and multiplication and the bracket operation defined by $[D, D']K = D(D'K) - D'(DK)$. Two derivations D and D' of $\mathscr{T}(M)$ coincide if they coincide on $\mathscr{T}^0_0(M)$ and $\mathscr{T}^1_0(M)$, i.e., on the functions and the vector fields on M. Every derivation D of $\mathscr{T}(M)$ can be descomposed uniquely as follows:

$$D = \mathscr{L}_X + i_F$$

where $\mathscr{L}_X$ is the Lie derivative with respect to a vector field X and i_F is the derivation defined by a tensor field F of type $(1, 1)$ on M. The set $\mathscr{L}(M)$ of Lie derivatives $\mathscr{L}_X$, forms a subalgebra of the Lie algebra $\mathscr{D}(M)$. On the other hand, the set $\mathscr{E}(M)$ of derivations i_F is an ideal of the Lie algebra $\mathscr{D}(M)$.

The following lemma will be useful

Lemma 2.1. (a) *Let* $\tilde{X}$, $\tilde{Y}$ *be vector fields on* $\mathscr{F}M$ *such that* $\tilde{X}f^V = \tilde{Y}f^V$, $\tilde{X}f^{(\alpha)} = \tilde{Y}f^{(\alpha)}$, $1 \leqq \alpha \leqq n$, *for an arbitrary function* f *on* M. *Then* $\tilde{X} = \tilde{Y}$.

(b) *Let* $\tilde{S}$, $\tilde{T}$ *be tensor fields of type* (r, s), $s > 0$ *on* $\mathscr{F}M$ *such that* $\tilde{S}(X_1^C, \dots, X_s^C) = \tilde{T}(X_1^C, \dots, X_s^C)$ *for any arbitrary vector fields* $X_1, \dots, X_s$ *on* M. *Then* $\tilde{S} = \tilde{T}$.

Proof. Part (b) can be found in Mok [6] and Part (a) follows easily through a simple computation in local coordinates.

Proposition 2.2. *Two derivations* D *and* D' *of* $\mathscr{T}(\mathscr{F}M)$ *coincide if and only if*

(a) $Df^V = D'f^V$, $Df^{(\alpha)} = D'f^{(\alpha)}$, $1 \leqq \alpha \leqq n$, *for any function* f *on* M.

(b) $DY^C = D'Y^C$, *for any vector field* Y *on* M.

Proof. It is sufficient to show that if $Df^V = 0$, $Df^{(\alpha)} = 0$, $1 \leqq \alpha \leqq n$, $DY^C = 0$, for any function f and any vector field Y on M, then $D = 0$. If $D = \mathscr{L}_{\tilde{X}} + i_{\tilde{F}}$, then

$$Df^V = \mathscr{L}_{\tilde{X}} f^V = \tilde{X}f^V = 0\,,$$

$$Df^{(\alpha)} = \mathscr{L}_{\tilde{X}} f^{(\alpha)} = \tilde{X}f^{(\alpha)} = 0\,, \quad 1 \leqq \alpha \leqq n$$

for every function f on M. Taking into account lemma 2.1, we deduce $\tilde{X} = 0$. Thus, $D = i_{\tilde{F}}$ and hence

$$DY^C = i_{\tilde{F}} Y^C = \tilde{F} Y^C = 0$$

for every vector field Y on M. Again, from lemma 2.1, we deduce $\tilde{F} = 0$.

Let $D = \mathscr{L}_X + i_F$ be a derivation of $\mathscr{T}(M)$, where X is a vector field and F a tensor field of type $(1, 1)$ on M. We define the *complete lift* D^C of D tp $\mathscr{F}M$ by

$$D^C = \mathscr{L}_{X^C} + i_{F^C}\,.$$

Taking into account (1.6) and (1.8), we have

Proposition 2.3.

$$D^C f^V = (Df)^V\,, \quad D^C f^{(\alpha)} = (Df)^{(\alpha)}\,, \quad 1 \leqq \alpha \leqq n\,, \quad D^C f^C = (Df)^C$$

$$D^C Y^C = (DY)^C\,,$$

for any function f *on* M *and any vector field* Y *on* M.

As a direct consequence of propositions 2.2 and 2.3, we easily obtain

Proposition 2.4. *The mapping* $D \to D^C$ *is a Lie algebra homomorphism of* $\mathscr{D}(M)$ *into* $\mathscr{D}(\mathscr{F}M)$.

Now, we consider the complete lifts of covariant differentations. Let ∇ be a linear connection on M. Then the covariant differentiation ∇_X with respect to a vector field X on M is a derivation of $\mathscr{T}(M)$. Since $\nabla_X f = Xf^-$, for any function f on M, we have the decomposition

$$\nabla_X = \mathscr{L}_X + i_F$$

where F is a tensor field of type (1, 1) on M. We notice that $FY = \nabla_X Y - [X, Y] = \hat{\nabla}_Y X$, that is, $F = \hat{\nabla} X$, where $\hat{\nabla}$ denotes the opposite connection of ∇.

Let $(\nabla_X)^C$ be the complete lift of ∇_X to $\mathscr{F}M$. Taking into account proposition 2.3, we have

$$(\nabla_X)^C f^V = (\nabla_X f)^V = (Xf)^V,$$
$$(\nabla_X)^C f^{(\alpha)} = (\nabla_X f)^{(\alpha)} = (Xf)^{(\alpha)}, \quad 1 \leqq \alpha \leqq n,$$

for any function f on M, and

$$(\nabla_X)^C Y^C = (\nabla_X Y)^C, \quad \text{for any vector field } Y \text{ on } M.$$

On the other hand, we can consider the complete lift ∇^C of ∇ to $\mathscr{F}M$ and the covariant differentiation $\nabla^C_{X^C}$ with respect to the complete lift X^C of X to $\mathscr{F}M$. Taking into account (1.8) and (1.15), we have

$$\begin{aligned} &\nabla^C_{X^C} f^V = X^C f^V = (Xf)^V, \\ &\nabla^C_{X^C} f^{(\alpha)} = X^C f^{(\alpha)} = (Xf)^{(\alpha)}, \quad 1 \leqq \alpha \leqq n, \\ &\nabla^C_{X^C} Y^C = (\nabla_X Y)^C, \quad \text{for any function } f \text{ and any vector field } Y \text{ on } M. \end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2), and taking into account proposition 2.2, we have

Proposition 2.5. $(\nabla_X)^C = \nabla^C_{X^C}$, *for every vector field X on M.*

3. HORIZONTAL LIFTS OF DERIVATIONS

Let ∇ be a linear connection on M. We have

Proposition 3.1. *Two derivations D and D' of $\mathscr{T}(\mathscr{F}M)$ coincide if and only if*
(a) *$Df^V = D'f^V$, $Df^{(\alpha)} = D'f^{(\alpha)}$, $1 \leqq \alpha \leqq n$, for any function f on M.*
(b) *$DY^H = D'Y^H$, $D(\lambda A) = D'(\lambda A)$, for any vector field Y on M and any $A \in \mathrm{gl}(n, R)$.*

Proof. It is sufficient to show that if $Df^V = 0$, $Df^{(\alpha)} = 0$, $1 \leqq \alpha \leqq n$, for any functions f on M and $DY^H = 0$, $D(\lambda A) = 0$, for any vector field Y on M and any $A \in \mathrm{gl}(n, R)$, then $D = 0$.

If $D = \mathscr{L}_{\tilde{X}} + i_{\tilde{F}}$, we have

$$Df^V = \mathscr{L}_{\tilde{X}} f^V = \tilde{X} f^V = 0, \quad Df^{(\alpha)} = \mathscr{L}_{\tilde{X}} f^{(\alpha)} = \tilde{X} f^{(\alpha)} = 0, \quad 1 \leqq \alpha \leqq n,$$

for every function f on M. Then $\tilde{X} = 0$ from lemma 2.1.

Thus, $D = i_{\tilde{F}}$ and hence

$$DY^H = i_{\tilde{F}} Y^H = \tilde{F} Y^H = 0,$$
$$D(\lambda A) = i_{\tilde{F}}(\lambda A) = \tilde{F}(\lambda A) = 0,$$

for every vector field Y on M and any $A \in \mathrm{gl}(n, R)$. Consequently, $\tilde{F} = 0$.

Now, let $D = \mathscr{L}_X + i_F$ be a derivation of $\mathscr{T}(M)$, where X is a vector field and F a tensor field of type (1, 1) on M. We define the *horizontal lift* D^H of D to $\mathscr{F}M$ by

$$D^H = \mathscr{L}_{X^H} + i_{F^H}.$$

If one takes into account (1.12), (1.13) and (1.14), we have

Proposition 3.2.
(a) $D^H f^V = (Df)^V$
$D^H f^{(\alpha)} = (Df)^{(\alpha)} - (\hat{\nabla}_\gamma X) f^{(\alpha)}$, $1 \leqq \alpha \leqq n$
$D^H f^C = (Df)^C - (\hat{\nabla}_\gamma X) f^C$, *for any function f on M.*
(b) $D^H Y^H = (DY)^H - \gamma R(X, Y)$
$D^H(\lambda A) = \lambda(F^\circ A)$, *for any vector field Y on M and any $A \in \mathrm{gl}(n, R)$.*

Let $D = \mathscr{L}_X + i_F$, $\bar{D} = \mathscr{L}_{\bar{X}} + i_{\bar{F}}$ be two derivations of $\mathscr{T}(M)$. Then the commutator of D and $\bar{D}$ is given by

$$[D, \bar{D}] = \mathscr{L}_{[X,\bar{X}]} + i_{[F,\bar{F}]}\,.$$

Hence

$$[D, \bar{D}]^H = \mathscr{L}_{[X,\bar{X}]^H} + i_{[F,\bar{F}]^H}\,.$$

Taking into account (1.12), (1.13), (1.14) and proposition 3.2, we have

$$\begin{aligned}
(3.1)\qquad & ([D^H, \bar{D}^H] - [D, \bar{D}]^H) f^V = 0 \\
& ([D^H, \bar{D}^H] - [D, \bar{D}]^H) f^{(\alpha)} = -(\gamma R(X, \bar{X})) f^{(\alpha)}, \quad 1 \leqq \alpha \leqq n \\
& ([D^H, \bar{D}^H] - [D, \bar{D}]^H)\, Y^H = -\gamma\{R(X, [\bar{X}, Y]) - R(\bar{X}, [X, Y]) + \\
& + R(Y, [X, \bar{X}]) + R(X, \bar{F}Y) + FR(\bar{X}, Y) - R(\bar{X}, FY) - \bar{F}R(X, Y)\} \\
& ([D^H, \bar{D}^H] - [D, D]^H)(\lambda A) = \lambda((\nabla_X \bar{F} - \nabla_{\bar{X}} F)^\circ A)
\end{aligned}$$

for any function f and any vector field Y on M and any $A \in \mathrm{gl}(n, R)$.

Using (3.1) and proposition 3.1, we have

Proposition 3.3. (a) *The mapping $i_F \to i_{F^H}$ is a Lie algebra homomorphism of $\mathscr{E}(M)$ into $\mathscr{E}(\mathscr{F}M)$.*
(b) *If ∇ is flat, then the mapping $\mathscr{L}_X \to \mathscr{L}_{X^H}$ is a Lie algebra homomorphism of $\mathscr{L}(M)$ into $\mathscr{L}(\mathscr{F}M)$.*

Remark. However, the mapping $D \to D^H$ is not a Lie algebra homomorphism between $\mathscr{D}(M)$ and $\mathscr{D}(\mathscr{F}M)$ even if ∇ is flat.

Next, we shall study the horizontal lifts of covariant differentations. Let ∇_X be the covariant differentation with respect to a vector field X on M. So, $\nabla_X = \mathscr{L}_X + i_F$, where $F = \hat{\nabla} X$ is a tensor field of type (1, 1) on M. If one considers the horizontal lift $(\nabla_X)^H$ of ∇_X to $\mathscr{F}M$, we have

$$\begin{aligned}
(3.2)\qquad & (\nabla_X)^H f^V = (\nabla_X f)^V = (Xf)^V \\
& (\nabla_X)^H f^{(\alpha)} = (\nabla_X f)^{(\alpha)} - (\hat{\nabla}_\gamma X) f^{(\alpha)} = (Xf)^{(\alpha)} - (\hat{\nabla}_\gamma X) f^{(\alpha)}, \quad 1 \leqq \alpha \leqq n \\
& (\nabla_X)^H\, Y^H = (\nabla_X Y)^H - \gamma R(X, Y) \\
& (\nabla_X)^H (\lambda A) = \lambda(F^\circ A), \text{ for any function } f \text{ and any vector field } Y \text{ on } M \\
& \text{and any } A \in \mathrm{gl}(n, R).
\end{aligned}$$

On the other hand, we can consider the horizontal lift ∇^H of ∇ to $\mathscr{F}M$ and the covariant differentiation $\nabla^H_{X^H}$ with respect to the horizontal lift X^H of X to $\mathscr{F}M$. Taking into account (1.12) and (1.17), we have

$$
\begin{aligned}
(3.3)\quad & \nabla^H_{X^H} f^V = X^H f^V = (Xf)^V \\
& \nabla^H_{X^H} f^{(\alpha)} = X^H f^{(\alpha)} = (Xf)^{(\alpha)} - (\hat{\nabla}_\gamma X) f^{(\alpha)}, \quad 1 \leqq \alpha \leqq n \\
& \nabla^H_{X^H} Y^H = (\nabla_X Y)^H \\
& \nabla^H_{X^H} \lambda A = 0, \text{ for any function } f \text{ and any vector field } Y \text{ on } M \text{ and any} \\
& A \in \mathrm{gl}(n, R).
\end{aligned}
$$

By comparing (3.2) and (3.3) and taking into account proposition 3.1, we have

Proposition 3.4. *If ∇ is a flat connection, then $(\nabla_X)^H = \nabla^H_{X^H}$, for every parallel vector field X on M.*

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