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FIXED POINT THEOREM IN UNIFORM SPACES AND APPLICATIONS

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The present paper deals with fixed point theorems in uniform spaces (Section I) and their applications to the existence-uniqueness problem for nonlinear functional differential equations of neutral type, with unbounded deviations (Section II). Since the uniform spaces form a natural extension of the metric spaces, many results in this direction have appeared in the last years. We shall mention only some of them [1]–[8].

It is known that every topological vector space is completely regular and therefore uniformisable. If E is a locally convex space with a saturated family of seminorms $\{p_\alpha\}_{\alpha\in A}$ then we can define a family of pseudometrics $\varrho_\alpha(x, y) = p_\alpha(x - y)$. The uniform topology obtained coincides with the original topology of the space E. Therefore, as a corollary of our results, we obtain fixed point theorems in a locally convex space.

We note that the known results in metric spaces are not applicable to the problems in Section II (cf. the survey papers [9], [10], [11]).

I.

Further on we denote by X a Hausdorff sequentially complete uniform space with uniformity defined by a saturated family of pseudometrics $\{\varrho_\alpha(x, y)\}_{\alpha\in A}$, A being an index set (cf. [12]).

Let $\Phi = \{\Phi_\alpha(t): \alpha \in A\}$ be a family of functions $\Phi_\alpha(t): R^1_+ \to R^1_+$ ($R^1_+ = [0, \infty)$) with the properties

1) $\Phi_\alpha(t)$ is monotone non-decreasing and continuous from the right on R^1_+,

2) $\Phi_\alpha(t) < t$ for all $t > 0$, and $j: A \to A$ is a mapping on the index set A into itself, where $j^0(\alpha) = \alpha$, $j^k(\alpha) = j(j^{k-1}(\alpha))$; k is a positive integer.

Definition 1. The map $T: M \to M$ is said to be a *Φ-contraction* on M if

$$\varrho_\alpha(Tx, Ty) \leqq \Phi_\alpha(\varrho_{j(\alpha)}(x, y))$$

for every $x, y \in M$ and $\alpha \in A$, $M \subset X$.

Definition 2. The set $M \subset X$ *is bounded* if it is bounded in every pseudometric ϱ_α, that is, $\varrho_\alpha^0 = \sup\{\varrho_\alpha(x, y): x, y \in M\} < \infty$.

Definition 3. We shall say that $\Phi_\alpha(t)$ is a *Φ-function* if it belongs to the family Φ.

Theorem 1. *Let the following conditions hold:*

1. *The operator $T: M \to M$ is a Φ-contraction on the totally bounded and closed set $M \subset X$, where X is also quasicomplete* (cf. [12]).
2. *For each $\alpha \in A$ there exists a Φ-function $\bar{\Phi}_\alpha(t)$ such that* $\sup\{\Phi_{j^n(\alpha)}(t): n = 0, 1, 2, \ldots\} \leqq \bar{\Phi}_\alpha(t)$ *and* $\varrho_{j^n(\alpha)}^0 \leqq \varrho_\alpha^0$ $(n = 0, 1, 2, \ldots)$.

Then there exists a unique fixed point $x \in M$ of T, such that $x = \lim_{n\to\infty} T^n x_0$ independently of the choice of $x_0 \in M$.

Proof. We define the sequence $x_n = Tx_{n-1}$ $(n = 1, 2, 3, \ldots)$ with an arbitrary $x_0 \in M$. We shall show that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence.

Let ϱ_α^0 be the diameter of M in the pseudometric $\varrho_\alpha(x, y)$. By condition 1 of Theorem 1 we have

$$\varrho_\alpha(x_s, x_l) = \varrho_\alpha(Tx_{s-1}, Tx_{l-1}) \leqq \Phi_\alpha(\varrho_{j(\alpha)}(x_{s-1}, x_{l-1}))$$

for all $s, l \geqq 1$.

If we set $c_n^\alpha = \sup\{\varrho_\alpha(x_s, x_l): s, l \geqq n\}$, then by the monotonicity of Φ_α we obtain

$$c_n^\alpha \leqq \sup\{\Phi_\alpha(\varrho_{j(\alpha)}(x_{s-1}, x_{l-1})): s, l \geqq n\} \leqq$$
$$\leqq \Phi_\alpha(\sup\{\varrho_{j(\alpha)}(x_{s-1}, x_{l-1}): s, l \geqq n\}),$$

that is $c_n^\alpha \leqq \Phi_\alpha(c_{n-1}^{j(\alpha)})$.

Further on, condition 2 of Theorem 1 implies

$$c_n^\alpha \leqq \Phi_\alpha(c_{n-1}^{j(\alpha)}) \leqq \Phi_\alpha(\Phi_{j(\alpha)}(c_{n-2}^{j^2(\alpha)})) \leqq \ldots \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(c_0^{j^n(\alpha)}) \ldots)) \leqq$$
$$\leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(\varrho_\alpha^0) \ldots)) \leqq \bar{\Phi}_\alpha^n(\varrho_\alpha^0),$$

where

$$\bar{\Phi}_\alpha^n(t) = \bar{\Phi}_\alpha(\bar{\Phi}_\alpha(\ldots \bar{\Phi}_\alpha(t))) \quad \text{is the } n\text{-th iterate of} \quad \bar{\Phi}_\alpha(t).$$

Let us set $d_n^\alpha = \bar{\Phi}_\alpha^n(\varrho_\alpha^0)$. We obtain $d_n^\alpha = \bar{\Phi}_\alpha(d_{n-1}^\alpha) \leqq d_{n-1}^\alpha$, $d_n^\alpha \geqq 0$. Therefore the limit $\lim_{n\to\infty} d_n^\alpha = d^\alpha$ exists and $d^\alpha \geqq 0$.

The right continuity of $\bar{\Phi}_\alpha(t)$ implies $\lim_{n\to\infty} \bar{\Phi}_\alpha(d_{n-1}^\alpha) = \bar{\Phi}_\alpha(d^\alpha)$ and hence $d^\alpha \leqq \bar{\Phi}_\alpha(d^\alpha)$. But $\bar{\Phi}_\alpha(t) < t$ for $t > 0$, hence we obtain $d^\alpha = 0$. On the other hand, $c_n^\alpha \leqq d_n^\alpha$. Therefore $\lim_{n\to\infty} c_n^\alpha = 0$, i.e. $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence. Since the uniform space X is sequentially complete, there exists an element $x \in X$ such that $\lim_{n\to\infty} x_n = x$. But X is quasicomplete. Consequently, $x \in M$.

An element $x \in M$ is the required fixed point of the operator T. Indeed, $\varrho_\alpha(x, Tx) \leqq \leqq \varrho_\alpha(x, x_n) + \varrho_\alpha(x_n, Tx) \leqq \varrho_\alpha(x, x_n) + \Phi_\alpha(\varrho_{j(\alpha)}(x_{n-1}, x))$ and when $n \to \infty$, we have $\varrho_\alpha(x, Tx) = 0$ for every $\alpha \in A$ and consequently, $x = Tx$.

Now, let x and y be two solutions of the equation $x = Tx$. Then we have the inequalities

$$\varrho_\alpha(x, y) = \varrho_\alpha(Tx, Ty) \leqq \Phi_\alpha(\varrho_{j(\alpha)}(x, y)) =$$
$$= \Phi_\alpha(\varrho_{j(\alpha)}(Tx, Ty)) \leqq \ldots \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(\varrho_\alpha^0) \ldots)) \leqq \bar{\Phi}_\alpha^n(\varrho_\alpha^0),$$

Bearing in mind that $\lim_{n\to\infty} \bar{\Phi}_\alpha^n(\varrho_\alpha^0) = 0$ we conclude that $\varrho_\alpha(x, y) = 0$ for every $\alpha \in A$, i.e. $x = y$. Thus Theorem 1 is proved.

Corollary 1. *Let the conditions of Theorem* 1 *hold with the assumption* 1 *replaced by*

$$\varrho_\alpha(T^s x, T^s y) \leqq \Phi_\alpha(\varrho_{j(\alpha)}(x, y))$$

for some positive integer s. Then T has a unique fixed point.

Proof. The operator T^s satisfies all conditions of Theorem 1 and therefore T^s has a unique fixed point x, i.e. $T^s x = x$. But $T(T^s x) = T^s(Tx)$, $Tx = T^s(Tx)$ and Tx is a fixed point of T^s. Since x is unique, we obtain $x = Tx$, which completes the proof.

If the operator $T: X \to X$ maps all the space X into itself, then we have the following result:

Theorem 2. *Let us suppose*

1. *the operator $T: X \to X$ is a Φ-contraction;*
2. *for each $\alpha \in A$ there exists a Φ-function $\bar{\Phi}_\alpha(t)$ such that* $\sup\{\Phi_{j^n(\alpha)}(t): n = 0, 1, 2, \ldots\} \leqq \bar{\Phi}_\alpha(t)$ *and $\bar{\Phi}_\alpha(t)/t$ is non-decreasing;*
3. *there exists an element $x_0 \in X$ such that $\varrho_{j^n(\alpha)}(x_0, Tx_0) \leqq p(\alpha) < \infty$ $(n = 0, 1, 2, \ldots)$.*

Then T has at least one fixed point in X.

Theorem 3. *If, in addition, we suppose that*

4. *the sequence $\{\varrho_{j^k(\alpha)}(x, y)\}_{k=0}^\infty$ is bounded for each $\alpha \in A$ and $x, y \in X$, i.e.* $\varrho_{j^k(\alpha)}(x, y) \leqq q(x, y, \alpha) < \infty$ $(k = 0, 1, 2, \ldots)$.

Then the fixed point of T is unique.

Proof of Theorem 2: Let us introduce the sequence $c_n^\alpha = \varrho_\alpha(T_{x_0}^{n+1}, T_{x_0}^n)$ for $x_0 \in X$ $(n = 0, 1, \ldots)$. Then we obtain

$$c_n^\alpha \leqq \Phi_\alpha(\varrho_{j(\alpha)}(T_{x_0}^n, T_{x_0}^{n-1})) \leqq \ldots \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(\varrho_{j^n(\alpha)}(Tx_0, x_0) \leqq$$
$$\leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(p(\alpha)) \ldots)) \leqq \bar{\Phi}_\alpha^n(p(\alpha)) \stackrel{\text{def}}{=} b_n^\alpha,$$

and the inequalities

$$\varrho_\alpha(x_{m+p}, x_m) \leqq \sum_{i=1}^p \varrho_\alpha(x_{m+p-i+1}, x_{m+p-i}) = \sum_{i=1}^p c_{m+p-i}^\alpha \leqq \sum_{i=1}^p b_{m+p-i}^\alpha < \infty$$

together with

$$b_{n+1}^\alpha / b_n^\alpha = \bar{\Phi}_\alpha(\bar{\Phi}_\alpha^n(p(\alpha)))/\bar{\Phi}_\alpha^n(p(\alpha)) \leqq \bar{\Phi}_\alpha(p(\alpha))/p(\alpha) < 1$$

imply that $\{x_n = T^n x_0\}_{n=0}^\infty$ is a Cauchy sequence, which completes the proof of Theorem 2.

The proof of Theorem 3 is analogous to that of Theorem 1.

Corollary 2. *Under the assumptions of Theorems* 2 *and* 3, *the operator* $T: X \to X$ *has a unique fixed point* x *if for some positive integer* s, T^s *is a* Φ*-contraction* (*instead of* T).

Let us note that the condition $\bar{\Phi}_\alpha(t) < t$ is restrictive but implies $\sum_n b_n^\alpha < \infty$. Nevertheless, Theorems 2 and 3 are useful for the application (see Sec. II of the present paper). In Theorem 4 we shall show that if $j: A \to A$ is surjective and $\varrho_\alpha(x_{m+n}, x_m) \geqq \varrho_{j(\alpha)}(x_{m+n}, x_m)$ for all $\alpha \in A$ $(m, n \geqq 0)$ and some $x_0 \in X$, then condition 2 of Theorem 2 may be weakened.

Theorem 4. *Let us suppose*:

1. *the operator* $T: X \to X$ *is a* Φ*-contraction*;
2. *for each* $\alpha \in A$, $\lim_{n\to\infty} \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(t) \ldots)) = 0$, $t > 0$;
3. *the mapping* $j: A \to A$ *is surjective and* $\varrho_\alpha(x_{m+1}, x_m) \geqq \varrho_{j(\alpha)}(x_{m+n}, x_m)$ *for some* $x_0 \in X$ $(\alpha \in A;\ m, n \geqq 0)$.

Then there exists at least one fixed point of T, i.e. $x = Tx$. If we add the conditions of Theorem 3, then x is unique.

Corollary 3. *Under the assumptions of Theorem* 4, *if* T^s *is a* Φ*-contraction, then* T *has a unique fixed point.*

Proof of Theorem 4. Introduce the sequence $c_n^\alpha = \varrho_\alpha(x_{n+1}, x_n)$ for an element $x_0 \in X$ and set $p(\alpha) = \varrho_\alpha(x_0, Tx_0)$. Then we obtain

$$c_n^\alpha \leqq \Phi_\alpha(\varrho_{j(\alpha)}(T_{x_0}^n, T_{x_0}^{n-1})) \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(\varrho_{j^n(\alpha)}(x_0, Tx_0)) \ldots)) \leqq$$
$$\leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(p(\alpha)) \ldots)) .$$

Consequently, $\lim_{n\to\infty} c_n^\alpha = 0$ for all $\alpha \in A$.

If we suppose that $\{x_n\}_{n=0}^\infty$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ and a finite number of pseudometrics $\{\varrho_{\alpha'}\}$ such that for every ν we can find $m(\nu) > \nu$ and $p(\nu) > 0$ for which $\varrho_{\alpha'}(x_{m+p}, x_m) \geqq \varepsilon_0$. But j is surjective and we conclude that there exists α such that $\alpha' = j(\alpha)$ and

$$\varrho_\alpha(x_{m+p}, x_m) \geqq \varrho_{j(\alpha)}(x_{m+p}, x_m) \geqq \varepsilon_0 .$$

Let $\bar{p}$ be the smallest positive integer for which

$$\varrho_{j(\alpha)}(x_{m+\bar{p}}, x_m) \geqq \varepsilon_0 , \quad \text{i.e.} \quad \varrho_{j(\alpha)}(x_{m+\bar{p}-1}, x_m) < \varepsilon_0 .$$

Let us set $h_\nu^{j(\alpha)} = \varrho_{j(\alpha)}(T^{m+\bar{p}}x_0, T^m x_0)$. Then

$$\varepsilon_0 \leqq h_\nu^{j(\alpha)} = \varrho_{j(\alpha)}(T^{m+\bar{p}}x_0, T^m x_0) \leqq \varrho_{j(\alpha)}(T^{m+\bar{p}}x_0, T^{m+\bar{p}-1}x_0) +$$
$$+ \varrho_{j(\alpha)}(T^{m+\bar{p}-1}x_0, T^m x_0) \leqq c_{m+\bar{p}-1}^{j(\alpha)} + \varepsilon_0 .$$

Passing to the limit in the last inequality for $\nu \to \infty$, we obtain $\lim_{\nu\to\infty} h_\nu^{j(\alpha)} = \varepsilon_0$.

On the other hand, we have

$$\varepsilon_0 \leqq \varrho_\alpha(T^{m+\bar{p}}x_0, T^m x_0) \leqq \varrho_\alpha(T^{m+\bar{p}+1}x_0, T^{m+\bar{p}}x_0) + \\ + \varrho_\alpha(T^{m+\bar{p}+1}x_0, T^{m+1}x_0) + \varrho_\alpha(T^{m+1}x_0, T^m x_0) \leqq c^\alpha_{m+\bar{p}} + \\ + \Phi_\alpha(\varrho_{j(\alpha)}(T^{m+\bar{p}}x_0, T^m x_0)) + c^\alpha_m \leqq c^\alpha_{m+\bar{p}} + \Phi_\alpha(h_\nu^{j(\alpha)}) + c^\alpha_m$$

which yields (by $\nu \to \infty$) $\varepsilon_0 \leqq \Phi_\alpha(\varepsilon_0)$. The contradiction obtained proves the existence of a fixed point of T.

If x and y are two fixed points of T we have

$$\varrho_\alpha(x, y) \leqq \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^{n-1}(\alpha)}(q(x, y, \alpha)) \ldots)), \quad \text{i.e.} \quad x = y .$$

Theorem 4 is thus proved.

Remark 1. Theorems 1–4 with generalized contraction conditions are analogues of the theorems of Krasnoselskii [9], Browder [13] and Boyd-Wong [14] in metric spaces.

Remark 2. Theorems 1–4 generalize the known results in uniform ([4], [8]) and locally convex spaces ([15]–[18]).

Finally, we shall prove the following theorems:

Theorem 5. *Let us suppose*

1) *for each $\alpha \in A$ and n (positive integer) there exists $\Phi_{\alpha,n}(t) \in \Phi$ such that*

$$\varrho_\alpha(T^n x, T^n y) \leqq \Phi_{\alpha,n}(\varrho_{j(\alpha,n)}(x, y)) \quad \textit{for every} \quad x, y \in X ;$$

2. *there exists an element $x_0 \in X$ such that $\varrho_{j(\alpha,n)}(x_0, Tx_0) \leqq p(x) < \infty$ $(n = 0, 1, \ldots)$, $\sum_n \Phi_{\alpha,n}(p(\alpha)) < \infty$ and $j: A \times N \to A$.*

Then T has at least one fixed point in X.

Theorem 6. *If, in addition, we suppose that for every $\alpha \in A$ and $x, y \in X$ there exists $0 < q(x, y, \alpha) < \infty$ such that*

$$\varrho_{j_n}(x, y) \leqq q(x, y, \alpha) < \infty ; \quad \sup \{\Phi_{j_n,1}(t): n = 0, 1, \ldots\} \leqq \bar{\Phi}_\alpha(t) \in \Phi$$

where $j_1 = j(\alpha, 1)$, $j_2 = j(j_1, 1), \ldots, j_n = j(j_{n-1}, 1), \ldots$, then the fixed point x is unique.

Proof of Theorem 5. The fact that $\{T^n x_0\}_{n=0}^\infty$ is a Cauchy sequence follows from the inequalities

$$\varrho_\alpha(T^{m+n}x_0, T^n x_0) \leqq \sum_{i=1}^m \varrho_\alpha(T^{n+i-1}(Tx_0), T^{n+i-1}x_0) \leqq \\ \leqq \sum_{i=1}^m \Phi_{\alpha,n+i-1}(\varrho_{j(\alpha,n+i-1)}(x_0, Tx_0)) \leqq \sum_{i=1}^m \Phi_{\alpha,n+i-1}(p(\alpha)) .$$

Then $x = \lim_{n\to\infty} T^n x_0$ is the required fixed point. Indeed, $\varrho_\alpha(x, Tx) \leqq \varrho_\alpha(Tx, x_{n+1}) + \varrho_\alpha(x_{n+1}, x) \leqq \Phi_{\alpha,1}(\varrho_{j(\alpha,1)}(x, x_n)) + \varrho_\alpha(x_{n+1}, x)$ which completes the proof.

Proof of Theorem 6. If we assume, by way of contradiction, that x and y are two fixed points of T, then

$$\varrho_\alpha(x, y) = \varrho_\alpha(Tx, Ty) \leqq \Phi_{\alpha,1}(\varrho_{j(\alpha,1)}(x, y)) \leqq \Phi_{\alpha,1}(\Phi_{j_1,1}(\varrho_{j_2}(x, y))) \leqq \dots$$
$$\dots \leqq \Phi_{\alpha,1}(\Phi_{j_1,1}(\dots \Phi_{j_{n-1},1}(\varrho_{j_n}(x, y)) \dots)) \leqq \bar{\Phi}_\alpha^n(q(x, y, \alpha))$$

for every $\alpha \in A$, or $x = y$. Thus Theorem 6 is proved.

II.

In this section we shall apply Theorems 2 and 3 in order to obtain existence-uniqueness results for neutral functional differential equations.

Let us consider the following initial value problem (IVP):

$$\text{(3)} \qquad \varphi'(t) = F(t, \varphi(\Delta_1(t)), \dots, \varphi(\Delta_m(t)), \varphi'(\tau_1(t)), \dots, \varphi'(\tau_n(t))), \quad t > 0,$$
$$\varphi(t) = \psi(t), \quad \varphi'(t) = \psi'(t), \quad t \leqq 0,$$

where the unknown function $\varphi(t)$ takes values in the Banach space B with a norm $\|\cdot\|$. The deviations $\Delta_i(t)$, $\tau_l(t)$ $(i = 1, \dots, m;\ l = 1, \dots, n)$ are of mixed type and, in the general case, unbounded. The derivative is taken in the strong sense [19]. By the substitution $x(t) = \varphi'(t)$ for $t > 0$ and $\theta(t) = \psi'(t)$ for $t \leqq 0$, assuming $\psi(0) = 0$ (cf. [20]), we obtain the equivalent IVP:

$$\text{(3')} \quad x(t) = F\left(t, \int_0^{\Delta_1(t)} x(s)\,ds, \dots, \int_0^{\Delta_m(t)} x(s)\,ds,\ x(\tau_1(t)), \dots, x(\tau_n(t))\right), \quad t > 0$$
$$x(t) = \theta(t), \quad t \leqq 0.$$

Introduce the notations $R^1 = (-\infty, \infty)$, $R^1_+ = [0, \infty)$, $R^1_- = (-\infty, 0]$,

$$R^n_+ = \underbrace{R^1_+ \times \dots \times R^1_+}_{n}, \quad B^n = \underbrace{B \times \dots \times B}_{n}.$$

We shall adopt the following assumptions:

(C1) The functions $\Delta_i(t)\colon R^1_+ \to R^1$ $(i = 1, \dots, m)$; $\tau_l(t)\colon R^1_+ \to R^1$ $(l = 1, \dots, n)$ are continuous and $\Delta_i(0) \leqq 0$, $\tau_l(0) \leqq 0$.

We shall look for a solution of the IVP (3′) in the space $C(R^1; B)$ consisting of all continuous functions $f(t)\colon R^1 \to B$. It is known that the family of seminorms $p_K(f) = \sup\{\|f(t)\|\colon t \in K\}$ (where K runs over all compact subsets of R^1) defines a locally convex Hausdorff topology of the space.

Let us first define the map $j\colon A \mapsto A$. In this case the index set A consists of all compact subsets of R^1. Let $K \subset R^1$ be an arbitrary compact set. Then the set $j(K)$ is defined in the following way: if $K_+ = K \cap (0, \infty) \neq \emptyset$ we set $j(K) = (\bigcup_{i=1}^m K_{\Delta_i}) \cup (\bigcup_{l=1}^n K_{\tau_l})$ and if $K_+ = \emptyset$, then $j(K) = K$. Here we have

$$K_{\Delta_i} = \begin{cases} [\bar{\Delta}_i, \bar{\bar{\Delta}}_i] & \text{when } 0 \in [\bar{\Delta}_i, \bar{\bar{\Delta}}_i], \\ [0, \bar{\bar{\Delta}}_i] & \text{when } \bar{\Delta}_i \geqq 0, \\ [\bar{\Delta}_i, 0] & \text{when } \bar{\bar{\Delta}}_i \leqq 0, \end{cases}$$

where $\bar{\Delta}_i = \inf\{\Delta_i(t): t \in K \cap R^1_+\}$, $\bar{\bar{\Delta}}_i = \sup\{\Delta_i(t): t \in K \cap R^1_+\}$ $(i = 1, 2, \ldots, m)$, $K_{\tau l} = \{\tau_l(t): t \in K \cap R^1_+\}$ $(l = 1, 2, \ldots, n)$. Since the functions $\Delta_i(t)$, $\tau_l(t)$ are continuous the set $j(K)$ is also compact. The map $j^n(K)$ is defined inductively, i.e. $j^n(K) = j(j^{n-1}(K))$, n a positive integer.

(C2) The function $F(t, u_1, \ldots, u_m, v_1, \ldots, v_n): R^1_+ \times B^{m+n} \to B$ is continuous and satisfies the conditions

$$\|F(t, u_1, \ldots, u_m, v_1, \ldots, v_n) - F(t, \bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n)\| \leqq$$
$$\leqq \Omega(t, \|u_1 - \bar{u}_1\|, \ldots, \|u_m - \bar{u}_m\|, \|v_1 - \bar{v}_1\|, \ldots, \|v_n - \bar{v}_n\|)$$

where the function $\Omega(t, x_1, \ldots, x_m, y_1, \ldots, y_n): R^{m+n+1}_+ \to R^1_+$ is continuous in t, non-decreasing and continuous from the right in x_i, y_l, $\Omega(t, ay, \ldots, ay, y, \ldots, y) < y$ for every constant $a > 0$ and $\Omega(t, ay, \ldots, ay, y, \ldots, y)/y$ is non-decreasing in y.

It follows from (C2) that

$$\Phi_K(y) = \begin{cases} \sup\{\Omega(t, ay, \ldots, ay, y, \ldots, y): t \in K \cap R^1_+ \neq \emptyset\}, \\ 0 \quad \text{when} \quad K \cap R^1_+ = \emptyset \end{cases}$$

is continuous from the right, non-decreasing and $\Phi_K(y) < y$ for $y > 0$ for any compact $K \subset R^1$, and $\Phi_K(y)/y$ is non-decreasing.

(C3) The initial function $O(t)$: $R^1_- \to B$ is continuous and satisfies the conformity condition

$$\theta(0) = F\left(0, \int_0^{\Delta_1(0)} \theta(s)\, ds, \ldots, \int_0^{\Delta_m(0)} \theta(s)\, ds, \theta(\tau_1(0)), \ldots, \theta(\tau_n(0))\right).$$

(C4) The functions $\Delta_i(t)$, $\tau_l(t)$ have the following property: for every compact $K \subset R^1$ there exists a compact $\bar{K}$ such that $j^n(K) \subseteq \bar{K}$ $(n = 0, 1, 2, \ldots)$.

Remark 3. As can readily be seen, assumption (C4) implies $\Phi_{j^n(K)}(y) \leqq \Phi_{\bar{K}}(y)$, i.e. condition 2 of Theorem 2 is satisfied.

Remark 4. Assumption (C4) has an implicit form. It is easy to verify that if the functions $\Delta_i(t)$, $\tau_l(t)$ are delays, i.e. $\Delta_i(t) \leqq t$, $\tau_l(t) \leqq t$, then (C4) is satisfied. For example, if $\Delta_1(t) = -t$, $\tau_1(t) = t - 2$, then $j([0, 2]) = [-2, 0]$ and $j^n([0, 2]) = [-2, 0]$, $n = 0, 1, 2, \ldots$ since $[-2, 0] \cap (0, \infty) = \emptyset$. Assumption (C4) also allows for more complicated functions $\Delta_i(t)$, $\tau_l(t)$ as for instance

$$\tau_l(t) = \begin{cases} \sqrt{t}, & 0 \leqq t \leqq 1 \\ 1 + \sqrt{(t-1)}, & 1 \leqq t \leqq 2 \\ \cdots\cdots\cdots & \\ n + \sqrt{(t-n)}, & n \leqq t \leqq n+1 \\ \cdots\cdots\cdots & \end{cases}$$

Theorem 7. *If assumptions* (C1)–(C4) *are satisfied, then there exists a unique continuous solution* $x(t)$ *of* IVP (3′).

Proof. Let X be the uniform sequentially complete Hausdorff space consisting of all continuous functions $f(t)$: $R^1 \to B$ which are equal to $\theta(t)$ for $t \in R^1_-$, with a saturated family of pseudometrics

$$\varrho_K(f, g) = \sup \{\|f(t) - g(t)\| : t \in K\},$$

where K runs over the compact subsets of R^1.

The operator $N: X \to X$ is defined by the formulas

$$(Nf)(t) = \begin{cases} F\left(t, \int_0^{\Delta_1(t)} f(s)\,ds, \ldots, \int_0^{\Delta_m(t)} f(s)\,ds, f(\tau_1(t)), \ldots, f(\tau_n(t))\right), & t > 0, \\ \theta(t), \quad t \leqq 0 \end{cases}$$

where $f \in X$.

Since the function $(Nf)(t)$ is continuous (as a composition of continuous functions), the operator N maps the space X into itself.

By assumption (C4) we have

$$\varrho_{j^n(K)}(\sigma, N\sigma) = \sup \{\|F(t, 0, \ldots, 0, 0, \ldots, 0)\| : t \in j^n_+(K)\} \leqq$$
$$\leqq \sup \{\|F(t, \ldots)\| : t \in \bar{K}_+\} = \varrho_{\bar{K}}(\sigma, N\sigma)$$

where

$$\sigma(t) = \begin{cases} 0, & t > 0, \\ \theta(t), & t \leqq 0, \end{cases}$$

that is, condition 3 of Theorem 2 is fulfilled; $j^n_+(K) = R^1_+ \cap j^n(K)$, $\bar{K}_+ = R^1_+ \cap \bar{K}$.

We already gave an explicit form of the mapping $j: A \to A$ so that we are now able to show that the operator N is a Φ-contraction.

Let $K \subset R^1$ be an arbitrary compact set and $f, g \in X$. Then for $t \in K \cap R^1_+$ we obtain

$$\|(Nf)(t) - (Ng)(t)\| \leqq \Omega(t, |\Delta_1(t)| \sup \{\|f(s) - g(s)\| : s \in K_{\Delta_1}\}, \ldots,$$
$$|\Delta_m(t)| \sup \{\|f(s) - g(s)\| : s \in K_{\Delta_m}\}, \quad \sup \{\|f(t) - g(t)\| : t \in K_{\tau_1}\}. \ldots,$$
$$\sup \{\|f(t) - g(t)\| : t \in K_{\tau_n}\}) \leqq \Omega(t, \bar{\Delta}_1 \sup \{\|f(s) - g(s)\| : s \in j(K)\}, \ldots,$$
$$\bar{\Delta}_m \sup \{\|f(s) - g(s)\| : s \in j(K)\}, \quad \sup \{\|f(t) - g(t) :\| t \in j(K)\}, \ldots,$$
$$\sup \{\|f(t) - g(t)\| : t \in j(K)\}) \leqq \Omega(t, \bar{\Delta}\varrho_{j(K)}(f, g), \ldots,$$
$$\bar{\Delta}\varrho_{j(K)}(f, g), \varrho_{j(K)}(f, g), \ldots, \varrho_{j(K)}(f, g)) \leqq \Phi_K(\varrho_{j(K)}(f, g)),$$

where

$$\bar{\Delta} = \max \{\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m\}, \quad \bar{\Delta}_i = \sup \{|\Delta_i(t)| : t \in K\} \quad (i = 1, 2, \ldots, m).$$

For $t \in K \cap R^1_-$ we have

$$\|(Nf)(t) - (Ng)(t)\| = 0.$$

Having in mind the definition of the function $\Phi_K(t)$ (cf. (C2)) we conclude that

$$\varrho_K(Nf, Ng) \leqq \Phi_K(\varrho_{j(K)}(f, g)).$$

Assumption (C4) implies that

$$\Phi_{j^n(K)}(y) \leqq \Phi_{\bar{K}}(y) = \bar{\Phi}_K(y) \in \Phi$$

and

$$\varrho_{j^n(K)}(f, g) \leqq \varrho_{\bar{K}}(f, g) \equiv q(f, g, K) < \infty .$$

Since all conditions of Theorems 2 and 3 are satisfied, we may assert that there is a unique solution $x(t) \in X$ of IVP (3′).

Theorem 7 is thus proved.

Let us compare it with some related results. It is well known (cf. [20]) that the Lipschitz constant l in the equation $y'(t) = l\, y'(\tau(t)) + h(t)$, $y(0) = 0$, $\tau(t) \leqq t$ must satisfy the condition $|l| < 1$ in the case $\tau(t) = t$ for some values of t. If we seek a global solution of the IVP

$$y'(t) = l(t)\, y'(\tau(t)) + h(t)\,, \quad t > 0\,, \quad y'(t) = \psi'(t)\,, \quad t \leqq 0$$

with a deviation $\tau(t)$ of mixed type and unbounded, the results of the paper [21] imply that if $\psi'(t)$ is bounded and continuous and $\bar{l} < 1$, where $\bar{l} = \sup\{|l(t)|: t \in R^1_+\}$, then there exists a unique solution $y(t)$ with a continuous and bounded derivative. Theorem 7 of the present paper guarantees existence and uniqueness of a global solution with a continuous derivative, which is not necessarily bounded, i.e. the solution belongs to a more general class of functions. Besides, we have existence and uniqueness even in the case when $|l(t)| < 1$, but $\bar{l} = 1$. For example, the IVP

$$\begin{aligned} y'(t) &= (1 - e^{-t})\, y'(-t) + e^{t} - e^{-t} + e^{-2t}\,, \quad t > 0 \\ y(t) &= e^{t}\,, \quad y'(t) = e^{t}\,, \quad t \leqq 0 \end{aligned}$$

has a unique solution although $\bar{l} = \sup\{1 - e^{-t}: t \in R^1_+\}$. The solution is $y(t) = e^t$.

Let us note that condition (C4) restricts the class of the deviations $\Delta_i(t)$, $\tau_l(t)$ when they are of advanced type, $\Delta_i(t) \geqq t$, $\tau_l(t) \geqq t$. But it is known [22] that without a restriction on the magnitude of the advancement we have neither existence nor uniqueness.

It is easy to formulate theorems for existence and uniqueness of the solution for nonlinear functional equation

$$\begin{aligned} \varphi(t) &= F(t, \varphi(\tau_1(t)), \ldots, \varphi(\tau_n(t)))\,, \quad t > 0\,, \\ \varphi(t) &= \theta(t)\,, \quad t \leqq 0 \end{aligned}$$

because this equation is a particular case of (3′).

As another application we shall seek a generalized solution of IVP (3′) in the space $L^\infty_{\mathrm{loc}}(R^1; B)$, consisting of all strongly measurable functions $f(t): R^1 \to B$, which are locally essentially bounded. It is a locally convex Hausdorff space with a topology defined by the neighbourhoods of zero

$$U_{\varepsilon,n} = \{f \in L^\infty_{\mathrm{loc}}(R^1; B): \|f\|_1 < \varepsilon, \ldots, \|f\|_n < \varepsilon\}\,,$$

where

$$\|f\|_i = \operatorname{ess\,sup}\{\|f(t)\|: t \in E_i\} \ (i = 1, \ldots, n)\,, \quad \{E_i\}_{i=1}^n$$

is a finite system of compact subsets of R^1.

We introduce the following assumptions:

(M1) The functions $\Delta_i(t)$, $\tau_l(t)$: $R^1_+ \to R^1$ ($i = 1, \ldots, m$; $l = 1, \ldots, n$) are measurable and map every bounded set into a bounded set. Since the index set A coincides with the totality of all compact subsets $E \subset R^1$, we are now going to define the map $j: A \to A$. The set $j(E)$ is defined in the following way: if $E_+ = E \cap (0, \infty) \neq \emptyset$, then we set $j(E) = (\bigcup_{i=1}^{m} E_{\Delta_i}) \cup (\bigcup_{l=1}^{n} E_{\tau_l})$, and if $E_+ = \emptyset$, then $j(E) = E$ where

$$E = \begin{cases} [\bar{\Delta}_i, \bar{\bar{\Delta}}_i] & \text{when} \quad 0 \in [\bar{\Delta}_i, \bar{\bar{\Delta}}_i]\,, \\ [0, \bar{\bar{\Delta}}_i] & \text{when} \quad \bar{\Delta}_i \geqq 0\,, \\ [\bar{\Delta}_i, 0] & \text{when} \quad \bar{\bar{\Delta}}_i \leqq 0\,, \end{cases}$$

$\bar{\Delta}_i = \operatorname{ess\,inf}\{\Delta_i(t): t \in E\}$, $\bar{\bar{\Delta}}_i = \operatorname{ess\,sup}\{\Delta_i(t): t \in E\}$, $E_{\tau_l} = \tau_l(E)$ ($i = 1, \ldots, m$; $l = 1, \ldots, n$). If the set $j(E)$ is not closed, then we set $j(E) = \operatorname{cl}[(\bigcup_{i=1}^{m} E_{\Delta_i}) \cup (\bigcup_{l=1}^{n} E_{\tau_l})]$ (i.e. $j(E)$ becomes a compact). The map $j^n(E)$ is defined inductivively, i.e. $j^n(E) = j(j^{n-1}(E))$, n positive integer.

(M2) The function $F(t, u_1, \ldots, u_m, v_1, \ldots, v_n)$: $R^1_+ \times B^{m+n} \to B$ satisfies the Carathéodory condition (measurable in t and continuous in $u_1, \ldots, u_m, v_1, \ldots, v_n$) and the conditions

$$\|F(t, u_1, \ldots, u_m, v_1, \ldots, v_n)\| \leqq \omega(t, \|u_1\|, \ldots, \|u_m\|, \|v_1\|, \ldots, \|v_n\|)\,,$$
$$\|F(t, u_1, \ldots, u_m, v_1, \ldots, v_n) - F(t, \bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n)\| \leqq$$
$$\leqq \Omega(t, \|u_1 - \bar{u}_1\|, \ldots, \|u_m - \bar{u}_m\|, \|v_1 - \bar{v}_1\|, \ldots, \|v_n - \bar{v}_n\|)\,,$$

where the functions $\omega(t, x_1, \ldots, x_m, y_1, \ldots, y_n)$, $\Omega(t, x_1, \ldots, x_m, y_1, \ldots, y_n)$: $R^{m+n+1}_+ \to R^1_+$ satisfy the Carathéodory condition. They are nondecreasing in x_i; y_l, and for any fixed $(x_1, \ldots, x_m, y_1, \ldots, y_n) \in R^{m+n}_+$,

$$\omega(\cdot, x_1, \ldots, x_m, y_1, \ldots, y_n),\ \Omega(\cdot, x_1, \ldots, x_m, y_1, \ldots, y_n) \in L^\infty_{\mathrm{loc}}(R^1_+)\,.$$

Besides, the function

$$\Omega_E(y) = \operatorname{ess\,sup}\{\Omega(t, ay, \ldots, ay, y, \ldots, y): t \in E\}$$

(when $\operatorname{mes} E_+ > 0$) is continuous from the right and $\Omega_E(y) < y$, $y > 0$ for any compact set $E \subset R^1$ and for any constant $a > 0$; $\Omega_E(y) = 0$ when $E_+ = \emptyset$ or $\operatorname{mes} E_+ = 0$, and $\Omega_E(y)/y$ is non-decreasing.

(M3) The initial function $O(t) \in L^\infty_{\mathrm{loc}}(R^1_-; B)$.

(M4) For every compact set E, there exists a compact interval $\bar{E} \subset R^1$ such that $j^n(E) \subseteq \bar{E}$, $n = 0, 1, 2, \ldots$.

Theorem 8. *If the assumptions* (M1)–(M4) *are satisfied, then there exists a unique solution* $x(t) \in L^\infty_{\mathrm{loc}}(R^1; B)$ *of IVP* (3').

Proof. Let X be the uniform space which consists of all functions $f \in L^\infty_{\mathrm{loc}}(R^1; B)$

equal to $O(t)$ for $t \in R^1_-$, with a saturated family of pseudometrics $\varrho_E(f, g) = $ $= \operatorname{ess\,sup} \{\|f(t) - g(t)\| : t \in E\}$ where E is an arbitrary compact subset of R^1.

The space $\mathscr{D} = \mathscr{D}(R^1; B)$ (consisting of all infinitely differentiable functions with compact support) is dense in X.

The operator $N: \mathscr{D} \to X$ is defined by the formula

$$(Nf)(t) = \begin{cases} F\left(t, \int_0^{\Delta_1(t)} f(s)\,\mathrm{d}s, \ldots, \int_0^{\Delta_m(t)} f(s)\,\mathrm{d}s, f(\tau_1(t)), \ldots, f(\tau_n(t))\right), & t > 0, \\ O(t), \quad t \leqq 0 \end{cases}$$

where $f \in \mathscr{D}$.

Since the function $f(t) \in \mathscr{D}$ is continuous, the compositions $f(\tau_l(t))$ $(l = 1, \ldots, n)$ are strongly measurable functions and therefore $(Nf)(t)$ is also a strongly measurable function. The estimate

$$\|(Nf)(t)\| \leqq \omega(t, \bar{\Delta}_1 \operatorname{ess\,sup} \{\|f(t)\| : t \in E\}, \ldots, \operatorname{ess\,sup} \{\|f(t)\| : t \in E_{\tau_n}\})$$

shows that the operator N maps $\mathscr{D}$ into X.

The operator N is a Φ-contraction. Indeed, if $E \subset R^1$ and $f, g \in \mathscr{D}$, then for $t \in E \cap R^1_+$ we obtain

$$\|(Nf)(t) - (Ng)(t)\| \leqq \Omega(t, |\Delta_1(t)| \operatorname{ess\,sup} \{\|f(s) - g(s)\| : s \in E_{\Delta_1}\}, \ldots,$$
$$|\Delta_m(t)| \operatorname{ess\,sup} \{\|f(s) - g(s)\| : s \in E_{\Delta_m}\}, \quad \operatorname{ess\,sup} \{\|f(t) - g(t)\| : t \in E_{\tau_1}\}, \ldots,$$
$$\operatorname{ess\,sup} \{\|f(t) - g(t)\| : t \in E_{\tau_n}\}) \leqq \Omega(t, \bar{\Delta}\varrho_{j(E)}(f, g), \ldots, \bar{\Delta}\varrho_{j(E)}(f, g),$$
$$\varrho_{j(E)}(f, g), \ldots, \varrho_{j(E)}(f, g)) \leqq \Omega_E(\varrho_{j(E)}(f, g)),$$

where $\bar{\Delta} = \max \{\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m\}$, $\bar{\Delta}_i = \operatorname{ess\,sup} \{|\Delta_i(t)| : t \in E\}$ $(i = 1, \ldots, m)$.

Therefore

$$\varrho_E(Nf, Ng) \leqq \Omega_E(\varrho_{j(E)}(f, g)).$$

The operator N is uniformly continuous. Since it is defined on a dense set $\mathscr{D}$, we may employ Theorem 4 [12], p. 33. The resulting extension on X we denote again by N. The operator N satisfies the conditions of Theorems 2 and 3. The conclusion of the present theorem is obtained in the same way as that of Theorem 7.

Remark 5. In [23] Zverkin has proved that the existence of an absolutely continuous solution of the neutral equation implies measurability of the functions $\Delta_i(t)$, $\tau_l(t)$. In the known results (see [20], [24] and references theirein), $\tau_l(t)$ has the following additional property: the inverse image of every null set is measurable. Here this additional condition is superfluous. As the proof of Theorem 8 shows, a basic role is played by the uniform continuity of the operator defined by the right-hand side of the equation (3′), and in fact, the extended operator yields the solution.

In order to illustrate the generalized solutions of (3′) which may be obtained by means of Theorem 8, we give a simple example. Its solution may be constructed in an explicit form by the step method.

Indeed, the IVP

$$x(t)=\begin{cases}\alpha(t)\,x(t-1)+1\,, & t>0\,,\quad \alpha(t)=1-\dfrac{1}{n}\,,\quad n-2\leqq t<n-1\ (n=2,3,\ldots),\\ 0\,, & -1\leqq t<0\,,\\ 1\,, & t=0\end{cases}$$

possesses the solution

$$x(t)=\begin{cases}1\,, & 0\leqq t<1\,,\\ (1-\tfrac{1}{3})+1\,, & 1\leqq t<2\,,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ (1-\tfrac{1}{3})(1-\tfrac{1}{4})\ldots\left(1-\dfrac{1}{n+2}\right)+(1-\tfrac{1}{4})(1-\tfrac{1}{5})\ldots\left(1-\dfrac{1}{n+2}\right)+\ldots & \\ \ldots+\left(1-\dfrac{1}{n+2}\right)+1=\dfrac{n(n+3)}{2(n+2)}+1\,, & n\leqq t<n+1\,.\end{cases}$$

In this example the discontinuity of the initial function induces discontinuities of the solution. Let us note that $\alpha(t)<1$ but ess sup $\{\alpha(t): t\in R^1_+\}=1$.

It is easy to verify that Theorem 8 implies existence and uniqueness of the solution $x(t)\in L^\infty_{\mathrm{loc}}(R^1; B)$.

As a consequence of Theorem 8 we obtain new existence – uniqueness results for the nonlinear functional equation (4) in the space $L^\infty_{\mathrm{loc}}(R^1; B)$. It is known ([25], pp. 44–45) that the problem of uniqueness of the solution is very important in the theory of functional equations.

Finally, we formulate conditions for existence and uniqueness of a solution of IVP (3′) belonging to $L^1_{\mathrm{loc}}(R^1; B)$.

Let us suppose

(L1) The functions $\Delta_i(t), \tau_l(t): R^1_+\to R^1$ are measurable and map bounded sets into bounded sets; $\tau_l(t)$ have the property

$$\int_E \|f(\tau_l(t))\|\,\mathrm{d}t\leqq k\int_{\tau(E)}\|f(t)\|\,\mathrm{d}t$$

for any continuous and bounded function $f(t)$ and any compact $E\subset R^1_+$, k is a constant. The mapping $j: A\to A$ is defined as in (M1).

(L2) The function $F(t,u_1,\ldots,v_n): R^1_+\times B^{m+n}\to B$ satisfies the Carathéodory condition and

$$\|F(t,u_1,\ldots,u_m,v_1,\ldots,v_n)\|\leqq\omega_0(t)+\omega_1\Big(\sum_{i=1}^m\|u_i\|+\sum_{l=1}^n\|v_l\|\Big)\,,$$

$$\|F(t,u_1,\ldots,u_m,v_1,\ldots,v_n)-F(t,\bar u_1,\ldots,\bar u_m,\bar v_1,\ldots,\bar v_n)\|\leqq$$

$$\leqq\alpha(t)\sum_{i=1}^m\|u_i-\bar u_i\|+\beta\sum_{l=1}^n\|v_l-\bar v_l\|$$

where $\omega_0(t),\ \alpha(t)\in L^1_{\mathrm{loc}}(R^1_+; R^1_+)$, ω_1, $\beta>0$ are constants.

Besides, for every compact $E \subset R^1_+$,

$$m \int_E \alpha(t)\, dt + n\beta k < 1 .$$

(L3) $O(t) \in L^1_{loc}(R^1_-; B)$.

(L4) For every compact $E \subset R^1$ there is a compact $\bar{E}$ such that $j^n(E) \subseteq E$ $(n = 0, 1, 2, \ldots)$.

Theorem 9. *Under the assumptions* (L1)–(L4), IVP (3′) *has a unique solution* $x(t) \in L^1_{loc}(R^1; B)$.

Proof. Let X be the uniform space of all $f \in L^1_{loc}(R^1; B)$ which equal $O(t)$ for a.e. $t \in R^1_-$, with a saturated family of pseudometrics $\varrho_E(f, g) = \int_E \|f(t) - g(t)\|\, dt$ where E runs over compact subsets of R^1.

The set $\mathscr{BC} = \{f \in L^1_{loc}(R^1; B): f$ is bounded and continuous$\}$ is dense in X. The operator $N\colon \mathscr{BC} \to X$ can be defined as in the proof of Theorem 8.

The estimate

$$\|(Nf)(t)\| \leqq \omega_0(t) + \omega_1 \left[\sum_{i=1}^{m} \int_{E\Delta_i} \|f(t)\|\, dt + \sum_{l=1}^{n} \|f(\tau_l(t))\| \right]$$

shows that $Nf \in X$.

The operator N is a Φ-contraction. Indeed,

$$\int_E \|(Nf)(t) - (Ng)(t)\|\, dt \leqq \sum_{i=1}^{m} \int_E \alpha(t) \left| \int_0^{\Delta_i(t)} \|f(s) - g(s)\|\, ds \right| dt +$$

$$+ \beta \sum_{l=1}^{n} k \int_{\tau_l(E)} \|f(t) - g(t)\|\, dt \leqq \left[m \int_E \alpha(t)\, dt + n\beta k \right] \varrho_{j(E)}(f, g) .$$

Further on, the proof can proceed as that of Theorem 8.

Example:

$$x(t) = \begin{cases} \alpha(t)\, x(t-1)\,, \quad t \geqq 0\,, \quad \alpha(t) = 1 - \dfrac{1}{n+1} & (n = 1, 2, \ldots)\,, \\ & n - 1 \leqq t \leqq n\,. \\ 1/\sqrt{|t|}\,, \quad -1 \leqq t < 0\,, & \\ 1\,, \quad t = 0\,. & \end{cases}$$

Then the solution $x(t) \in L^1_{loc}(R^1_+)$ has the form

$$x(t) = \begin{cases} \dfrac{1}{2\sqrt{|t-1|}}\,, & 0 \leqq t < 1\,, \\ \dfrac{2}{3}\dfrac{1}{2}\dfrac{1}{\sqrt{|t-2|}}\,, & 1 \leqq t < 2\,, \\ \dots\dots\dots\dots\dots\dots & \\ \dfrac{n}{n+1}\,\dfrac{n-1}{n} \cdots \dfrac{1}{2}\,\dfrac{1}{\sqrt{|t-n|}}\,, & n - 1 \leqq t < n\,, \\ \dots\dots\dots\dots\dots\dots & \end{cases}$$

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