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QUASIGROUPS DETERMINED BY BALANCED IDENTITIES OF LENGTH $\leqq 6$

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A universal algebra (Q, f_1, f_2, f_3) is called a *primitive quasigroup* if f_2 and f_3 is respectively the left and the right division operation of the operation f_1; if f_1 is denoted by $(\cdot)$ then put $f_2 = /$, $f_3 = \backslash$, thus $(Q, \cdot, /, \backslash)$ means a primitive quasigroup.

An identity $w \doteq w'$ on a primitive quasigroup is called *balanced* if each variable appears exactly twice in $w \doteq w'$, once on each side. An identity on a primitive quasigroup $(Q, \cdot, /, \backslash)$ is called *strictly balanced* if it is balanced and contains neither $/$ nor $\backslash$.

In [4] J. Ježek and T. Kepka found all varieties of quasigroups determined by a set of strictly balanced identities of length $\leqq 6$; there are eleven such varieties. In this paper we find all varieties of quasigroups determined by an identity of the set of all balanced identities of length $\leqq 6$ on a primitive quasigroup $(Q, \cdot, /, \backslash)$.

1. NOTATIONS AND PRELIMINARIES

Let $(Q, \cdot, /, \backslash)$ be a primitive quasigroup; we shall denote $L_a x = a \,.\, x$, $R_a x = x \,.\, a$, $T_a x = x \backslash a$. Then $L_a^{-1} x = a \backslash x$, $R_a^{-1} x = a / x$. Further we denote $\mathcal{T} = \{L, R, T, L^{-1}, R^{-1}, T^{-1}\}$ and say that for each $X \in \mathcal{T}$ and $a \in Q$, X_a is a translation of $(Q, \cdot)$. If a quasigroup operation is denoted, say, by $\Box$, then write $L_a^{\Box}, R_a^{\Box}, \ldots$ $\ldots, \mathcal{T}^{\Box} = \{L^{\Box}, R^{\Box}, \ldots\}$. For $(Q, \cdot, /, \backslash)$ put $\Sigma\mathcal{T} = \mathcal{T} \cup \mathcal{T}^{/} \cup \mathcal{T}^{\backslash}$.

If (Q, A) is a quasigroup and A is denoted by $(\cdot)$ then put ${}^{-1}A = /$, $A^{-1} = \backslash$, ${}^{-1}(A^{-1}) = \nabla$, $({}^{-1}A)^{-1} = \Delta$, $({}^{-1}(A^{-1}))^{-1} = *$ and $\Sigma(\cdot) = \{\cdot, /, \nabla, \Delta, *, \backslash\}$. Relations between translations of $\Sigma\mathcal{T}$ are given in Table 1.

From this table we have, for example, $(L^{-1})^{\nabla} = R$, $T^{\nabla} = L$ etc., $x \,.\, y = z$ iff $y * x = z$ iff $z / y = x$ etc. Thus each translation of a quasigroup $(Q, \Box)$, where $\Box \in \Sigma(\cdot)$, is a translation of $(Q, \cdot)$.

1.1. Lemma. *Let $w \doteq w'$ be a balanced identity of length $\leqq 6$ on a primitive quasigroup $(Q, \cdot, /, \backslash)$. Then there exist operations $\Box_1, \Box_2, \Box_3, \Box_4 \in \Sigma(\cdot)$*

such that $w \simeq w'$ *is equivalent with at least one of the identities*

(I) $$x \square_1 (y \square_2 z) = x \square_3 (y \square_4 z),$$

(II) $$x \square_1 (y \square_2 z) = (x \square_3 y) \square_4 z.$$

Every primitive quasigroup $(Q, \cdot, /, \backslash)$ *that satisfies the identity* (II) *is transitive* (i.e. $(Q, \cdot)$, $(Q, /)$, $(Q, \backslash)$ *are all transitive quasigroups*).

Proof. See [2, Lemma 1.1].

Table 1

	$\cdot$	$*$	$/$	∇	$\backslash$	Δ
L	L	R	T^{-1}	R^{-1}	L^{-1}	T
R	R	L	R^{-1}	T^{-1}	T	L^{-1}
T	T	T^{-1}	L^{-1}	L	R	R^{-1}
L^{-1}	L^{-1}	R^{-1}	T	R	L	T^{-1}
R^{-1}	R^{-1}	L^{-1}	R	T	T^{-1}	L
T^{-1}	T^{-1}	T	L	L^{-1}	R^{-1}	R
	$x \cdot y = z$	$y * x = z$	$z / y = x$	$y \nabla z = x$	$x \backslash z = y$	$z \Delta x = y$

1.2. Lemma. *Each of the identities* I, II *on* $\Sigma(\cdot)$ *is equivalent to a balanced identity on a primitive quasigroup* $(Q, \cdot, /, \backslash)$.

Proof. It is a consequence of the following statement: For $\square_i \in \{*, \nabla, \Delta\}$ there exists $\square \in \{\cdot, /, \backslash\}$ such that for all x, y, $x \square_i y = y \square x$ (see Table 1).

2. QUASIGROUPS DETERMINED BY IDENTITY I

Let $A, B, C, D \in \Sigma T$; denote by $\mathrm{Mod}(AB \simeq CD)$ the class of all quasigroups $(Q, \cdot)$ such that $A_x B_x y = C_z D_z y$ for all x, y, z in Q; if $A = C$ and $B = D$ put $\mathrm{Mod}(AB \simeq CD) = \mathrm{M}(AB)$.

2.1. Lemma. *For every* $A, B \in \Sigma\mathscr{T}$ *there exist* $C, D \in \mathscr{T}$ *such that* $\mathrm{M}(AB) = \mathrm{M}(CD)$.

Proof. See Table 1.

2.2. Lemma. *Let* $(Q, \cdot, /, \backslash)$ *be a primitive quasigroup. Then the following relations are equivalent*:

(i) *there exists a balanced identity of length* ≤ 6 *of type* I *that is valid on* $(Q, \cdot, /, \backslash)$;

(ii) *there exist* $A, B, C, D \in \mathscr{T}$ *such that* $(Q, \cdot) \in \mathrm{Mod}(AB \simeq CD)$.

Proof. It suffices to rewrite I with translations and to use Table 1.

2.3. Lemma. *A quasigroup* $(Q, \cdot) \in \mathrm{M}(AB)$ *iff there exists a permutation* φ *of* Q *such that* $A_x B_x = \varphi$ *for all* $x \in Q$.

Proof. Easy.

A quasigroup $(Q, \cdot)$ is called an LIP (or RIP)-*quasigroup* if there exists a permutation I_l (or I_r) of Q such that for all $x, y \in Q$, $I_l x \, . \, xy = y$ ($yx \, . \, I_r x = y$, respectively). A quasigroup is called an IP-*quasigroup* if it is both an LIP- and an RIP-quasigroup. A commutative IP-quasigroup is called a CIP-*quasigroup*. Let $(Q, \cdot)$ be a loop, e the identity of $(Q, \cdot)$ and $x \, . \, I_r x = e$ for all $x \in Q$; a loop $(Q, \cdot)$ is called a WIP- or a CI-*loop* if for all $x, y \in Q$, $x \, . \, I_r(xy) = I_r y$ or $xy \, . \, I_r x = y$, respectively (see [1]).

2.4. Lemma. *For each* $A, B \in \mathcal{T}$ *let there exist a permutation* φ *of* Q *such that* $A_x B_x = \varphi$ *for all* $x \in Q$. *Then*

(1) $L_x L_x = \varphi \Leftrightarrow x \, . \, xy \bumpeq \varphi y$

(2) $T_x R_x^{-1} = \varphi \Leftrightarrow x \, . \, \varphi(xy) \bumpeq y$

(3) $T_x^{-1} R_x = \varphi \Leftrightarrow \varphi x \, . \, xy \bumpeq y$

(4) $L_x^{-1} R_x = \varphi \Leftrightarrow xy \bumpeq y \varphi x$

(5) $L_x R_x = \varphi \Leftrightarrow x \, . \, yx \bumpeq \varphi y$

(6) $R_x T_x = \varphi \Leftrightarrow x \, . \, yx \bumpeq \varphi y$

(7) $T_x R_x = \varphi \Leftrightarrow xy \, . \, \varphi x \bumpeq y$

(8) $T_x T_x = \varphi \Leftrightarrow xy \bumpeq y \, . \, \varphi x$

(9) $L_x R_x^{-1} = \varphi \Leftrightarrow xy \bumpeq \varphi(yx)$

(10) $R_x R_x = \varphi \Leftrightarrow yx \, . \, x \bumpeq \varphi y$

(11) $T_x^{-1} L_x^{-1} = \varphi \Leftrightarrow \varphi(yx) \, . \, x \bumpeq y$

(12) $T_x L_x = \varphi \Leftrightarrow yx \, . \, \varphi x \bumpeq y$

(13) $R_x^{-1} L_x = \varphi \Leftrightarrow xy \bumpeq \varphi y \, . \, x$

(14) $R_x L_x = \varphi \Leftrightarrow xy \, . \, x \bumpeq \varphi y$

(15) $L_x T_x^{-1} = \varphi \Leftrightarrow xy \, . \, x \bumpeq \varphi y$

(16) $T_x^{-1} L_x = \varphi \Leftrightarrow \varphi x \, . \, yx \bumpeq y$

(17) $T_x^{-1} T_x^{-1} = \varphi \Leftrightarrow \varphi x \, . \, y \bumpeq yx$

(18) $R_x L_x^{-1} = \varphi \Leftrightarrow \varphi(xy) \bumpeq yx$

(19) $A_x A_x^{-1} = \varphi \Rightarrow \varphi = 1$

(20) $A_x A_x = \varphi \Rightarrow \varphi = 1$

(21) $T_x R_x^{-1} = \varphi \Rightarrow \varphi = 1$

(22) $L_x^{-1} R_x = \varphi \Rightarrow \varphi = 1$

(23) $T_x^{-1} R_x = \varphi \Rightarrow \varphi^2 = 1$

(24) $T_x L_x = \varphi \Rightarrow \varphi^2 = 1$

(25) $L_x R_x^{-1} = \varphi \Rightarrow \varphi^2 = 1$

(26) $L_x L_x = \varphi \Leftrightarrow T_x R_x^{-1} = \varphi$

(27) $L_x^{-1} R_x = \varphi \Leftrightarrow T_x T_x = \varphi$

(28) $L_x R_x = \varphi \Leftrightarrow R_x T_x = \varphi$

(29) $T_x R_x = \varphi \Leftrightarrow L_x^{-1} T_x = \varphi$

Proof. The relations (1)–(9) are duals of (10)–(18); we prove only (8): $T_x T_x = \varphi$ iff for every $y \in Q$, $T_x y = T_x^{-1} \varphi y$ iff $x = y \, . \, T_x^{-1} \varphi y$ iff $T_x^{-1} \varphi y = z$, and $x = yz$ iff $x = z \varphi y$ and $x = yz$ iff $z \, . \, \varphi y = yz$. Further, we prove the implication: $L_x L_x = \varphi$ for all x implies $\varphi = 1$. If we put $xy = y$ in $x \, . \, xy = \varphi y$ then $y = xy = x \, . \, xy = \varphi y$, i.e. $\varphi = 1$. It follows from Table 1 that for each $A \in \mathcal{T}$ there exists $\square \in \Sigma(\cdot)$ such that $A = L^{\square}$. Therefore $A_x A_x = \varphi$ implies $L_x^{\square} L_x^{\square} = \varphi$, hence $\varphi = 1$; this proves (20). Now, we prove (21): Let $T_x R_x^{-1} = \varphi$, then by (2), $x \, . \, \varphi(xy) = y$ and if we write xy instead of y, $x \, . \, \varphi(x \, . \, xy) = xy$ whence $\varphi(x \, . \, xy) = y$, i.e. $x \, . \, xy = \varphi^{-1} y$ and by (20), $\varphi^{-1} = 1 = \varphi$. Further, let us prove (22): From Table 1 it follows that $L_x^{-1} R_x = \varphi$ iff $T_x^l (R_x^l)^{-1} = \varphi$ so that, by (21), $\varphi = 1$. (23) and (24) are known properties of IP-quasigroups. (25) follows from (9). (26) follows from (20), (21), (1) and (3). (27) follows from (4) and (8). (28) follows from (5) and (6). (29) follows from (7) and (16).

2.5. Lemma. *For each* $A, B \in \mathscr{T}$, $\mathrm{M}(AB) = \mathrm{M}(B^{-1}A^{-1})$.

Proof. Easy.

2.6. Theorem. *The following relations hold*:

$$\mathrm{M}(LL^{-1}) = \mathrm{M}(RR^{-1}) = \mathrm{M}(TT^{-1}) = \mathrm{M}(L^{-1}L) = \mathrm{M}(R^{-1}R) = \mathrm{M}(T^{-1}T),$$
$$\mathrm{M}(L^{-1}R) = \mathrm{M}(R^{-1}L) = \mathrm{M}(TT) = \mathrm{M}(T^{-1}T^{-1}),$$
$$\mathrm{M}(LL) = \mathrm{M}(L^{-1}L^{-1}) = \mathrm{M}(TR^{-1}) = \mathrm{M}(RT^{-1}),$$
$$\mathrm{M}(RR) = \mathrm{M}(R^{-1}R^{-1}) = \mathrm{M}(T^{-1}L^{-1}) = \mathrm{M}(LT),$$
$$\mathrm{M}(T^{-1}R) = \mathrm{M}(R^{-1}T),$$
$$\mathrm{M}(TL) = \mathrm{M}(L^{-1}T^{-1}),$$
$$\mathrm{M}(LR^{-1}) = \mathrm{M}(RL^{-1}),$$
$$\mathrm{M}(LR) = \mathrm{M}(R^{-1}L^{-1}) = \mathrm{M}(RT) = \mathrm{M}(T^{-1}R^{-1}),$$
$$\mathrm{M}(RL) = \mathrm{M}(L^{-1}R^{-1}) = \mathrm{M}(LT^{-1}) = \mathrm{M}(TL^{-1}),$$
$$\mathrm{M}(TR) = \mathrm{M}(R^{-1}T^{-1}) = \mathrm{M}(L^{-1}T) = \mathrm{M}(T^{-1}L).$$

Proof follows from Lemmas 2.3, 2.4, 2.5.

2.7. Lemma. *Let* $(Q, \cdot)$ *be a quasigroup and* φ *a permutation of* Q *such that* $x \,.\, yx = \varphi y$ *for all* $x, y \in Q$. *Then*

(i) $xy \,.\, \varphi x = \varphi y$ *for all* $x, y \in Q$;

(ii) φ *is an automorphism of* $(Q, \cdot)$;

(iii) $xy = \varphi yx$ *for all* $x, y \Leftrightarrow \varphi x \,.\, xy = y$ *for all* $x, y \Leftrightarrow R_x^2 = 1$ *for all* x;

(iv) $L_x R_x^{-1} = \varphi$ *for all* $x \Leftrightarrow T_x^{-1} R_x = \varphi$ *for all* x.

Proof. (i). From $x \,.\, yx = \varphi y$ we have $yx \,.\, (x \,.\, yx) = yx \,.\, \varphi y$, i.e. $L_{yx} R_{yx} x = yx \,.\, \varphi y$, therefore $\varphi x = yx \,.\, \varphi y$, i.e. $R_{\varphi y} L_y = \varphi$. (ii). From $x \,.\, yx = \varphi y$ we have $(x \,.\, yx) \,.\, \varphi x = \varphi y \,.\, \varphi x$, i.e. $R_{\varphi x} L_x yx = \varphi y \,.\, \varphi x$ and by (i), $\varphi yx = \varphi y \,.\, \varphi x$. (iii). We prove the implications $xy = \varphi yx \Rightarrow \varphi x \,.\, xy = y \Rightarrow yx \,.\, x = y \Rightarrow \varphi(xy) = yx$. Let $xy = \varphi yx$; then $\varphi^2 = 1$ so that from $x \,.\, yx = \varphi y$ we have $\varphi(x \,.\, yx) = \varphi(\varphi y) = y$, whence $\varphi x \,.\, \varphi yx = y$ and also $\varphi x \,.\, xy = y$, i.e. $L_{\varphi x} L_x = 1$. From (ii) we have $\varphi L_x = L_{\varphi x} \varphi$, whence $L_x = \varphi L_{\varphi x} \varphi = L_x R_x L_{\varphi x} L_x R_x = L_x R_x^2$ so that $R_x^2 = 1$. If we write yx instead of y in $x \,.\, yx = \varphi y$ then $x(yx \,.\, x) = \varphi yx$ and according to $R_x^2 = 1$, $xy = \varphi yx$. (iv) directly follows from (iii).

2.8. Lemma. $L_x^{-1} T_x = 1$ *for all* x *iff* $T_x R_x = 1$ *for all* x *iff* $R_x T_x = 1$ *for all* x *iff* $L_x R_x = 1$ *for all* x.

Proof. Directly follows from (29) and (28).

2.9. Lemma. $R_x = T_x$ *for all* x *iff* $L_x^2 = 1$ *for all* x.

Proof. Directly follows from (26).

A commutative quasigroup $(Q, \cdot)$ is called a TS-*quasigroup* if $x \,.\, xy = y$ for all $x, y \in Q$.

2.10. Lemma. *If for all* $x, y \in Q$, $L_x L_x = R_y R_y$, *then a quasigroup* $(Q, \cdot)$ *is a* TS-*quasigroup*.

Proof. From (20), (26) and the dual of (26) we obtain $L_xL_x = \varphi = R_yR_y = 1 = T_xR_x^{-1}$ and $T_x^{-1}L_x^{-1} = 1$, whence $L_x^{-1} = R_x$. Since $L_x^2 = 1$ (i.e. $L_x = L_x^{-1}$), $L_x = R_x$.

2.11. Lemma. *If $(Q, \cdot)$ is a TS-quasigroup then for every $\square \in \Sigma(\cdot)$, $(Q, \cdot) = (Q, \square)$ and $(Q, \square)$ is a TS-quasigroup.*

Proof. Obvious.

By Φ we shall denote the group of all central regular permutations of a quasigroup $(Q, \cdot)$.

2.12. Lemma. *The following relations hold:*

(i) $\mathrm{Mod}(LR \doteq R^{-1}T^{-1}) = \mathrm{Mod}(LR \doteq T^{-1}L)$,

(ii) $(Q, \cdot) \in \mathrm{Mod}(LR \doteq R^{-1}T^{-1})$ *and $\varphi = L_xR_x$ for all x implies $\varphi \in \Phi$, $\varphi^3 = 1$, φ is the dual of φ (as a central regular transformation), and for each $x \in Q$, $L_x^2R_x^2 = 1$.*

Proof. From (29) it follows that $R_xT_x^{-1} = \varphi \Leftrightarrow T_x^{-1}L_x = \varphi$; this proves (i). Now, (ii). From (5) and (7) we obtain (iii) $x \,.\, (\varphi^{-1}y)\, x = y$ and (iv) $x \,.\, y\varphi^{-1}x = y$ whence (v) $\varphi^{-1}y \,.\, x = y \,.\, \varphi^{-1}x$, therefore $\varphi^{-1} \in \Phi$ (as well as $\varphi \in \Phi$), $(\varphi^{-1})^* = \varphi^{-1}$ and $yx = \varphi y \,.\, \varphi^{-1}x$ so that $(\varphi, \varphi^{-1}, 1)$ is an autotopy of $(Q, \cdot)$. From (iv) we have (vi) $xy \,.\, \varphi^{-1}x = y$ and by (v), $\varphi^{-1}(xy) \,.\, x = y$ and (vii) $(\varphi x)\, y \,.\, x = y$; therefore $\varphi^{-1}(xy) = \varphi x \,.\, y$, i.e. $(\varphi, 1, \varphi^{-1})$ is an autotopy of $(Q, \cdot)$. Then $(\varphi, \varphi^{-1}, 1)^{-1} \,.\, (\varphi, 1, \varphi^{-1}) = (1, \varphi, \varphi^{-1})$, $(\varphi, 1, \varphi^{-1})(1, \varphi, \varphi^{-1}) = (\varphi, \varphi, \varphi^{-2})$ are autotopies of $(Q, \cdot)$. By Lemma 2.7 (ii), $(\varphi^{-1}, \varphi^{-1}, \varphi^{1})$ is an autotopy of $(Q, \cdot)$ so that $(\varphi^{-1}, \varphi^{-1}, \varphi^{-1})\ (\varphi, \varphi, \varphi^{-2}) = (1, 1, \varphi^{-3})$ is an autotopy of $(Q, \cdot)$. Therefore $\varphi^{-3} = 1$, i.e. $\varphi^3 = 1$. From (vi) we have $(\varphi x)\, y \,.\, x = y$ and since $\varphi \in \Phi$, $x\varphi y \,.\, x = y$, i.e. $R_xL_x = \varphi^{-1}$; therefore $1 = \varphi^{-1}\varphi = R_xL_xL_xR_x$ and so $L_x^2R_x^2 = 1$.

2.13. Theorem. *Let $(Q, \cdot)$ be a quasigroup and φ a permutation of Q such that $x \,.\, y\varphi x = y$ for all $x, y \in Q$. Then φ is an automorphism of $(Q, \cdot)$. If $(Q, \cdot)$ is a loop then $(Q, \cdot)$ is a* WIP-*loop and a* CI-*loop.*

Proof. Obviously, $x \,.\, y\varphi x = y$ is equivalent to $xy \,.\, \varphi x = y$. Therefore $(xy \,.\, \varphi x) \,.\, \varphi xy = \varphi x$, (i) $y \,.\, \varphi xy = \varphi x$, $(y \,.\, \varphi(xy))\, \varphi y = \varphi x \,.\, \varphi y$, $\varphi xy = \varphi x \,.\, \varphi y$. Let $x \,.\, I_rx = 1$ for all x. Since $xI_rx \,.\, \varphi x = I_rx$, $1.\ x = I_rx$, i.e. $\varphi = \mathrm{I}$ and so with respect to (i), the loop $(Q, \cdot)$ is a WIP-loop and by the assumption a CI-loop as well.

2.14. Theorem. *Let $(Q, \cdot) \in \mathrm{M}(LR)$. Then $(Q, \cdot) \in \mathrm{Mod}(LR \doteq T^{-1}L)$ iff $L_xR_x = \varphi$ is a central regular permutation of $(Q, \cdot)$ and $\varphi = \varphi^*$.*

Proof. Let $\varphi \in \Phi$, $\varphi^* = \varphi$. Then $\varphi^{-1} \in \Phi$, $(\varphi^*)^{-1} = (\varphi^{-1})^*$. From $L_xR_x = \varphi$ we have $x \,.\, (\varphi^{-1}y)\, x = y$: since $\varphi^{-1} \in \Phi$, $x \,.\, y(\varphi^{-1}x) = y$, whence $xy \,.\, \varphi^{-1}x = y$ and with respect to (7), $T_xR_x = \varphi^{-1}$, i.e. $R_x^{-1}T_x^{-1} = \varphi$ and by (29), $\varphi = T_x^{-1}L_x$. The converse follows from Lemma 2.12 (ii).

We shall denote by $\mathrm{Mod}(w \doteq w')$ and $\mathrm{Mod}(w \to w')$ the variety and the quasivariety of quasigroups determined by an identity $w \doteq w'$ and the quasiidentity $w \to w'$, respectively.

We define 17 varieties:

(30)

$$
\begin{array}{lll}
V_1 = M(LL^{-1}) & V_8 = \mathrm{Mod}\,(LL^{-1} \doteq LR) & V_{13} = \mathrm{Mod}\,(LR^{-1} \doteq LR) \\
V_2 = M(L^{-1}R) & V_9 = \mathrm{Mod}\,(L^{-1}R \doteq LL) & V_{14} = \mathrm{Mod}\,(LR \doteq R^{-1}L^{-1}) \\
V_3 = M(LL) & V_{10} = \mathrm{Mod}\,(T^{-1}R \doteq TL) & V_{15} = \mathrm{Mod}\,(LR \doteq L^{-1}R^{-1}) \\
V_4 = M(T^{-1}R) & V_{11} = \mathrm{Mod}\,(T^{-1}R \doteq LR^{-1}) & V_{16} = \mathrm{Mod}\,(LR \doteq R^{-1}T^{-1}) \\
V_5 = M(LR^{-1}) & V_{12} = \mathrm{Mod}\,(T^{-1}R \doteq TR) & V_{17} = \mathrm{Mod}\,(TR \doteq R^{-1}T^{-1}) \\
V_6 = M(LR) & & \\
V_7 = M(TR). & &
\end{array}
$$

By V_i^* we shall denote the dual variety of V_i for $i \in \{1, 2, \ldots, 17\}$.

2.15. Theorem. *Let $(Q, \cdot) = G$ and φ denote a quasigroup and a permutation of Q, respectively. The following relations hold:*

$V_1 = \mathrm{Mod}\,(x \,.\, yz \doteq x \,.\, yz) = V_1^*$; V_1 *is the variety of quasigroups.*

$V_2 = \mathrm{Mod}\,(x \,.\, yz \doteq yz \,.\, x) = V_2^*$; V_2 *is the variety of commutative quasigroups.*

$V_3 = \mathrm{Mod}\,(x \,.\, xy \doteq z \,.\, zy) = \mathrm{Mod}\,(x \,.\, xy \doteq y)$.

$V_4 = \mathrm{Mod}\,(x = t \,.\, zx \rightarrow y = t \,.\, zy)$; V_4 *is the variety of* LIP-*quasigroups.*

$V_5 = \mathrm{Mod}\,(xy = tz \rightarrow yx = zt) = V_5^*$; $G \in V_5$ *if there exists φ such that $yx = \varphi(xy)$ for all x, y.*

$V_6 = \mathrm{Mod}\,(x \,.\, zx \doteq y \,.\, zy)$; $G \in V_6$ *iff there exists φ such that $x \,.\, yx = \varphi y$ for all x, y.*

$V_7 = \mathrm{Mod}\,(x = tx \,.\, z \rightarrow y = ty \,.\, z) = V_7^*$; $G \in V_7$ *iff there exists φ such that $xy \,.\, \varphi x = y$ for all x, y.*

$V_8 = \mathrm{Mod}\,(x \,.\, yz)\, x \doteq yz) = V_8^* = \mathrm{Mod}\,(x \,.\, yx \doteq y)$.

$V_9 = \mathrm{Mod}\,(x \,.\, xy \doteq yz \,.\, z) = V_9^*$; V_9 *is the variety of* TS-*quasigroups.*

$V_{10} = \mathrm{Mod}\,(x = t \,.\, zx \leftrightarrow y = yz \,.\, t) = V_{10}$; V_{10} *is the variety of* IP-*quasigroups with $I_l = I_r$.*

$V_{11} = \mathrm{Mod}\,(xy \,.\, (yx \,.\, z) \doteq z)$; $G \in V_{11}$ *iff* $(Q, \backslash) \in V_{10}$.

$V_{12} = \mathrm{Mod}\,(x = tx \,.\, z \leftrightarrow y = z \,.\, ty) = V_{12}^*$; V_{12} *is the variety of* CIP-*quasigroups.*

$V_{13} = \mathrm{Mod}\,(x(zy \,.\, x \doteq yz)$; $G \in V_{13}$ *iff* $(Q, \backslash) \in V_{12}$.

$V_{14} = \mathrm{Mod}\,(x(yx \,.\, x) \,.\, y \doteq z)$; $G \in V_{14}$ *iff there exists φ such that for all x, y, $x \,.\, yx = \varphi y$ and $\varphi^2 = 1$.*

$V_{15} = \mathrm{Mod}\,(x \,.\, y(zx \,.\, y) \doteq z) = V_{15}^*$; $G \in V_{15}$ *iff there exists φ such that for all $x, y \in Q$, $x \,.\, yx = \varphi y$ and $xy \,.\, x = \varphi^{-1} y$.*

$V_{16} = \mathrm{Mod}\,((x \,.\, yx)\, z \,.\, y \doteq z)$; $G \in V_{16}$ *iff there exists φ such that for all x, y, $x \,.\, yx = \varphi y$, $\varphi \in \Phi$, $\varphi = \varphi^*$.*

$V_{17} = \mathrm{Mod}\,(x = tx \,.\, z \leftrightarrow y = zy \,.\, t) = V_{17}$; $G \in V_{17}$ *iff there exists such that for all x, y, $xy \,.\, \varphi x = y$ and $\varphi^2 = 1$.*

Proof. The relations on V_1 are obvious. The relations on V_2 follow from (4) and (22). From (20) and (1) we have the relations on V_3. The second relation on V_4

follows from (3) and (23); the first relation is obvious. The relations on $V_5 - V_8$ are easy. The relations on V_9 follows from Lemma 2.10. The relations on V_{10} follows from (3), (12) and (23). On V_{11}: $T_z^{-1}R_z y = L_x R_x^{-1} y \Leftrightarrow T_z^{-1} R_z R_x y = L_x y \Leftrightarrow xy . (yx . z) = z$. By Table 1, $T_z^{-1}R_z = L_x R_x^{-1} \Leftrightarrow (R_z^{\setminus})^{-1} T_z^{\setminus} = (L_x^{\setminus})^{-1} . (T_x^{\setminus})^{-1} \Leftrightarrow (T_z^{\setminus})^{-1} R_z^{\setminus} = T_x^{\setminus} L_x^{\setminus} \Leftrightarrow (Q, \setminus) \in V_{10}$. On V_{12}: Obviously $T_x^{-1} = T_x$, i.e. $T_x T_x = 1$ and by (27), $L_x = R_x$; therefore $\operatorname{Mod}(T^{-1}R \simeq TR) \subset \operatorname{Mod}(T^{-1}R \simeq TL) = V_{10}$, thus $V_{12} = V_{10} \cap V_2$. Further, $y = t . zy \Leftrightarrow T_y^{-1} R_y z = t \Leftrightarrow T_x R_x z = t \Leftrightarrow zx . t = x$. Analogously we prove the relations on $V_{13}, V_{14}, V_{15}, V_{17}$. Lemma 2.12 and Theorem 2.14 yield the relations on V_{16}.

2.16. Example. *Let $(C, +, \cdot)$ be the field of complex numbers, $a, b, c \in C$, $a . b \neq 0$ and $x \circ y = a . x + b . y + c$ for all $x, y \in C$. Then $(C, \circ) = Q$ is a quasigroup and the following relations hold:*

$Q \in V_2$ *iff* $a = b$, $\quad Q \in V_5$ *iff* $a^2 = b^2$,
$Q \in V_3$ *iff* $b = -1$, $\quad Q \in V_6$ *iff* $a = -b^2$,
$Q \in V_4$ *iff* $b^2 = 1$, $\quad Q \in V_7$ *iff* $a . b = 1$ *and* $c = 0$.

The varieties $V_1, \ldots, V_7, V_3^*, V_4^*, V_6^*$ are pairwise different.

Proof. Easy.

2.17. Lemma. *Let $(Q, \cdot)$ be a quasigroup and $A, B, C, D \in \mathcal{T}$. If $M(AB) = M(CD)$ then there exists $\delta \in \{1, -1\}$ such that*

$$(A_x B_x)^\delta = C_y D_y \quad \text{for all} \quad x, y \in Q .$$

Proof. If $A, B \in \mathcal{T}$ then there are $6.6 = 36$ varieties $M(AB)$. Each of them occurs in some relation of Theorem 2.6. The rest of the proof follows from Lemma 2.16, (30), Theorem 2.6 and (26)–(29).

2.18. Lemma. *Let $(Q, \cdot)$ be a quasigroup, $A, B, C, D, E, F, G, H \in \mathcal{T}$. If $(A_x B_x)^2 = 1$, $M(AB) = M(EF)$ and $M(CD) = M(GH)$ then $\operatorname{Mod}(AB \simeq CD) = \operatorname{Mod}(EF \simeq GH)$.*

Proof. By Lemma 2.17, $AB = (EF)^\delta$ and $CD = (GH)^\varepsilon$ (indices are omitted), where $\delta, \varepsilon \in \{1, -1\}$. If $AB = CD$ then with respect to $AB = (AB)^{-1}$ we have $CD = (CD)^{-1}$, $AB = EF$ and $CD = GH$, therefore $EF = GH$ and thus $\operatorname{Mod}(AB \simeq CD) \subset \operatorname{Mod}(EF \simeq GH)$. From the symmetry we obtain the converse.

2.19. Theorem. *For each $A, B, C, D \in \mathcal{T}$ there exists at most one $i \in \{1, 2, \ldots, 17\}$ such that $\operatorname{Mod}(AB \simeq CD) \in \{V_i, V_i^*\}$.*

Proof. In the proof we shall write $AB \simeq CD$ instead of $\operatorname{Mod}(AB \simeq CD)$. According to Theorem 2.6, Lemma 2.18 and (19)–(25), it suffices to consider all varieties given in Table 2.

From 61 varieties given in Table 2, the following pairs are dual:

(11, 11) (21, 21) (31, 41) (51, 61) (71, 71) (81, 91) (12, 12) (22, 32) (42, 52) (62, 62) (72, 82) (92, 92) (13, 14) (23, 23) (33, 34) (43, 24) (53, 44) (63, 64) (73, 54) (83, 74)

$(15, 61)$ $(25, 25)$ $(35, c1)$ $(45, e1)$ $(55, d1)$ $(65, f1)$ $(a1, 91)$ $(b2, b2)$ $(c2, d2)$ $(e2, e2)$ $(b3, b4)$ $(c3, c4)$ $(d3, d3)$ $(e3, e3)$ $(f3, e4)$ $(g3, d4)$ $(b5, b5)$ $(c5, c5)$.

Table 2

	1	2	3	4	5
1	$LL^{-1} \simeq LL^{-1}$	$L^{-1}R \simeq L^{-1}R$	$LL \simeq LL$	$RR \simeq RR$	$T^{-1}R \simeq T^{-1}R$
2	$L^{-1}R$	LL	RR	$T^{-1}R$	TL
3	LL	RR	$T^{-1}R$	TL	LR^{-1}
4	RR	$T^{-1}R$	TL	LR^{-1}	LR
5	$T^{-1}R$	TL	LR^{-1}	LR	RL
6	TL	LR^{-1}	LR	RL	TR
7	LR^{-1}	LR	RL	TR	
8	LR	RL	TR		
9	RL	TR			
a	TR				
b	$TL \simeq TL$	$LR^{-1} \simeq LR^{-1}$	$LR \simeq LR$	$RL \simeq RL$	$TR \simeq TR$
c	LR^{-1}	LR	$R^{-1}L^{-1}$	$L^{-1}R^{-1}$	$R^{-1}T^{-1}$
d	LR	RL	RL	TR	
e	RL	TR	$L^{-1}R^{-1}$	$R^{-1}T^{-1}$	
f	TR		TR		
g			$R^{-1}T^{-1}$		

Thus, for example, the pair $(35, c1)$ is the pair of the variety $\mathrm{Mod}\,(T^{-1}R \simeq LR^{-1})$ and its dual variety $\mathrm{Mod}\,(TL \simeq LR^{-1})$. The duality of these varieties follows from $\mathrm{Mod}\,(TL \simeq RL^{-1}) = \mathrm{Mod}\,(TL \simeq LR^{-1})$ (by (24), $RL^{-1} = (RL^{-1})^{-1} = (LR^{-1})$. Similarly we prove the rest of the above dualities. Therefore we investigate the first members of all the above pairs. The results are summarized in Table 3. We prove only the following equalities:

$42 = V_9$: From $\varphi = L_x^{-1}R_x = T_y^{-1}R_y$, for $y = x$, we have $L_x = T_x$ and according to Lemma 2.8, $\varphi = 1$ whence $L_x = R_x = T_x$.

$43 = V_9$: From $\varphi = L_xL_x = T_yL_y$, for $y = x$, we have $L_x = T_x$, therefore by (29), $T_x = R_x^{-1}$, hence $L_x = R_x^{-1}$, i.e. $L_x^{-1} = R_x$ and with respect to (20), $L_x = R_x$ so that $L_x = T_x = R_x$.

$45 = V_{13}$: This follows from Lemma 2.7(iv).

$55 = V_9$: From $\varphi = T_x^{-1}R_x = R_xL_x$ we have $\varphi x \,.\, xy = y$ (by (3)), $xy \,.\, x = \varphi y$ (by (14)), $\varphi^2 = 1$ (by (23)) and φ is an automorphism of $(Q, \cdot)$ (by the dual of Lemma 2.7). Therefore $L_{\varphi x}L_x = L_xL_{\varphi x} = 1$, $\varphi L_x = L_{\varphi x}\varphi$ so that $L_x = \varphi L_{\varphi x}\varphi = R_xL_xL_{\varphi x}R_xL_x = R_x^2L_x$, hence $R_x^2 = 1$; $\varphi = \varphi^{-1} = L_x^{-1}R_x^{-1} = L_x^{-1}R_x$ and by (22), $\varphi = 1$, therefore $L_x = R_x$.

$e2 = V_9$: By (25), $\varphi = L_xR_x^{-1}$ for all x implies $\varphi^2 = 1$. By Table 1, $\mathrm{Mod}\,(LR^{-1} \simeq TR) = \mathrm{Mod}\,((R^\Delta)^{-1}\,T^\Delta \simeq L^\Delta(T^\Delta)^{-1} = \mathrm{Mod}(T^\Delta)^{-1}\,R^\Delta \simeq L^\Delta(T^\Delta)^{-1}) = (55)^\Delta = V_9^\Delta = V_9$ (the meaning of V_9^Δ is analogous to $L^\Delta, R^\Delta, \ldots$).

$d3 = V_8$: By (28), $R_x T_x = R_x L_x$, therefore $L_x = T_x$, so by Lemma 2.8, $L_x R_x = 1$, i.e. $R_x L_x = 1$.

The other equalities are proved similarly or they are trivial.

Table 3

	1	2	3	4	5
1	V_1	V_2	V_3	V_3^*	V_4
2	V_2	V_9	V_9	V_9	V_{10}
3	V_3	V_9	V_3	V_3^*	V_{11}
4	V_3^*	V_9	V_9	V_9	V_{13}
5	V_3	V_9	V_9	V_9	V_9
6	V_3^*	V_2	V_9	V_9	V_{12}
7	V_2	V_9	V_9	V_9	
8	V_8	V_9	V_9		
9	V_8	V_9			
a	V_8				
b	V_4^*	V_5	V_6	V_6^*	V_7
c	V_{11}^*	V_{13}	V_{14}	V_{14}^*	V_{17}
d	V_9	V_{13}^*	V_8	V_{16}^*	
e	V_{13}^*	V_9	V_{15}	V_8	
f	V_{12}		V_8		
g			V_{16}		

2.20. Corollary. *Let a primitive quasigroup* $(Q, \cdot, /, \backslash)$ *satisfy a balanced identity of length* $\leqq 6$ *of type* I. *Then there exists* $i \in \{1, 2, \ldots, 17\}$ *such that* $(Q, \cdot) \in V_i$ *or* $(Q, \cdot) \in V_i^*$.

2.21. Corollary. *There are* 24 *varieties determined by a balanced identity of length* $\leqq 6$ *and of type* I. *They are* $V_1, V_2, \ldots, V_{17}, V_3^*, V_4^*, V_6^*, V_{11}^*, V_{13}^*, V_{14}^*, V_{16}^*$.

3. QUASIGROUPS DETERMINED BY IDENTITY II

In Section 1 we have proved that each quasigroup satisfying an identity of type II is a transitive quasigroup. Thus we shall deal with transitive quasigroups in this section.

We shall use some results on transitive quasigroups presented in [4].

A collection of mappings $\{\varphi_i;\ i \in S\}$, where S is a non-empty index set, will be called *disjoint* if $\varphi_i(a) = \varphi_j(a)$ implies $i = j$ (cf. [4], Definition 2.2).

Let $(Q, \cdot)$ be a quasigroup, $A, B \in T$; we shall denote

$$Q(AB) = \{A_x B_y;\ x, y \in Q\}\,.$$

If $(Q, \circ)$ is a group and φ (ψ) its automorphism (antiautomorphism) then $L_s^\circ \varphi$ ($L_s^\circ \psi$)

is called *a quasiautomorphism* (*antiquasiautomorphism*, respectively) of $(Q, \circ)$ for each $s \in Q$. For every quasiautomorphism $\gamma = L_s^\circ \varphi$ there exists an automorphism ξ of $(Q, \circ)$ such that $\gamma = R_s^\circ \xi$. Analogously, for any antiautomorphism ψ there exists an antiautomorphism η such that $L_s^\circ \psi = R_s^\circ \eta$ (see [2]).

3.1. Lemma. *Let $(Q, \cdot)$ be a quasigroup and let*

$Q(1) = \{Q(LL), Q(L^{-1}L^{-1}), Q(T^{-1}R), Q(R^{-1}T)\}$,
$Q(2) = \{Q(RR), Q(R^{-1}R^{-1}), Q(TL), Q(L^{-1}T^{-1})\}$,
$Q(3) = \{Q(LR), Q(R^{-1}L^{-1}), Q(T^{-1}L), Q(L^{-1}T)\}$,
$Q(4) = \{Q(RL), Q(L^{-1}R^{-1}), Q(TR), Q(R^{-1}T^{-1})\}$,
$Q(5) = \{Q(LT), Q(T^{-1}L^{-1}), Q(TR^{-1}), Q(RT^{-1}), Q(L^{-1}R), Q(R^{-1}L)\}$,
$Q(6) = \{Q(LL^{-1}), Q(RR^{-1}), Q(TT^{-1}), Q(L^{-1}L), Q(R^{-1}R), Q(T^{-1}T)\}$,
$Q(7) = \{Q(LR^{-1}), Q(RL^{-1}), Q(TT), Q(T^{-1}T^{-1})\}$,
$Q(8) = \{Q(LT^{-1}), Q(TL^{-1}), Q(RT), Q(T^{-1}R^{-1})\}$.

Let $i = \{1, 2, \ldots, 8\}$ and $M \in Q(i)$ then M is disjoint implies X is disjoint for all $X \in Q(i)$.

Proof. For $i = 2$ it suffices to use Lemma 2.4 and Theorem 2.2 in [4]. Analogously we do the rest of the proof.

3.2. Lemma. *Let $(Q, \cdot)$ be a quasigroup, α, β permutations of Q and let $x \,.\, y = \alpha x \circ \beta y$ for all $x, y \in Q$. If $(Q, \circ)$ is a loop then*

(1) *$Q(LL)$ is disjoint iff $(Q, \circ)$ is a group and β its quasiautomorphism,*
(2) *$Q(RR)$ is disjoint iff $(Q, \circ)$ is a group and α its quasiautomorphism,*
(3) *$Q(LR)$ is disjoint iff $(Q, \circ)$ is a group and β its antiquasiautomorphism,*
(4) *$Q(RL)$ is disjoint iff $(Q, \circ)$ is a group and α its antiquasiautomorphism,*
(5) *$Q(LT)$ is disjoint iff $(Q, \circ)$ is an abelian group,*
(6) *$Q(LL^{-1})$ is disjoint iff $(Q, \circ)$ is a group,*
(7) *$Q(LR^{-1})$ is disjoint iff $(Q, \circ)$ is a group and $\alpha\beta^{-1}$ its antiquasiautomorphism,*
(8) *$Q(LT^{-1})$ is disjoint iff $(Q, \circ)$ is a group and $\alpha\beta^{-1}$ its quasiautomorphism.*

Proof. For (2) it suffices to use Theorem 2.2(ii) $\leftrightarrow$ (iv) in [4]. Analogously we do the rest of the proof.

3.3. Lemma. *If α is a quasiautomorphism and antiquasiautomorphism of a group $(Q, \circ)$ then $(Q, \circ)$ is abelian.*

Proof. Let $\alpha = L_a^\circ \eta = L_b^\circ \xi$, where η is an automorphism and ξ an antiautomorphism of the group $(Q, \circ)$. Then $L_c^\circ \eta = \xi$ $(c = a \circ b^{-1})$ whence $L_c^\circ \eta x = \xi x$ for all $x \in Q$, therefore $L_c^\circ \eta 1 = 1$, hence $c = 1$. Thus for all $x, y \in Q$, $x \circ y = \eta^{-1}\eta(x \circ y) = \eta^{-1}(\eta x \circ \eta y) = \eta^{-1}(\xi x \circ \xi y) = \eta^{-1}\xi(y \circ x) = \eta^{-1}\eta(y \circ x) = y \circ x$.

3.4. Lemma. *If α is a quasiautomorphism and β an antiquasiautomorphism of a group $(Q, \circ)$ then $\alpha\beta$ is an antiquasiautomorphism of $(Q, \circ)$.*

Proof. Easy.

If we rewrite the identity II with translations of the quasigroup $(Q, \cdot)$ then

$$A_x B_y C_{x\Box y} z = z \tag{9}$$

for some $A, B, C \in \mathscr{T}$, $\Box \in \Sigma(\cdot)$ and all $x, y, z \in Q$. Let $(Q, \cdot)$, $(Q, \Box)$ be quasigroups and $A, B, C \in \mathscr{T}$ then $(Q, \cdot, /, \backslash)$ is called an $(ABC\,\Box)$-quasigroup if for all $x, y, z \in Q$, (9) holds ($\Box$ need not be in $\Sigma(\cdot)$).

3.5. Lemma. *Let $(Q, \cdot)$ be a quasigroup. The following statements are equivalent:*
(i) *$(Q, \cdot, /, \backslash)$ satisfies an identity of type* II.
(ii) *There exists $\Box$ in $\Sigma(\cdot)$ such that $(Q, \cdot, /, \backslash)$ is an $(ABC\,\Box)$-quasigroup.*

Proof. See Lemmas 1.1, 1.2, and Lemma 1.2 in [4].

Thus, to classify all primitive quasigroups that satisfy an identity of type II means to classify $(ABC\,\Box)$-quasigroups for $A, B, C \in \mathscr{T}$ and $\Box \in \Sigma(\cdot)$.

A primitive quasigroup $(Q, \cdot, /, \backslash)$ is called an *(ABC)-quasigroup* if there exists a quasigroup $(Q, \Box)$ such that $(Q, \cdot, /, \backslash)$ is an $(ABC\,\Box)$-quasigroup.

We order the set $\mathscr{T}$ by $L < R < T < L^{-1} < R^{-1} < T^{-1}$ and the set S of all ordered triads (ABC), $A, B, C \in \mathscr{T}$ by $(ABC) < (DEF)$ iff $A < D$ or $A = D$ and $B < E$ or $A = D$, $B = E$ and $C < F$. Let δA mean the dual symbol of A (i.e. $\delta L = R$, $\delta T = T^{-1}, \ldots$), $\delta(ABC) = (\delta A \delta B \delta C)$ and if H is a set, $\delta H = \{\delta X \;\; X \in H\}$. Let $\langle (ABC) \rangle = \{(ABC), (BCA), (CAB), (C^{-1}B^{-1}A^{-1}), (B^{-1}A^{-1}C^{-1}), (A^{-1}C^{-1}B^{-1})\}$ and if G is a set of triads, $\langle G \rangle = \cup \{\langle X \rangle, X \in G\}$.

3.6. Lemma. *If a primitive quasigroup $(Q, \cdot, /, \backslash)$ is a (DEF)-quasigroup for some $(DEF) \in \langle (ABC) \rangle$ then $(Q, \cdot, /, \backslash)$ is an (XYZ)-quasigroup for all $(XYZ) \in$ $\in \langle (ABC) \rangle$.*

Proof. Easy.

3.7. Lemma. *If $(Q, \cdot, /, \backslash)$ is a primitive (ABC)-quasigroup then $(ABC) \in$ $\in U \cup \delta U$, where*

$$U = \{(LLL), (LLR), (LLT), (LLL^{-1}), (LLR^{-1}), (LLT^{-1}), (LRT), (LRL^{-1}), (LRR^{-1}), (LRT^{-1}), (LTT), (LTL^{-1}), (LTR^{-1}), (LTT^{-1}), (LL^{-1}T), (LL^{-1}R^{-1}), (LR^{-1}T), (LR^{-1}T^{-1}), (LT^{-1}T), (LT^{-1}R^{-1}), LT^{-1}T^{-1}, (TTT), (TTT^{-1})\}.$$

Proof. The proof will be based on Lemma 3.6 and the following construction of ordered sets $S_1, S_2, \ldots, S_i, \ldots$. Let $S_1 = S \setminus \{(LLL)\}$. Let S_m, $m > 1$, have been constructed. Then $S_{m+1} = S_m \cup \{(ABC)\}$ where (ABC) is the smallest element of the set $S \setminus (\langle S_m \rangle \cup \delta \langle S_m \rangle)$. Since S is finite, there exists a positive integer k such that for all $i < k < j$, $S_i \neq S_k = S_j$. By this construction, we obtain $k = 23$ and $S_{23} = U$.

3.8. Lemma. *Let $(Q, \cdot, /, \backslash)$ be a primitive (ABC)-quasigroup, $\alpha\beta$ permutations of Q, $x \,.\, y = \alpha x \circ \beta y$ for all $x, y \in Q$ and let $(Q, \circ)$ be a loop. Then $(Q, \circ)$ is a group and the relations presented in Table 4 are fulfilled.*

Proof. We prove only the relations for (LLR) (analogously we do the rest of the

Table 4

	quasiautomorphism			antiquasiautomorphism			abelian group
	α	β	$\alpha\beta^{-1}$	α	β	$\alpha\beta^{-1}$	$\circ$
LLL		×					
LLR	×	×	×	×	×	×	×
LLT	×	×	×	×	×	×	×
LLL^{-1}		×					
LLR^{-1}	×	×	×	×	×	×	×
LLT^{-1}	×	×	×	×	×	×	×
LRT	×	×	×	×	×	×	×
LRL^{-1}	×				×	×	
LRR^{-1}		×			×		×
LRT^{-1}		×			×		×
LTT	×	×	×	×	×	×	×
LTL^{-1}			×			×	×
LTR^{-1}							×
LTT^{-1}		×			×		×
$LL^{-1}T$	×				×	×	
$LL^{-1}R^{-1}$	×			×			×
$LR^{-1}T$	×	×	×	×	×	×	×
$LR^{-1}T^{-1}$	×	×	×	×	×	×	×
$LT^{-1}T$	×	×	×				
$LT^{-1}R^{-1}$			×			×	×
$LT^{-1}T^{-1}$	×	×	×	×	×	×	×
TTT						×	
TTT^{-1}						×	

proof). Thus we must prove that if $(Q, \cdot, /, \backslash)$ is an (LLR)-quasigroup then $(Q, \circ)$ is an abelian group and $\alpha, \beta, \alpha\beta^{-1}$ are its quasiautomorphisms and antiautomorphisms. By [4, Lemma 2.3(iii)], $Q(LL)$, $Q(LR)$ and $Q(RL)$ are disjoint, therefore by the dual of [4, Theorem 2.2, Theorem 2.4] and by [4, Theorem 2.4], β is a quasiautomorphism and an antiquasiautomorphism and α is an antiquasiautomorphism of the group $(Q, \circ)$. Thus $(Q, \circ)$ is an abelian group and α, β its automorphisms therefore $\alpha\beta^{-1}$ is also an automorphism of $(Q, \circ)$.

3.9. Lemma. *Let $(Q, \square)$ be a quasigroup and let $(Q, \cdot, /, \backslash)$ be a primitive quasigroup. The following conditions are equivalent:*

(i) $(Q, \cdot, /, \backslash)$ *is an* $(LLL\square)$*-quasigroup.*

(ii) $(x \square y) . (x . yz) \simeq z$.

(iii) *There exists a group $(Q, \circ)$, its automorphism ξ and a permutation γ of Q such that $\xi^3 = 1$, $x . y = \gamma x \circ \xi y$, $I\xi^2\gamma(x \square y) = \gamma x \circ \xi\gamma y$ for all $x, y \in Q$ where $x \circ Ix = 1$ for all $x \in Q$.*

Proof. (i) $\Leftrightarrow$ (ii) is evident. (i) $\rightarrow$ (iii). By Table 4, $x . y = \alpha x \circ \beta y$ where β is a quasiautomorphism of a group $(Q, \circ)$. There exists $s \in Q$ and an automorphism ξ

Table 5

$LLL\square$	$(x \square y)(x \,.\, yz) \bumpeq z$	$\circ$	$(\gamma, \xi, 1)$	$(\gamma, \xi\gamma, \mathrm{I}\xi^2\gamma)$	$\xi^3 = 1$
$LLR\square$	$(x \,.\, yz)(x \square y) \bumpeq z$	$+$	$(\xi^{-2}, \xi, L^{+}_{-k})$	$(1, \xi, L^{+}_{h}\mathrm{I}\xi^5)$	$-h = \xi^3 k + \xi^2 k + \xi^4 k$
$LLT\square$	$x \square y \bumpeq (x \,.\, yz)\, z$	$+$	$(\mathrm{I}\xi^{-1}, \xi, L^{+}_{-k})$	$(\xi^{-2}, \xi^{-1}, L^{+}_{\xi^{-1}k})$	
$LLL^{-1}\square$	$(x \square y)\, z \bumpeq x \,.\, yz$	$\circ$	$(\gamma, 1, 1)$	(γ, γ, γ)	
$LLR^{-1}\square$	$z \,.\, (x \square y) \bumpeq x \,.\, yz$	$+$	(ξ^2, ξ, L^{+}_{-k})	$x \square y \bumpeq y \,.\, x$	
$LLT^{-1}\square$	$x \square y \bumpeq z(x \,.\, yz)$	$+$	$(\mathrm{I}\xi^3, \xi, L^{+}_{-k})$	$(\mathrm{I}\xi^4, \mathrm{I}\xi^5, L^{+}_{h})$	$-h = \xi^2 k + \xi k + k$
$LRT\square$	$x \square y \bumpeq (x \,.\, zy)\, z$	$+$	(ψ, ξ, L^{+}_{-k})	$(\psi^2, \psi\xi^2, L^{+}_{h})$	$-h = \psi\xi k + \psi k + k,\ \psi\xi\psi = \mathrm{I}\xi$
$LRL^{-1}\square$	$(x \square y) \,.\, z \bumpeq x \,.\, zy$	$\circ$	$(R^{\circ}_{c}, \eta, 1)$	$(R^{\circ}_{c}, \eta^2, L^{\circ}_{\eta c \circ \mathrm{I}c})$	
$LRR^{-1}\square$	$z \,.\, (x \square y) \bumpeq x \,.\, zy$	$+$	$(\gamma, 1, 1)$	$x \square y \bumpeq x \,.\, y$	
$LRT^{-1}\square$	$x \square y \bumpeq z \,.\, (x \,.\, zy)$	$+$	$(\gamma, \xi, 1)$	$(\xi\gamma, \mathrm{I}\xi, 1)$	$\xi^2 = \mathrm{I}$
$LTT\square$	$x \square (z \,.\, y) \bumpeq xy \,.\, z$	$\oplus$	$(\psi, \psi^2, L^{\oplus}_{-k}$	$(\psi^2, \psi, L^{\oplus}_{-k})$	
$LTL^{-1}\square$	$(x \square (z \,.\, y)) \,.\, z = xy$	$+$	$(\mathrm{I}\beta, \beta, L^{+}_{-k})$	$(\beta, \mathrm{I}, L^{+}_{k}\beta)$	
$LTR^{-1}\square$	$z \,.\, (x \square (z \,.\, y)) \bumpeq xy$	$\oplus$	$(\alpha, \beta, 1)$	$(\alpha, 1, \beta)$	
$LTT^{-1}\square$	$x \square (z \,.\, y) \bumpeq z \,.\, xy$	$+$	$(\alpha, 1, 1)$	$(\alpha, 1, 1)$	
$LL^{-1}T\square$	$x \square y \bumpeq xz \,.\, yz$	$\circ$	$(\mathrm{I}\eta, \eta, L^{\circ}_{\mathrm{I}k})$	$(L^{\circ}_{k\circ \mathrm{I}\eta k}\eta^2, R^{\circ}_{\eta k}\mathrm{I}\eta^2, 1)$	
$LL^{-1}R^{-1}\square$	$yz \,.\, (x \square y) \bumpeq xz$	$+$	$(1, \beta, 1)$	$(1, \mathrm{I}, \beta)$	
$LR^{-1}T\square$	$x \square y \bumpeq xz \,.\, zy$	$+$	$(\gamma, \xi, L^{+}_{-k}$	$(\psi^2, \xi^2, L^{+}_{h})$	$-h = \psi k + \xi k + k,\ \psi\xi = \mathrm{I}\xi\psi$
$LR^{-1}T^{-1}\square$	$x \square y \bumpeq zy \,.\, xz$	$+$	(ψ, ξ, L^{+}_{-k})	$(\xi\psi, \psi\xi, L^{+}_{h})$	$-h = \psi k + \xi k + k,\ \psi^2 = \mathrm{I}\xi^2$
$LT^{-1}T\square$	$x \square (y \,.\, z) \bumpeq xy \,.\, z$	$\circ$	$(L^{\circ}_{k}\psi, 1, 1)$	$(L^{\circ}_{k\circ\psi k}\psi^2, L^{\circ}_{\mathrm{I}k}, 1)$	
$LT^{-1}R^{-1}\square$	$z \,.\, (x \square (y \,.\, z)) \bumpeq xy$	$+$	$(\xi\beta, \beta, L^{+}_{-k})$	$(\xi\beta, \mathrm{I}\xi, L^{+}_{\mathrm{I}\xi k}\beta)$	$\xi^2 = \mathrm{I}$
$LT^{-1}T^{-1}\square$	$x \square yz \bumpeq z \,.\, xy$	$+$	(ψ, ξ, L^{+}_{-k})	$(\xi\psi, \psi\xi^{-1}, L^{+}_{-k})$	$k + \xi k - \xi^2\psi^{-1}k = 0,\ h = k + \xi k - \xi^2\psi k$
$TTT\square$	$zx \square yz \bumpeq xy$	$\circ$	$(\eta\beta, L^{\circ}_{a}\beta, 1)$	$(\eta, \eta^{-1}, 1)$	$\eta^3 = \mathrm{I},\ \eta^{-1}a \circ \mathrm{I}a \circ \eta a = 1$
$TTT^{-1}\square$	$zx \square yz \bumpeq yx$	$\circ$	$(\mathrm{I}\gamma, \gamma, 1)$	$x \square y \bumpeq y \circ x$	

of $(Q, \circ)$ such that $\beta y = s \circ \xi y$ if we denote $\alpha x \circ s = \gamma x$ then $x . y = \gamma x \circ \xi y$. If $x(y . (x \square y) z \doteq z$, which is equivalent with (ii), is rewritten by $(\circ)$ then

(iv) $$\gamma x \circ \xi \gamma y \circ \xi^2 \gamma(x \square y) \circ \xi^3 z = z ,$$

whence for $z = 1$

(v) $$\gamma x \circ \xi \gamma y \circ \xi^2 \gamma(x \square y) = 1 .$$

If we apply (v) to (iv), we obtain $\xi^3 = 1$. Obviously (v) implies $\mathrm{I}\xi^2\gamma(x \square y) = \gamma x \circ \xi \gamma y$. The proof of (iii) $\Rightarrow$ (ii) is easy.

Similarly we can prove analogous theorems for the remaining triads of the set U (see Lemma 3.7). The results are summarized in Table 5, where $(Q, \circ)$ is a group, $(Q, +)$ is an abelian group, $(Q, \oplus)$ is a 2-group, η is an antiautomorphism of $(Q, \circ)$, ψ, ξ are automorphisms of $(Q, \circ)$ or $(Q, +)$ or $(Q, \oplus)$, $x \circ \mathrm{I}x = 1$ or $x + \mathrm{I}x = 0$ for all $x \in Q$ and α, β, γ are permutations of Q.

By Table 6, where the same symbols as in Table 5 are used, we define some classes of quasigroups.

Table 6

V_{18}	The variety of 2-groups
V_{19}	$x . y = x - y + k$
V_{20}	$x . y = -x + \xi y + k,\ \xi^2 = \mathrm{I}$
V_{21}	$x . y = \xi^5 x \oplus \xi y \oplus k,\ \xi^7 = 1,\ k + \xi^2 k + \xi^3 k + \xi^4 k = 0$
V_{22}	$x . y = x \oplus \xi y \oplus k,\ \xi^2 = 1$
V_{23}	$x . y = \xi^2 x \oplus \xi y \oplus k,\ \xi^3 = 1,\ \xi k = k$
V_{24}	$x . y = \xi x + \xi y + k,\ \xi^2 = \mathrm{I}$
V_{25}	$x . y = -\xi^{-1} x + \xi y + k$
V_{26}	The variety of groups
V_{27}	The variety of abelian groups
V_{28}	$x . y = k \circ x^{-1} \circ y$
V_{29}	$x . y = \xi^2 x + \xi y + k$
V_{30}	$x . y = \xi^3 x \oplus \xi y \oplus k,\ \xi^7 = 1,\ \xi k = k$
V_{31}	$x . y = \xi x \oplus \xi y \oplus k,\ \xi^2 = 1$
V_{32}	$x . y = \gamma x + y$
V_{33}	$x . y = \gamma x \oplus y$
V_{34}	$x . y = \gamma x + \xi y,\ \xi^2 = \mathrm{I}$
V_{35}	$x . y = -\beta x + \beta y + k$
V_{36}	$x . y = \beta x \oplus \beta y \oplus k$
V_{37}	$x . y = \alpha x \oplus \beta y$
V_{38}	$x . y = \alpha x + \xi \alpha y + k,\ \xi^2 = \mathrm{I}$

Table 6 reads like this: For example, V_{30} is the class of all quasigroups $(Q, \cdot)$ that are isotopes of a 2-group $(Q, \oplus)$ by the rule $x . y = \xi x \oplus \xi y \oplus k$, where ξ is an automorphism of $(Q, \oplus)$, ξ^2 is the identity map Q onto Q and $k \in Q$ V_{32} is the class of all quasigroups $(Q, \cdot)$ that are isotopes of an abelian group $(Q, +)$ by the rule $x . y = \gamma x + y$ where γ is a permutation of Q.

3.10. Lemma. $V_i = V_i^*$ (V_*^i *is the dual of* V_i) *for all* $i \in \{18, 23, 24, 25, 26, 27, 31, 35, 36, 37\}$.

Proof. Easy.

If Lemma 3.9 is applied to $\square \in \Sigma(\cdot)$ then we obtain the following theorem.

3.11. Theorem. *Let* $(Q, \square)$ *be a quasigroup and let* $(Q, \cdot, /, \backslash)$ *be a primitive quasigroup. The following conditions are equivalent:*

(i) $(Q, \cdot, /, \backslash)$ *is an* $(LLL\square)$*-quasigroup and* $\square \in \Sigma(\cdot)$.

(ii) $(Q, \cdot, /, \backslash)$ *is an* $(LLL\square)$*-quasigroup and* $\square \in \{\cdot, *\}$.

(iii) $x \,.\, y(xy \,.\, z) \doteq z$.

(iv) $x \,.\, y(yx \,.\, z) \doteq z$.

(v) $(Q, \cdot)$ *is a 2-group.*

Proof. (i) $\Rightarrow$ (ii). Let $\square \in \Sigma(\cdot)$ and $x \,\square\, y = t$. Then at least one of the following relations holds: $y = xt$, $y = tx$, $x = yt$, $x = ty$, $t = xy$, $t = yx$. If we apply these equalities to $L_x L_y L_{x\square y} = 1$ and use the identity $L_a L_b L_c = L_c L_b L_a = L_a L_c L_b$ we obtain $L_x L_y L_{xy} = 1$ or $L_x L_y L_{yx} = 1$. (iii) $\Rightarrow$ (v). From Lemma 3.9 we have $x \,.\, y = \gamma x \circ \xi y$, $I\xi^2\gamma(x \,.\, y) = \gamma x \circ \xi\gamma y$, therefore $I\xi^2\gamma(\gamma x \circ \xi y) = \gamma x \circ \xi\gamma y$, i.e. $I\xi^2\gamma(x \circ \xi y) = x \circ \xi\gamma y$; for $x = 1$ we obtain $I\xi^2\gamma\xi = \xi\gamma$ and for $y = 1$, $I\xi^2\gamma = R_a^\circ$, i.e. $\gamma = I\xi R_a^\circ$. Thus $I\xi^2 I\xi R_a^\circ \xi = \xi I\xi R_a^\circ$, i.e. $R_a^\circ \xi = I\xi^2 R_a^\circ$, whence $\xi R_a^\circ \xi = IR_a^\circ$. Consequently, for all $x \in Q$, $\xi^2 x \circ \xi a = Ia \circ Ix$; $x = 1$ implies $\xi a = Ia = b$ so that $\xi^2 x = b \circ Ix \circ Ib$ and also $\xi^2 Ix = b \circ x \circ Ib$, i.e. $\xi^2 I$ is an inner automorphism of $(Q, \circ)$. Therefore I is an automorphism of $(Q, \circ)$ and consequently $(Q, \circ)$ is an abelian group. Then $\xi^2 = I$ and since $\xi^3 = 1$, $I = 1$, so $(Q, \circ)$ is a 2-group. Similarly we prove (iv) $\Rightarrow$ (v). The rest of the proof is easy.

Similarly we can prove analogous theorems for the remaining triads of the set U (see Lemma 3.7). The results are summarized in Table 7, where the same symbols as in the tables 5 and 6 are used.

3.12. Theorem. *Let a primitive quasigroup* $(Q, \cdot, /, \backslash)$ *satisfy a balanced identity of length* $\leqq 6$ *of type* II. *Then there exists* $i \in \{18, 19, \ldots, 38\}$ *such that* $(Q, \cdot) \in V_i \cup V_i^*$.

Proof. See Table 7.

3.13. Corollary. *There are* 31 *varieties determined by a balanced identity of length* $\leqq 6$ *and of type* II. *They are* $V_{18}, V_{19}, \ldots, V_{38}, V_{19}^*, V_{20}^*, V_{21}^*, V_{22}^*, V_{28}^*, V_{29}^*, V_{30}^*, V_{32}^*, V_{33}^*, V_{34}^*$.

4. MAIN RESULTS

4.1. Theorem. *Let a primitive quasigroup* $(Q, \cdot, /, \backslash)$ *satisfy a balanced identity of length* $\leqq 6$. *Then there exists* $i \in \{1, 2, \ldots, 38\}$ *such that* $(Q, \cdot) \in V_i \cup V_i^*$.

Proof. See 2.20 and 3.12.

4.2. Theorem. *There are 55 varieties of quasigroups determined by a balanced identity (on a primitive quasigroup) of length* $\leqq 6$.

Proof. See 2.21 and 3.13.

4.3. Corollary. *Every balanced identity (on a primitive quasigroup) of length* $\leqq 6$ *is equivalent to at least one of the identities or quasiidentities listed in Theorem* 2.15 *or Table* 7.

Table 7

LLL.	$xy . (x . yz) \bumpeq z$	V_{18}	*LLR.*	$(x . yz) . xy \bumpeq z$	V_{19}
*	$yx . (x . yz) \bumpeq z$	V_{18}	*	$(x . yz) . yx \bumpeq z$	V_{20}
/	$yx . (x . yz) \bumpeq z$	V_{18}	/	$(xy . yz)\, x \bumpeq z$	V_{21}
∇	$yx . (x . yz) \bumpeq z$	V_{18}	∇	$x(yx . z) . y \bumpeq z$	V_{20}
$\backslash$	$xy . (x . yz) \bumpeq z$	V_{18}	$\backslash$	$x(xy . z) . y \bumpeq z$	V_{19}
Δ	$xy . (x . yz) \bumpeq z$	V_{18}	Δ	$x(yx . z) . y \bumpeq z$	V_{22}
LLT.	$(y . zx)\, x \bumpeq yz$	V_{19}	LLL^{-1}.	$x . yz \bumpeq xy . z$	V_{26}
*	$(z . yx)\, x \bumpeq yz$	V_{23}	*	$x . yz \bumpeq yx . z$	V_{27}
/	$(zy . yx)\, x \bumpeq z$	V_{23}	/	$yx . xz \bumpeq yz$	V_{18}
∇	$x(zx . y) . y \bumpeq z$	V_{24}	∇	$x(yx . z) \bumpeq yz$	V_{18}
$\backslash$	$x . (x . zy)\, y \bumpeq z$	V_{25}	$\backslash$	$x(xy . z) \bumpeq yz$	V_{19}^*
Δ	$x . (z . xy)\, y \bumpeq z$	V_{19}^*	Δ	$xy . xz \bumpeq yz$	V_{28}
LLR^{-1}.	$x . yz \bumpeq z . xy$	V_{27}	LLT^{-1}.	$x(y . zx) \bumpeq yz$	V_{18}
*	$x . yz \bumpeq z . yx$	V_{29}	*	$x(z . yx) \bumpeq yz$	V_{19}
/	$zx . xy \bumpeq yz$	V_{18}	/	$x(z . yx) . y \bumpeq z$	V_{30}
∇	$x(zx . y) \bumpeq yz$	V_{20}	∇	$x(y . zx) . y \bumpeq z$	V_{18}
$\backslash$	$x(xz . y) \bumpeq yz$	V_{23}	$\backslash$	$x . y(x . zy) \bumpeq z$	V_{31}
Δ	$xz . xy \bumpeq yz$	V_{18}	Δ	$x . y(z . xy) \bumpeq z$	V_{19}
LRT.	$(y . xz)\, x \bumpeq yz$	V_{18}	LRL^{-1}.	$x . zy \bumpeq xy . z$	V_{27}
*	$(z . xy)\, x \bumpeq yz$	V_{20}^*	*	$x . zy \bumpeq yx . z$	V_{27}
/	$(z . xy)\, x . y \bumpeq z$	V_{18}	/	$yx . zx \bumpeq yz$	V_{28}^*
∇	$(x . yz)\, y . x \bumpeq z$	V_{24}	∇	$x(z . yx) \bumpeq yz$	V_{19}
$\backslash$	$x . (x . yz)\, y \bumpeq z$	V_{31}	$\backslash$	$x(z . xy) \bumpeq yz$	V_{18}
Δ	$x . (z . yx)\, y \bumpeq z$	V_{22}^*	Δ	$xy . zx \bumpeq yz$	V_{18}
LRR^{-1}.	$x . yz \bumpeq y . xz$	V_{32}	LRT^{-1}.	$x(y . xz) \bumpeq yz$	V_{32}
*	$x . yz \bumpeq y . zx$	V_{27}	*	$x(z . xy) \bumpeq yz$	V_{18}
/	$zx . yx \bumpeq yz$	V_{19}^*	/	$x(zy . xy) \bumpeq z$	V_{18}
∇	$x(y . zx) \bumpeq yz$	V_{18}	∇	$x . y(x . zy) \bumpeq z$	V_{24}
$\backslash$	$x(y . xz) \bumpeq yz$	V_{33}	$\backslash$	$x . y(x . yz) \bumpeq z$	V_{34}
Δ	$xz . yx \bumpeq yz$	V_{18}	Δ	$x(yz . xy) \bumpeq z$	V_{20}

Continued tab. 7

LTT.	$x \,.\, yz \bumpeq xz \,.\, y$	V_{27}	*LTL^{-1}.*	$(y \,.\, xz) \,.\, x \bumpeq yz$	V_{18}
*	$xy \,.\, z \bumpeq zy \,.\, x$	V_{29}	*	$(xz \,.\, y)\, x \bumpeq yz$	V_{19}
/	$(zy \,.\, x) \,.\, xy \bumpeq z$	V_{23}	/	$(y \,.\, zx)\, x \bumpeq yz$	V_{19}
∇	$(xz \,.\, y)\, x \bumpeq yz$	V_{19}	∇	$x_1y_1 = x_2y_2 \rightarrow x_2x_1 = y_2y_1$	V_{35}
\	$x(xz \,.\, y) \bumpeq yz$	V_{23}	\	$x_1y_1 = x_2y_2 \rightarrow x_2y_1 = y_2x_1$	V_{36}
Δ	$xy \,.\, (zy \,.\, x) \bumpeq z$	V_{20}	Δ	$(zx \,.\, y)\, x \bumpeq yz$	V_{18}
LTR^{-1}.	$x(y \,.\, xz) \bumpeq yz$	V_{33}	*LTT^{-1}.*	$x \,.\, yz \bumpeq y \,.\, xz$	V_{32}
*	$x(xz \,.\, y) \bumpeq yz$	V_{23}	*	$x \,.\, yz \bumpeq xz \,.\, y$	V_{27}
/	$(z \,.\, yx)\, x \bumpeq yz$	V_{23}	/	$(x \,.\, zy) \,.\, xy \bumpeq z$	V_{19}
∇	$x_1y_1 = x_2y_2 \rightarrow x_2y_1 = y_2x_1$	V_{36}	∇	$(y \,.\, xz)\, x \bumpeq yz$	V_{18}
\	$x_1y_1 = x_2y_2 \rightarrow x_1y_2 = x_2y_1$	V_{37}	\	$x(y \,.\, xz) \bumpeq yz$	V_{33}
Δ	$(yx \,.\, z)\, x \bumpeq yz$	V_{33}	Δ	$xy \,.\, (x \,.\, zy) \bumpeq z$	V_{18}
$LL^{-1}T$.	$yx \,.\, zx \bumpeq yz$	V_{28}	*$LL^{-1}R^{-1}$.*	$xz \,.\, zx \bumpeq yz$	V_{18}
*	$zx \,.\, yx \bumpeq yz$	V_{18}	*	$xz \,.\, xy \bumpeq yz$	V_{18}
/	$(zx \,.\, yx)\, y \bumpeq z$	V_{18}	/	$xy \,.\, z \bumpeq yz \,.\, x$	V_{27}
∇	$(xy \,.\, zy)\, x \bumpeq z$	V_{18}	∇	$(xy \,.\, z)\, x \bumpeq yz$	V_{18}
\	$x(xy \,.\, zy) \bumpeq z$	V_{18}	\	$(yx \,.\, z)\, x \bumpeq yz$	V_{33}
Δ	$x(zy \,.\, xy) \bumpeq z$	V_{18}	Δ	$xy \,.\, z \bumpeq xz \,.\, y$	V_{27}
$LR^{-1}T$.	$yx \,.\, xz \bumpeq yz$	V_{18}	*$LR^{-1}T^{-1}$.*	$xz \,.\, yx \bumpeq yz$	V_{18}
*	$zx \,.\, xy \bumpeq yz$	V_{18}	*	$xy \,.\, zx \bumpeq yz$	V_{18}
/	$(zx \,.\, xy)\, y \bumpeq z$	V_{23}	/	$(xy \,.\, zx)\, y \bumpeq z$	V_{20}
∇	$(xy \,.\, yz)\, x \bumpeq z$	V_{21}	∇	$(xz \,.\, yx)\, y \bumpeq z$	V_{20}
\	$x(xy \,.\, yz) \bumpeq z$	V_{23}	\	$x(yz \,.\, xy) \bumpeq z$	V_{20}
Δ	$x(zy \,.\, yx) \bumpeq z$	V_{21}	Δ	$x(yx \,.\, zy) \bumpeq z$	V_{20}
$LT^{-1}T$.	$xy \,.\, z \bumpeq x \,.\, yz$	V_{26}	*$LT^{-1}R^{-1}$.*	$x(y \,.\, zx) \bumpeq yz$	V_{18}
*	$xy \,.\, z \bumpeq yz \,.\, x$	V_{27}	*	$x(zx \,.\, y) \bumpeq yz$	V_{20}
/	$(zx \,.\, y) \,.\, xy \bumpeq z$	V_{18}	/	$(z \,.\, xy)\, x \bumpeq yz$	V_{38}
∇	$(xy \,.\, z)\, x \bumpeq z$	V_{18}	∇	$x_1y_1 = x_2y_2 \rightarrow y_1x_2 = y_2x_1$	V_{39}
\	$x(xy \,.\, z) \bumpeq yz$	V_{19}	\	$x_1y_1 = x_2y_2 \rightarrow x_1x_2 = y_2y_1$	V_{36}
Δ	$xy \,.\, (zx \,.\, y) \bumpeq z$	V_{19}	Δ	$(xy \,.\, z)\, x \bumpeq yz$	V_{18}
$LT^{-1}T^{-1}$.	$x \,.\, yx \bumpeq z \,.\, xy$	V_{27}	*TTT.*	$xy \,.\, zx \bumpeq yz$	V_{18}
*	$xy \,.\, y \bumpeq y \,.\, zx$	V_{27}	*	$zx \,.\, xy \bumpeq yz$	V_{18}
/	$(x \,.\, zy) \,.\, yx \bumpeq z$	V^*_{20}	/	$zx \,.\, xy \bumpeq yz$	V_{18}
∇	$(z \,.\, xy)x \bumpeq yz$	V^*_{20}	∇	$xy \,.\, zx \bumpeq yz$	V_{18}
\	$x(z \,.\, xy) \bumpeq yz$	V_{18}	\	$zx \,.\, xy \bumpeq yz$	V_{18}
Δ	$xy \,.\, (y \,.\, zx) \bumpeq z$	V_{21}	Δ	$xy \,.\, zx \bumpeq yz$	V_{18}
TTT^{-1}.	$xz \,.\, yx \bumpeq yz$	V_{18}	∇	$yx \,.\, zx \bumpeq yz$	V_{18}
*	$yx \,.\, xz \bumpeq yz$	V_{18}	\	$zx \,.\, yx \bumpeq yz$	V_{19}
/	$xz \,.\, xy \bumpeq yz$	V_{18}	Δ	$xy \,.\, xz \bumpeq yz$	K_{28}

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