Jarosław Werbowski

Oscillations of all solutions of functional differential inequalities

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 4, 586–598

Persistent URL: `http://dml.cz/dmlcz/102118`

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

OSCILLATIONS OF ALL SOLUTIONS OF FUNCTIONAL DIFFERENTIAL INEQUALITIES

JAROSŁAW WERBOWSKI, Poznań

(Received June 28, 1985)

1. INTRODUCTION

In this paper we are concerned with the oscillatory behavior of the functional differential inequality of the form

$$(\mathrm{E}_z) \qquad (-1)^z\, x^{(n)}(t) \operatorname{sgn} x(t) \geqq p(t) \prod_{i=1}^{m} |x(g_i(t))|^{\alpha_i}\,, \quad n \geqq 2\,,$$

where z is a natural number, $\alpha_i \in R_+ = [0, \infty)$ with $\alpha_1 + \ldots + \alpha_m = 1$, the function $p\colon R_+ \to R_+$ is integrable on each finite segment and is not identically zero in every neighbourhood of infinity, the functions $g_i\colon R_+ \to R_+$ $(i = 1, \ldots, m)$ are continuous and $\lim_{t\to\infty} g_i(t) = \infty$.

By a solution of (E_z) we mean a function $x\colon [t_0, \infty) \to R$, $t_0 \in R_+$, such that
(a) $x^{(k)}$ $(k = 0, 1, \ldots, n-1)$ is absolutely continuous on $[t_0, \infty)$,
(b) $\sup\{|x(s)|\colon t \leqq s < \infty\} > 0$ for any $t \geqq t_0$,
(c) there exists a $t_1 \in (t_0, \infty)$ such that x satisfies (E_z) almost everywhere on $[t_1, \infty)$.

A solution of (E_z) is called *oscillatory*, if it has an infinite sequence of zeros tending to infinity. Otherwise it is called *nonoscillatory*.

The purpose of this paper is to study oscillations of solutions of (E_z) generated by general deviating arguments g_i (not necessarily delay or advanced arguments). Some results on oscillation of functional differential equations and inequalities with general deviating arguments have been obtained in the papers [2, 5, 7, 9, 15, 19, 22, 24, 26–28]. The main results of this paper are new and are independent of the analogous ones known for delay and advanced differential equations. Some specific comparisons to known results will be made in the text of the paper.

The following notation will be used throughout this paper:

$$D = \{t \in R_+\colon g_i(t) \leqq t \ (i = 1, \ldots, m)\}\,, \quad A = \{t \in R_+\colon g_i(t) \geqq t \ (i = 1, \ldots, m)\}\,.$$

Let $g_i^*, d_i, a_i\colon R_+ \to R_+$ $(i = 1, \ldots, m)$ be nondecreasing functions such that

$$g_i^*(t) \leqq \min(t, g_i(t)) \quad \text{and} \quad d_i(t) \leqq t \leqq a_i(t) \quad \text{for} \quad t \in R_+\,,$$
$$g_i(t) \leqq d_i(t) \quad \text{for} \quad t \in D \quad \text{and} \quad a_i(t) \leqq g_i(t) \quad \text{for} \quad t \in A\,.$$

Let $D_i(t) = D \cap [d_i(t), t]$ and $A_i(t) = A \cap [t, a_i(t)]$ for $t \in R_+$.

To obtain our results we need the following two lemmas.

Lemma 1 [6]. *Let x be a nonoscillatory solution of (E_z). Then there exist an integer $l \in \{0, 1, \ldots, n\}$ with $n + l + z$ even and a number $t_1 \in [t_0, \infty)$ such that*

$$(\mathrm{N}_l) \qquad x(t)\, x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, l - 1),$$
$$(-1)^{k+l}\, x(t)\, x^{(k)}(t) > 0 \quad (k = l, \ldots, n - 1)$$

for $t \geqq t_1$.

Lemma 2 [7]. *Let x be a nonoscillatory solution of (E_z) satisfying the inequality (N_l) with $l \in \{1, \ldots, n - 1\}$ and $n + l + z$ even. In addition, let*

$$\int^{\infty} t^{n-l} |x^{(n)}(t)| \,\mathrm{d}t = \infty \,. \tag{1}$$

Then the following inequalities hold for sufficiently large $t \geqq t_1$:

$$|x^{(l-1)}(t)| \geqq \frac{t}{(n - l)!} \int_t^{\infty} s^{n-l-1} |x^{(n)}(s)| \,\mathrm{d}s \tag{2}$$

and

$$k|x^{(l-k)}(t)| \geqq t|x^{(l-k+1)}(t)| \,, \quad (k = 1, \ldots, l) \,. \tag{3}$$

2. MAIN RESULTS

Theorem 1. *Consider the differential inequality (E_z) subject to the condition*

$$\limsup_{t \to \infty} \prod_{j=1}^{m} \Big[\int_{g_j^*(t)}^{t} p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-1)\alpha_i} \,\mathrm{d}s + \prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i} \int_t^{\infty} p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i} \,\mathrm{d}s \Big]^{\alpha_j} > (n - 1)! \,. \tag{4}$$

Then

(i) *for n even, every solution of (E_1) is oscillatory,*

(ii) *for n odd, every solution of (E_1) is either oscillatory or $\lim_{t\to\infty} x^{(k)}(t) = 0$ $(k = 0, 1, \ldots, n - 1)$ monotonically,*

(iii) *for n odd, every solution of (E_2) is either oscillatory or $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n - 1)$ monotonically,*

(iv) *for n even, every solution of (E_2) is either oscillatory or $\lim_{t\to\infty} x^{(k)}(t) = 0$ or $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n - 1)$ monotonically.*

Proof. Suppose that the inequality (E_z) has a nonoscillatory solution $x(t) \neq 0$ for $t \geqq t_0$. Therefore for sufficiently large $t \geqq t_1$, by Lemma 1, there exists an integer $l \in \{0, 1, \ldots, n\}$ with $n + l + z$ even, such that $x(t)$ satisfies the inequalities (N_l).

Case (i). Then we have n even, $z = 1$ and an odd $l \in \{1, 3, \ldots, n - 1\}$. We observe that (4) and (N_1) $(1 \leqq l \leqq n-1)$ imply that the condition (1) of Lemma 2 is satisfied. Therefore (2) and (E_1) yield

$$|x^{(l-1)}(t)| \geqq \frac{t}{(n-l)!} \int_t^\infty s^{n-l-1} \, p(s) \prod_{i=1}^m |x(g_i(s))|^{\alpha_i} \, ds \geqq$$

$$\geqq \frac{t}{(n-l)!} \int_t^\infty s^{n-l-1} \, p(s) \prod_{i=1}^m |x(g_i^*(s))|^{\alpha_i} \, ds \,,$$

which yields, by (3),

$$|x^{(l-1)}(t)| \geqq \frac{t}{l!\,(n-l)!} \int_t^\infty s^{n-l-1} \, p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)\alpha_i} \, |x^{(l-1)}(g_i^*(s))|^{\alpha_i} \, ds \,.$$

From the above inequality for $j \in \{1, \ldots, m\}$ and $t \geqq t_1$ we get

$$\begin{aligned} l!\,(n-l)! \frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} &\geqq \int_{g_j^*(t)}^t s^{n-l-1} \, p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)\alpha_i} \, |x^{(l-1)}(g_i^*(s))|^{\alpha_i} \, ds \, + \\ &+ \int_t^\infty s^{n-l-1} \, p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)\alpha_i} \, |x^{(l-1)}(g_i^*(s))|^{\alpha_i} \, ds \,. \end{aligned} \tag{5}$$

Since $|x^{(l-1)}(t)| \, t^{-1}$ is nonincreasing for $1 \leqq l \leqq n-1$ and $t \geqq t_1$ by (3), we obtain for $s \in [g_j^*(t), t]$ and $i \in \{1, \ldots, m\}$

$$|x^{(l-1)}(g_i^*(s))| \geqq \frac{g_i^*(s)}{g_i^*(t)} \, x^{(l-1)}(g_i^*(t))| \,. \tag{6}$$

Therefore from (5) and (6), in view of the increasing character of $|x^{(l-1)}(t)|$, we derive for $t \geqq t_1$

$$\begin{aligned} l!\,(n-l)! \frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} &\geqq \prod_{i=1}^m \left[\frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)}\right]^{\alpha_i} \int_{g_j^*(t)}^t s^{n-l-1} p(s) \prod_{i=1}^m [g_i^*(s)]^{l\alpha_i} \, ds \, + \\ &+ \prod_{i=1}^m |x^{(l-1)}(g_i^*(t))|^{\alpha_i} \int_t^\infty s^{n-l-1} \, p(s) \prod_{i=1}^m [g_i^*(s)]^{(l-1)\alpha_i} \, ds \,. \end{aligned} \tag{7}$$

For $l \in \{1, \ldots, n-1\}$ the following inequalities hold for $s \geqq t_1$:

$$\begin{aligned} s^{n-l-1} \prod_{i=1}^m [g_i^*(s)]^{l\alpha_i} &= s^{n-1} \prod_{i=1}^m \left[\frac{g_i^*(s)}{s}\right]^{l\alpha_i} \geqq \\ &\geqq s^{n-1} \prod_{i=1}^m \left[\frac{g_i^*(s)}{s}\right]^{(n-1)\alpha_i} = \prod_{i=1}^m [g_i^*(s)]^{(n-1)\alpha_i} \end{aligned} \tag{8}$$

and

$$s^{n-l-1} \prod_{i=1}^m [g_i^*(s)]^{(l-1)\alpha_i} \geqq \prod_{i=1}^m [g_i^*(s)]^{(n-2)\alpha_i} \,. \tag{9}$$

Using now (8) and (9) in (7) we get

$$(n-1)!\,\frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} \geqq \prod_{i=1}^{m}\left[\frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)}\right]^{\alpha_i}\left\{\int_{g_j^*(t)}^{t} p(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-1)\alpha_i}\,\mathrm{d}s + \right.$$

$$\left. + \prod_{i=1}^{m}[g_i^*(t)]^{\alpha_i}\int_t^{\infty} p(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-2)\alpha_i}\,\mathrm{d}s\right\}.$$

Raising both sides of the above inequality to α_j and then multiplying the resulting inequalities we obtain

$$(n-1)!\prod_{j=1}^{m}\left[\frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)}\right]^{\alpha_j} \geqq$$

$$\geqq \prod_{i=1}^{m}\left[\frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)}\right]^{\alpha_i}\prod_{j=1}^{m}\left[\int_{g_j^*(t)}^{t} p(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-1)\alpha_i}\,\mathrm{d}s + \right.$$

$$\left. + \prod_{i=1}^{m}[g_i^*(t)]^{\alpha_i}\int_t^{\infty} p(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-2)\alpha_i}\,\mathrm{d}s\right]^{\alpha_j},$$

which contradicts (4).

Case (ii). Then $x(t)$ satisfies the inequalities (N_l) for $l \in \{0, 2, 4, \ldots, n-1\}$. By arguments similar to those in the proof of (i), we prove that the case $l \in \{2, 4, \ldots, n-1\}$ is impossible. Therefore $x(t)$ satisfies (N_l) for $l = 0$, i.e.

$$(10)\qquad (-1)^k\,x(t)\,x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n-1) \quad \text{for} \quad t \geqq t_1\,.$$

We shall prove that $\lim_{t\to\infty} x(t) = 0$. Suppose to the contrary that $\lim_{t\to\infty} x(t)| = C > 0$. Then $|x(g_i(t))| \geqq C\ (i = 1, \ldots, m)$ for $t \geqq t_2 \geqq t_1$. From (10) it follows that (see [7])

$$\int_{t_2}^{\infty} t^{n-1}|x^{(n)}(t)|\,\mathrm{d}t < \infty\,,$$

which implies, by (E_1),

$$\infty > \int_{t_2}^{\infty} t^{n-1}|x^{(n)}(t)|\,\mathrm{d}t \geqq \int_{t_2}^{\infty} t^{n-1}\,p(t)\prod_{i=1}^{m}|x(g_i(t))|^{\alpha_i}\,\mathrm{d}t \geqq C\int_{t_2}^{\infty} t^{n-1}\,p(t)\,\mathrm{d}t\,.$$

But this gives a contradiction, since (4) implies that

$$(11)\qquad \int_{t_2}^{\infty} p(t)\prod_{i=1}^{m}[g_i^*(t)]^{(n-1)\alpha_i}\,\mathrm{d}t = \infty\,.$$

Case (iii) and (iv). Then $x(t)$ satisfies (N_l) for $l \in \{0, 2, 4, \ldots, n\}$. The case $l = 0$ holds only when n is even. Then, by arguments analogous to those in the proof of (ii), we have $\lim_{t\to\infty} x^{(k)}(t) = 0\ (k = 0, 1, \ldots, n-1)$. Similarly as in the proof of (i), we prove that the case $l \in \{2, 4, \ldots, n-2\}$ is impossible. In the case $l = n$ we have

$$(12)\qquad x(t)\,x^{(n)}(t) \geqq 0 \quad \text{and} \quad x(t)\,x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n-1)$$

for $t \geqq t_2$. We shall prove that $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n-1)$. From (12) it follows that there exist a point $t_3 \geqq t_2$ and a positive constant γ such that

$$|x(g_i(t))| \geqq \gamma\, g_i^{n-1}(t) \quad (i = 1, \ldots, m) \quad \text{for} \quad t \geqq t_3\,. \tag{13}$$

Integrating now (E_2) from t_3 to t, by (13) we obtain

$$|x^{(n-1)}(t)| \geqq |x^{(n-1)}(t_3)| + \int_{t_3}^{t} p(s) \prod_{i=1}^{m} |x(g_i(s))|^{\alpha_i}\, ds \geqq$$

$$\geqq \gamma \int_{t_3}^{t} p(s) \prod_{i=1}^{m} [g_i(s)]^{(n-1)\alpha_i}\, ds \geqq \gamma \int_{t_3}^{t} p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-1)\alpha_i} ds\,.$$

From the above inequality and (11) we get $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ $(k = 0, 1, \ldots, n-1)$. Thus the proof is complete.

Corollary 1. *Consider the linear differential equation with general deviating argument*

$$x^{(n)}(t) + p(t)\, x(g(t)) = 0\,, \quad n \geqq 2\,, \tag{14}$$

where p is the same as in (E_2), $g: R_+ \to R_+$ is continuous and $\lim_{t\to\infty} g(t) = \infty$. *Let the function $g^*(t) \leqq \min(t, g(t))$ be nondecreasing on R_+. If*

$$\limsup_{t\to\infty} \left\{ \int_{g^*(t)}^{t} [g^*(s)]^{n-1} p(s)\, ds + g^*(t) \int_{t}^{\infty} [g^*(s)]^{n-2} p(s)\, ds \right\} > (n-1)!\,, \tag{15}$$

then every solution of (14) *is oscillatory, if n is even, and every solution of* (14) *is either oscillatory or* $\lim_{t\to\infty} x^{(k)}(t) = 0$ $(k = 0, 1, \ldots, n-1)$ *monotonically if n is odd.*

Remark 1. From Corollary 1, in the case of ordinary linear differential equations $(g(t) = t)$, we obtain the result of Chanturia [1, Th. 2.3]. In the case of advanced differential equations $(g(t) \geqq t)$, Corollary 1 gives the result of Kusano [8, Th. 3].

Remark 2. Recently, conditions of nature similar to that in Corollary 1 have been obtained by Oláh [15, Th. 1], [16, Th. 2] and [17, Th. 1] (cf. also [2, C. 2.17] and [14, Th. 8]). These conditions for the equation (14) have the following forms:

$$\limsup_{t\to\infty} [g^*(t)]^{n-1} \int_{t}^{\infty} p(s)\, ds > (n-1)! \tag{16}$$

for the general deviating argument $g(t)$ and

$$\limsup_{t\to\infty} g(t) \int_{t}^{\infty} g^{n-2}(s)\, p(s)\, ds > (n-1)!\,, \tag{17}$$

$$\limsup_{t\to\infty} \left[\frac{1}{t} \int_{T}^{t} s g^{n-1}(s)\, p(s)\, ds + t \int_{t}^{\infty} \frac{g^{n-1}(s)}{s}\, p(s)\, ds \right] > (n-1)! \tag{18}$$

in the case $g(t) \leqq t$.

We note that the condition (15) of Corollary 1 is independent of the above conditions (16)–(18) for oscillations. For example, if we put $p(t) = (n-1)!\, e^{3n-4} t^{-n}$ and $g(t) = e^{-3} t$ in the equation (14), then the condition (15) of Corollary 1 is satisfied. In this case none of the conditions (16)–(18) is satisfied.

Corollary 2. *Suppose that* $\prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i}$ *has a nonnegative derivative on* R_+. *If there exists a positive nonincreasing function* ψ *on* R_+, *such that*

$$\int^{\infty} t^{-1} \psi(t)\, dt < \infty \quad \text{and} \quad \int^{\infty} p(t) \prod_{i=1}^{m} [g_i^*(t)]^{(n-1)\alpha_i} \psi\Big(\prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i}\Big)\, dt = \infty,$$

then the conclusion of Theorem 1 *holds.*

Proof. Proceeding identically as in the proof of Corollary of [15], we prove that

$$\limsup_{t \to \infty} \prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i} \int_t^{\infty} p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i}\, ds = \infty.$$

Therefore the condition (4) is satisfied and the conclusion of Theorem 1 holds.

Remark 3. Similar conditions as in Corollary 2 can be found in the papers [7, 15, 17, 21, 23].

Theorem 2. *If*

(19)

$$\limsup_{t \to \infty} \prod_{j=1}^{m} \left[\int_{D_j(t)} [s - d_j(t)]^{n-\nu-1} \prod_{i=1}^{m} [d_i(t) - g_i(s)]^{\nu \alpha_i} p(s)\, ds\right]^{\alpha_j} > \nu!\,(n-\nu-1)!$$

for some $\nu \in \{0, 1, \ldots, n-1\}$ *then every bounded solution of* (E_n) *is oscillatory.*

Proof. Assume that (E_n) has a bounded nonoscillatory solution $x(t) \neq 0$ for $t \geqq t_0$. Then for sufficiently large $t \geqq t_1 \geqq t_0$ we have, by (E_n),

$$(20) \qquad (-1)^n x(t)\, x^{(n)}(t) \geqq 0 \quad \text{and} \quad (-1)^k x(t)\, x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n-1).$$

From the equality

$$(21) \qquad x^{(\nu)}(t) = \sum_{k=\nu}^{n-1} \frac{(t-u)^{k-\nu}}{(k-\nu)!} x^{(k)}(u) + \int_u^t \frac{(t-s)^{n-\nu-1}}{(n-\nu-1)!} x^{(n)}(s)\, ds$$

we obtain, in view of (20), for $\nu \in \{0, 1, \ldots, n-1\}$ and $u \geqq t \geqq t_1$

$$(22) \qquad |x^{(\nu)}(t)| \geqq \int_t^u \frac{(s-t)^{n-\nu-1}}{(n-\nu-1)!} |x^{(n)}(s)|\, ds$$

and

$$(23) \qquad |x(t)| \geqq \frac{(u-t)^{\nu}}{\nu!} |x^{(\nu)}(u)|.$$

From (23) we have for $s \in D_j(t)$ $(j = 1, \ldots, m)$ and $t \geqq t_1$

$$(24) \qquad |x(g_i(s))| \geqq \frac{[d_i(t) - g_i(s)]^{\nu}}{\nu!} |x^{(\nu)}(d_i(t))| \quad (i = 1, \ldots, m).$$

Therefore from (22), (E_n) and (24) we get for $t \geqq t_1$ and $\nu \in \{0, 1, \ldots, n-1\}$, $j \in \in \{1, \ldots, m\}$

$$|x^{(\nu)}(d_j(t))| \geqq \int_{d_j(t)}^{t} \frac{[s - d_j(t)]^{n-\nu-1}}{(n-\nu-1)!} |x^{(n)}(s)| \, ds \geqq$$

$$\geqq \int_{D_j(t)} \frac{[s - d_j(t)]^{n-\nu-1}}{(n-\nu-1)!} p(s) \prod_{i=1}^{m} |x(g_i(s))|^{\alpha_i} \, ds \geqq$$

$$\geqq \prod_{i=1}^{m} |x^{(\nu)}(d_i(t))|^{\alpha_i} \int_{D_j(t)} \frac{[s - d_j(t)]^{n-\nu-1}}{\nu!\,(n-\nu-1)!} p(s) \prod_{i=1}^{m} [d_i(t) - g_i(s)]^{\nu\alpha_i} \, ds \, .$$

Raising both sides of the above inequality to α_j and then multiplying, we obtain

$$\prod_{j=1}^{m} |x^{(\nu)}(d_j(t))|^{\alpha_j} \geqq$$

$$\geqq \prod_{i=1}^{m} |x^{(\nu)}(d_i(t))|^{\alpha_i} \prod_{j=1}^{m} \left[\int_{D_j(t)} \frac{[s - d_j(t)]^{n-\nu-1}}{\nu!\,(n-\nu-1)!} p(s) \prod_{i=1}^{m} [d_i(t) - g_i(s)]^{\nu\alpha_i} \, ds \right]^{\alpha_j},$$

which contradicts (19). Thus the proof is complete.

Corollary 3. *Suppose that in the equation* (E_n), $g_i(t) \leqq t$ $(i = 1, \ldots, m)$ *on* R_+. *Then every bounded solution of* (E_n) *is oscillatory, if*

$$\limsup_{t \to \infty} \prod_{i=1}^{m} \left[\int_{d_i(t)}^{t} [s - d_i(t)]^{n-1} p(s) \, ds \right]^{\alpha_i} > (n-1)! \, . \tag{25}$$

Remark 4. The oscillatory character of the bounded solutions of delay differential equations and inequalities has been considered in many papers, see for example [3, 4, 7, 10–13, 18, 20, 25, 28]. A conditions similar to that in Corollary 3 for delay differential equations and inequalities can be found in the papers [13, Th. 2], [18, Th. 1] and [25, Th. 2]. From these results it follows that every bounded solution of (E_n) is oscillatory, if

$$\limsup_{t \to \infty} \int_{g(t)}^{t} [s - g(t)]^{n-1} p(s) \, ds > (n-1)! \, , \tag{26}$$

where $g(t) = \max(g_1(t), \ldots, g_m(t))$ and $g_i(t)$ are nondecreasing. We note that the condition (25) of Corollary 3 is better than (26).

Theorem 3. *Let* $n \geqq 3$ *be odd. Consider the differential inequality* (E_1) *subject to the conditions* (4) *and* (19). *Then every solution of* (E_1) *is oscillatory.*

Proof. The above theorem follows from Theorem 1 and 2.

Theorem 4. *Let* $n \geqq 3$ *be odd. Consider the differential inequality* (E_2) *subject to the conditions*

$$\limsup_{t \to \infty} \prod_{j=1}^{m} \left[\int_{g_j^*(t)}^{t} s p(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i} \, ds + \right. \tag{27}$$

$$+ \prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i} \int_t^\infty sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-3)\alpha_i} \, ds \Bigg]^{\alpha_j} > (n-1)!$$

and

(28)

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\int_{A_j(t)} [a_j(t) - s]^{n-\nu-1} \prod_{i=1}^{m} [g_i(s) - a_i(t)]^{\nu\alpha_i} p(s) \, ds \right]^{\alpha_j} > \nu!\,(n-\nu-1)!$$

for some $\nu \in \{0, 1, \ldots, n-1\}$. *Then every solution of* (E_2) *is oscillatory.*

Proof. Suppose that the inequality (E_2) has a nonoscillatory solution $x(t) \neq 0$ for $t \geqq t_0$. Then Lemma 1 implies that either

$$x(t)\, x^{(n)}(t) \geqq 0\,, \quad x(t)\, x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, n-1)\,, \tag{29}$$

or there exists an odd $l \in \{1, 3, \ldots, n-2\}$ such that

$$\begin{gathered} x(t)\, x^{(k)}(t) > 0 \quad (k = 0, 1, \ldots, l-1)\,, \\ (-1)^{k+1} x(t)\, x^{(k)}(t) > 0 \quad (k = l, \ldots, n-1) \quad \text{for} \quad t \geqq t_1 \geqq t_0\,. \end{gathered} \tag{30}$$

Let (29) hold. Then from (21) we have for $u \geqq t \geqq t_1$ and $\nu \in \{0, 1, \ldots, n-1\}$

$$|x^{(\nu)}(u)| \geqq \int_t^u \frac{(u-s)^{n-\nu-1}}{(n-\nu-1)!} |x^{(n)}(s)| \, ds \tag{31}$$

and

$$|x(u)| \geqq \frac{1}{\nu!} (u-t)^\nu |x^{(\nu)}(t)|\,. \tag{32}$$

From (32) for $s \in A_j(t)$ $(j = 1, \ldots, m)$ and $t \geqq t_1$ we obtain

$$|x(g_i(s))| \geqq \frac{1}{\nu!} [g_i(s) - a_i(t)]^\nu |x^{(\nu)}(a_i(t))| \quad i = 1, \ldots, m)\,. \tag{33}$$

Then from (31), (E_2) and (33) we derive for $j \in \{1, \ldots, m\}$, $\nu \in \{1, \ldots, n-1\}$ and $t \geqq t_1$

$$\begin{aligned} |x^{(\nu)}(a_j(t))| &\geqq \int_t^{a_j(t)} \frac{[a_j(t) - s]^{n-\nu-1}}{(n-\nu-1)!} |x^{(n)}(s)| \, ds \geqq \\ &\geqq \int_{A_j(t)} \frac{[a_j(t) - s]^{n-\nu-1}}{(n-\nu-1)!} p(s) \prod_{i=1}^{m} |x(g_i(s))|^{\alpha_i} \, ds \geqq \\ \geqq \prod_{i=1}^{m} |x^{(\nu)}(a_i(t))|^{\alpha_i} &\int_{A_j(t)} \frac{[a_j(t) - s]^{n-\nu-1}}{\nu!\,(n-\nu-1)!} p(s) \prod_{i=1}^{m} [g_i(s) - a_i(t)]^{\nu\alpha_i} \, ds\,. \end{aligned}$$

Raising both sides of the above inequality to α_j and then multiplying the resulting inequalities we get

$$\prod_{j=1}^{m} |x^{(\nu)}(a_j(t))|^{\alpha_j} \geqq$$

$$\geqq \prod_{i=1}^{m} |x^{(\nu)}(a_i(t))|^{\alpha_i} \prod_{j=1}^{m} \left[\int_{A_j(t)} \frac{[a_j(t) - s]^{n-\nu-1}}{\nu!\,(n - \nu - 1)!} p(s) \prod_{i=1}^{m} [g_i(s) - a_i(t)]^{\nu\alpha_i} \, ds \right]^{\alpha_j},$$

which contradicts (28). Thus, the case (29) is impossible.

Suppose now that (30) holds. Then, in view of (27) and (30), the assumptions of Lemma 2 are satisfied. Therefore, by arguments similar to those in the proof of Theorem 1, we obtain the inequality (7). Since $l \in \{1, 3, \ldots, n - 2\}$, we get for $s \geqq t_1$

$$(34) \qquad s^{n-l-1} \prod_{i=1}^{m} [g_i^*(s)]^{l\alpha_i} = s^{n-1} \prod_{i=1}^{m} \left[\frac{g_i^*(s)}{s}\right]^{l\alpha_i} \geqq$$

$$\geqq s^{n-1} \prod_{i=1}^{m} \left[\frac{g_i^*(s)}{s}\right]^{(n-2)\alpha_i} = s \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i}$$

and

$$(35) \qquad s^{n-l-1} \prod_{i=1}^{m} [g_i^*(s)]^{(l-1)\alpha_i} = s^{n-2} \prod_{i=1}^{m} \left[\frac{g_i^*(s)}{s}\right]^{(l-1)\alpha_i} \geqq$$

$$\geqq s^{n-2} \prod_{i=1}^{m} \left[\frac{g_i^*(s)}{s}\right]^{(n-3)\alpha_i} = s \prod_{i=1}^{m} [g_i^*(s)]^{(n-3)\alpha_i}.$$

Using now (34) and (35) in (7) we obtain

$$(n - 1)! \frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} \geqq \prod_{i=1}^{m} \left[\frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)}\right]^{\alpha_i} \left\{ \int_{g_j^*(t)}^{t} sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i} \, ds + \right.$$

$$\left. + \prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i} \int_t^{\infty} sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-3)\alpha_i} \, ds \right\}.$$

Proceeding as in the corresponding part of the proof of Theorem 1, we get a contradiction with the assumption (27). Thus, the inequalities (30) cannot hold. This completes the proof.

Theorem 5. *Let n be even. Consider the differential inequality* (E_2) *subject to the conditions* (19) *and* (28). *In addition, let for* $n \geqq 4$

$$(36) \qquad \limsup_{t\to\infty} \prod_{j=1}^{m} \left[\int_{g_j^*(t)}^{t} sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-2)\alpha_i} \, ds + \right.$$

$$\left. + \prod_{i=1}^{m} [g_i^*(t)]^{\alpha_i} \int_t^{\infty} sp(s) \prod_{i=1}^{m} [g_i^*(s)]^{(n-3)\alpha_i} \, ds \right]^{\alpha_j} > 2(n - 2)!\,.$$

Then every solution of (E_2) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (E_2) on an interval $[t_0, \infty)$. Thus, by Lemma 1, x satisfies the inequalities (N_l) with $l \in \{0, 2, \ldots, n - 2, n\}$. The cases $l = 0$ and $l = n$ are impossible, by the assumptions (19), (28) and Theorem 2 and 4, respectively.

Suppose now that $l \in \{2, \ldots, n - 2\}$, which is possible only if $n \geqq 4$. Therefore, by arguments similar to those in the corresponding part of the proof of Theorems 1 and 4, we obtain the inequalities (7), (34) and (35). Combining (34) and (35) with (7) and using the fact that $2 \leqq l \leqq n - 2$, we have

$$2(n-2)!\,\frac{|x^{(l-1)}(g_j^*(t))|}{g_j^*(t)} \geqq \prod_{i=1}^{m}\left[\frac{|x^{(l-1)}(g_i^*(t))|}{g_i^*(t)}\right]^{\alpha_i}\int_{g_j^*(t)}^{t} sp(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-2)\alpha_i}\,\mathrm{d}s + \\ + \prod_{i=1}^{m}|x^{(l-1)}(g_i^*(t))|^{\alpha_i}\int_t^{\infty} sp(s)\prod_{i=1}^{m}[g_i^*(s)]^{(n-3)\alpha_i}\,\mathrm{d}s\,.$$

From this inequality, similarly as in the proof of Theorem 4, we obtain a contradiction with the assumption (36). Thus $l \notin \{2, \ldots, n-2\}$ and the proof is complete.

3. FINAL REMARKS

For simplicity, we consider the linear differential equation with a deviating argument

$$\text{(L)} \qquad x^{(n)}(t) = p(t)\,x(g(t))\,,$$

where n is even, $p\colon R_+ \to (0, \infty)$ and $g\colon R_+ \to R_+$ are continuous, $g(t)$ is nondecreasing and $\lim_{t\to\infty} g(t) = \infty$. Let $g_*(t) = \min(t, g(t))$, $g^*(t) = \max(t, g(t))$, $D = \{t \in R_+ : g(t) < t\}$ and $A = \{t \in R_+ : g(t) > t\}$.

It is known that in the case of ordinary differential equation, i.e. $g(t) = t$, the equation (L) always has nonoscillatory solutions satisfying the inequalities (N_0) and (N_n). The situation is different when $g(t) \neq t$. For example, in view of Theorems 1 and 2, every solution x of (L) is either oscillatory or $\lim_{t\to\infty} |x^{(k)}(t)| = \infty$ ($k = 0, 1, \ldots, n-1$) monotonically, if the following conditions hold:

$$\text{(37)} \qquad \limsup_{t\to\infty}\left\{\int_{g^*(t)}^{t} g_*^{n-1}(s)\,p(s)\,\mathrm{d}s + g_*(t)\int_t^{\infty} g_*^{n-2}(s)\,p(s)\,\mathrm{d}s\right\} > (n-1)!\,,$$

and for some $\nu \in \{0, 1, \ldots, n-1\}$,

$$\text{(38)} \qquad \limsup_{t\to\infty}\int_{D\cap[g_*(t),t]} [s - g_*(t)]^{n-\nu-1}\,[g_*(t) - g(s)]^{\nu}\,p(s)\,\mathrm{d}s > \nu!\,(n-\nu-1)!\,.$$

On the basis of Theorems 1 and 4 we can prove that every solution x of (L) is either oscillatory or $\lim_{t\to\infty} x^{(k)}(t) = 0$ ($k = 0, 1, \ldots, n-1$) monotonically, if (37) holds and

$$\text{(39)} \qquad \limsup_{t\to\infty}\int_{A\cap[t,g^*(t)]} [g^*(t) - s]^{n-\nu-1}\,[g(s) - g^*(t)]^{\nu}\,p(s)\,\mathrm{d}s > \nu!\,(n-\nu-1)!$$

for some $\nu \in \{0, 1, \ldots, n-1\}$.

From Theorem 5 it follows that every solution of (L) is oscillatory if (38) and (39) hold. In addition, when $n \geqq 4$, the following inequality is satisfied:

$$(40) \quad \limsup_{t\to\infty} \left\{ \int_{g^*(t)}^{t} s g_*^{n-2}(s)\, p(s)\, ds + g_*(t) \int_t^\infty s g_*^{n-3}(s)\, p(s)\, ds \right\} > 2(n-2)!\,.$$

Recently, Kusano [9, Th. 1] has proved that every solution of (L) is oscillatory if there exist two sequences $\{t_k\}, \{\tau_k\}$ such that $t_k \in A$, $t_k \to \infty$ as $k \to \infty$, $\tau_k \in D$, $\tau_k \to \infty$ as $k \to \infty$,

(41)

$$\min \left\{ \int_{g(t_k)}^{\tau_k} [g(\tau_k) - g(s)]^{n-1}\, p(s)\, ds,\ \int_{t_k}^{g(t_k)} [g(s) - g(t_k)]^{n-1}\, p(s)\, ds \right\} \geqq (n-1)!$$

for all $k = 1, 2, \ldots$ and

$$(42) \qquad \int^\infty [g_*(t)]^{n-1-\varepsilon}\, p(t)\, dt = \infty \quad \text{for some} \quad \varepsilon > 0\,.$$

In some cases the conditions (38)–(40) are better than the conditions (41) and (42). For example, consider the differential equation with general deviating argument

$$(43) \qquad x^{(n)}(t) = p\, x(t + \sin t)\,, \quad p \text{ a positive constant}\,.$$

Then

$$D = \bigcup_{k=0}^{\infty} ((2k+1)\pi, (2k+2)\pi)\,, \quad A = \bigcup_{k=0}^{\infty} (2k\pi, (2k+1)\pi)$$

and

$$g_*(t) = \begin{cases} t + \sin t & \text{for} \quad t \in D\,, \\ t & \text{for} \quad t \notin D\,, \end{cases} \qquad g^*(t) = \begin{cases} t + \sin t & \text{for} \quad t \in A\,, \\ t & \text{for} \quad t \notin A\,. \end{cases}$$

If we choose $t_k = (2k+1)\pi + \frac{1}{2}\pi$ $(k = 1, 2, \ldots)$, then $D \cap [g_*(t_k), t_k] = [g_*(t_k), t_k]$ and

$$\int_{D\cap[g_*(t_k),t_k]} [s - g_*(t_k)]^{n-1}\, p(s)\, ds = p \int_{g_*(t_k)}^{t_k} [s - g_*(t_k)]^{n-1}\, ds =$$

$$= \frac{p}{n} [t_k - g_*(t_k)]^n = \frac{p}{n}\,.$$

Thus, the condition (38) is satisfied for $p > n!$. If we choose $t_k = 2k\pi + \frac{1}{2}\pi$ $(k = 1, 2, \ldots)$, then we can prove by a similar argument as above that also the condition (39) is satisfied for $p > n!$. Therefore for $p > n!$ all solutions of (43) are oscillatory. The conditions (41) and (42) (cf. [9]) imply that every solution of (43) is oscillatory for $p \geqq (n-1)!\,(\sin 1 - \frac{1}{2})^{1-n}$. Finally, we remark that $(n-1)!\,(\sin 1 - \frac{1}{2})^{1-n} > n!$ for $n \geqq 2$.

References

[1] *Chanturia, T. A.:* Integral conditions of oscillation of solutions of higher-order linear differential equations, Differencialny'e Uravnenija *16* (1980), 470—481. (Russian)

[2] *Chiou, K. L.:* Oscillation and nonoscillation theorems for second order functional differential equations, J. Math. Anal. Appl. *45* (1974), 382—403.

[3] *Dahiya, R. S.:* Oscillation generating delay terms in even order retarded equations, J. Math. Anal. Appl. *49* (1975), 158—165.

[4] *Grammatikopoulos, M. K.:* On the effect of deviating arguments on the behavior of bounded solutions of nonlinear differential equations, Ukrain. Mat. Z. *30* (1978), 462—473. (Russian)

[5] *Ivanov, A. F., Kitamura, Y., Kusano, T.* and *Shevelo, V. N.:* Oscillatory solutions of functional differential equations generated by deviation of arguments of mixed type, Hiroshima Math. J. *12* (1982), 645—655.

[6] *Kiguradze, I. T.:* Some singular boundary value problems for ordinary differential equations, Tbilisi Univ. Press, Tbilisi, 1975. (Russian)

[7] *Koplatadze, R. G.* and *Chanturia, T. A.:* On oscillatory properties of differential equations with a deviating argument, Tbilisi Univ. Press, Tbilisi, 1977. (Russian)

[8] *Kusano, T.:* On strong oscillation of even order differential equations with advanced arguments, Hiroshima Math. J. *11* (1981), 617—620.

[9] *Kusano, T.:* Oscillation of even order linear functional differential equations with deviating arguments of mixed type, J. Math. Anal. Appl. *98* (1984), 341—347.

[10] *Ladas. G., Ladde. G.* and *Papadakis, J. S.:* Oscillations of functional-differential equations generated by delays, J. Differential Equations *12* (1972), 385—395.

[11] *Mahfoud, W. E.:* Remarks on some oscillation theorems for *n*th-order differential equations with a retarded argument, J. Math. Anal. Appl. *65* (1978), 68—80.

[12] *Marušiak, P.* and *Werbowski, J.:* Oscillation and asymptotic behavior of differential equations generated by delays, Fasc. Math. *13* (1981), 61—72.

[13] *Naito, M.:* Oscillations of differential inequalities with retarded arguments, Hiroshima Math. J. *5* (1975), 187—192.

[14] *Ohirska, J.:* Oscillation of *n*-th order linear and nonlinear delay differential equations, Czechoslovak Math. J. *32* (1982), 271—274.

[15] *Oláh, R.:* Note on the oscillation of differential equations with deviating argument. Časopis pěst. mat. *107* (1982), 380—387.

[16] *Oláh, R.:* On oscillation of solutions of nonlinear retarded differential equation of Emden-Fowler type, Czechoslovak Math. J. *34* (1984), 417—423.

[17] *Oláh, R.:* Oscillation of linear retarded differential equation, Czechoslovak Math. J. *34* (1984), 371—377.

[18] *Philos, Ch. G.:* Bounded oscillations generated by retardations for differential equations of arbitrary order, Utilitas Math. *15* (1979), 161—182.

[19] *Philos, Ch. G.:* Some results on the oscillatory and asymptotic behavior of the solutions of differential equations with deviating arguments, Arch. Math. *15* (1979), 19—36.

[20] *Sficas, Y. G.* and *Staikos, V. A.:* Oscillations of differential equations with deviating arguments, Funkcial. Ekvac. *19* (1976), 35—43.

[21] *Sficas, Y. G.* and *Staikos, V. A.:* Oscillations of retarded differential equations, Proc. Camb. Phil. Soc. *75* (1974), 95—101.

[22] *Sficas, Y. G.* and *Stavroulakis, I. P.:* On the oscillatory and asymptotic behavior of a class of differential equations with deviating arguments, SIAM J. Math. Anal. *9* (1978), 956—966.

[23] *Shevelo, V. N.* and *Vareh, N. V.:* On the oscillation of solutions of higher order linear differential equations with retarded argument, Ukrain. Mat. Ž. *24* (1972), 513—520. (Russian)

[24] *Staikos, V. A.:* Basic results on oscillation for differential equations with deviating arguments, Hiroshima Math. J. *10* (1980), 495—516.
[25] *Staikos, V. A.* and *Stavroulakis, I. P.:* Bounded oscillations under the effect of retardations for differential equations of arbitrary order, Proc. Royal Soc. Ed. *77* (1977), 129—136.
[26] *Stavroulakis, I. P.:* Some oscillatory and asymptotic properties of differential equations with deviating arguments, Bull. Math. Soc. Sci. Math. Roum. *22* (1978), 191—204.
[27] *Werbowski, J.:* On oscillation of solutions of differential inequallities with deviating arguments of mixed type, Trudy Inst. Pr. Mat. im. I. N. Vekua *14* (1983), 5—12. (Russian)
[28] *Werbowski, J.:* Oscillations of nonlinear differential equations caused by deviating arguments, Funkcial. Ekvac. *25* (1982), 295—301.

Author's address: Poznań Technical University, Institute of Mathematics, 60-965 Poznań, Poland.