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ON THE STIEFEL-WHITNEY CLASSES AND THE SPAN OF REAL GRASSMANNIANS

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1. INTRODUCTION AND STATEMENT OF RESULTS

This paper is a contribution to the study of vector fields on the Grassmann manifold $G_{n,r}$ of r-planes in the real n-space. As vector fields we always consider continuous cross-sections of the tangent bundle.

Recall (Thomas [11]) that the span of a closed connected smooth manifold M is defined as the maximal number of linearly independent vector fields on M. Obviously, if span $M \geqq j$, then

$$w_{m-j+1}(M) = w_{m-j+2}(M) = \ldots = w_m(M) = 0$$

for Stiefel-Whitney classes of M, where $m = \dim M$.

This fact, producing an estimate span $M \leqq m - k$ provided $w_k(M) \neq 0$ for some k, motivates our interest in the Stiefel-Whitney classes of Grassmann manifolds.

Namely, it is known, see for instance [5], that

$$TG_{n,r} \oplus \gamma_{n,r} \otimes \gamma_{n,r} \cong n\gamma_{n,r} \tag{1.1}$$

where $TG_{n,r}$ denotes the tangent bundle and $\gamma_{n,r}$ the canonical r-plane bundle over $G_{n,r}$.

By Borel [4],

$$H^*(G_{n,r}; Z_2) \cong Z_2[w_1(\gamma_{n,r}), \ldots, w_r(\gamma_{n,r})]/J_{n,r} \tag{1.2}$$

for the cohomology algebra, where all the ideal $J_{n,r}$ is determined by the only equation:

$$(1 + w_1(\gamma_{n,r}) + \ldots + w_r(\gamma_{n,r}))(1 + \bar{w}_1(\gamma_{n,r}) + \ldots + \bar{w}_{n-r}(\gamma_{n,r})) = 1\,. \tag{1.3}$$

Here $\bar{w}_i(\gamma_{n,r})$ denotes the i-th dual Stiefel-Whitney class of $\gamma_{n,r}$ and, moreover, $J_{n,r}^{(k)} = 0$ for the k-th homogeneous component, if $k \leqq n - r$ (cf. [9], [3]).

Hence, if one overcomes difficulties arising when computing the Stiefel-Whitney classes of the tensor square $\gamma_{n,r} \otimes \gamma_{n,r}$ in terms of $w_i(\gamma_{n,r})$, $i = 1, \ldots, r$, then one can explicitly express $w_k(G_{n,r})$ in the same terms (dealing with $n\gamma_{n,r}$ is not hard) and, in addition, decide whether or not $w_k(G_{n,r}) \neq 0$.

This is, essentially, what has been done in [3], for $k \leqq 9$. The present paper provides an improvement of the method used there, making the induction in computing the dependence of $w_k(\gamma_{n,r} \otimes \gamma_{n,r})$ on $w_i(\gamma_{n,r})$, $i = 1, \ldots, r$, transparent. So, one can verify the following extension of [3, 1.1], where, as always in this paper, p_i $(i \geqq 0)$ means the i-th dyadic coefficient of the positive integer p:

(1.4) **Theorem.** *Let* w_i *abbreviate* $w_i(\gamma_{n,r}) \in H^i(G_{n,r}; Z_2)$. *Let* $a = n_1 + r_0$, $b = n_2 + r_1$, $c = n_1 + n_2 + r_1$, $d = n_2 + r_0 + r_1$, $e = n_3 + r_2$ and $f = n_4 + r_3$. *Then*

$$\begin{aligned} w_{10}(G_{n,r}) = {} & (1 + a)(br_1 + e)\, w_1^{10} + n_0((an_1 + r_1)\, b + e)\, w_1^8 w_2 + \\ & + n_0(1 + a)\, bw_1^6 w_4 + n_0 adw_1^4 w_2^3 + (1 + ac)\, w_1^4 w_3^2 + n_0(an_1 + b)\, w_1^4 w_6 + \\ & + (1 + a)(1 + b)\, w_1^2 w_2^4 + n_0(1 + a)\, w_1^2 w_8 + adw_1^6 w_2^2 + \\ & + n_0(1 + a)\, w_1^2 w_2 w_3^2 + n_0(1 + ar_0 + b)\, w_2^5 + n_0 aw_2 w_4^2 + n_0 aw_1^2 w_2^2 w_4 + \\ & + n_0 aw_3^2 w_4 + n_0 aw_2^2 w_6 + aw_1^2 w_4^2 + aw_5^2 + n_0 w_{10}\,; \end{aligned}$$

$$\begin{aligned} w_{12}(G_{n,r}) = {} & ((1 + an_1)\, br_1 + (c + n_1 r_0)\, e)\, w_1^{12} + n_0(1 + a)(br_1 + e)\, w_1^{10} w_2 + \\ & + (a(e + r_0) + cn_2 r_0 + dn_1 r_1)\, w_1^8 w_2^2 + n_0((an_1 + r_1)\, b + e)\, w_1^8 w_4 + \\ & + n_0 adw_1^6 w_2^3 + (1 + a)(1 + b)\, w_1^6 w_3^2 + n_0(1 + a)\, bw_1^6 w_6 + \\ & + (1 + a)(1 + b)\, w_1^4 w_2^4 + n_0 adw_1^4 w_2^2 w_4 + n_0(1 + ac)\, w_1^4 w_2 w_3^2 + \\ & + (1 + ad)\, w_1^4 w_4^2 + n_0(an_1 + b)\, w_1^4 w_8 + n_0(1 + a)(1 + b)\, w_1^2 w_2^5 + \\ & + acw_2^6 + n_0(1 + ar_0 + b)\, w_2^4 w_4 + n_0 aw_2^2 w_8 + n_0(1 + a)\, w_1^2 w_3^2 w_4 + \\ & + n_0 aw_4^3 + (1 + a)\, w_1^2 w_2^2 w_3^2 + (1 + a)\, w_1^2 w_5^2 + n_0(1 + a)\, w_1^2 w_{10} + \\ & + (1 + ar_0 + b)\, w_3^4 + n_0 aw_1^2 w_2^2 w_6 + n_0 aw_3^2 w_6 + n_0 aw_1^2 w_2 w_4^2 + \\ & + n_0 aw_2 w_5^2 + aw_6^2 + n_0 w_{12}\,; \end{aligned}$$

$$\begin{aligned} w_{14}(G_{n,r}) = {} & (1 + a)\, b(e + r_1)\, w_1^{14} + n_0(an_1 + b)(br_1 + e)\, w_1^{12} w_2 + \\ & + n_0(1 + a)(br_1 + e)\, w_1^{10} w_4 + ((1 + a)(1 + b) + (a + bn_2)\, r_0 + \\ & + a(e + n_1 r_1) + (1 + d)\, n_1 n_2)\, w_1^8 w_3^2 + n_0((an_1 + r_1)\, b + e)\, w_1^8 w_6 + \\ & + (1 + a)(1 + b)\, w_1^6 w_2^4 + n_0(1 + a)(1 + b)\, w_1^6 w_2 w_3^2 + acw_1^6 w_4^2 + \\ & + n_0(1 + a)\, bw_1^6 w_8 + n_0(1 + a)(1 + b)\, w_1^4 w_2^5 + \\ & + n_0(1 + a)(1 + b)\, w_1^2 w_2^4 w_4 + n_0(1 + ac)\, w_1^4 w_3^2 w_4 + acw_1^4 w_5^2 + \\ & + n_0(an_1 + b)\, w_1^4 w_{10} + (1 + a)\, bw_1^2 w_3^4 + n_0(1 + a)\, w_1^2 w_{12} + \\ & + a(b(d + n_2) + e + r_0)\, w_1^{10} w_2^2 + n_0 a(b(d + n_2) + e + r_0)\, w_1^8 w_2^3 + \\ & + n_0(1 + ad)\, w_1^4 w_2 w_4^2 + acw_1^2 w_2^6 + n_0 acw_2^7 + n_0(1 + a)\, w_1^2 w_2^3 w_3^2 + \\ & + n_0(1 + a)\, w_1^2 w_2 w_5^2 + n_0(1 + ar_0 + b)\, w_2 w_3^4 + n_0 aw_2 w_6^2 + \\ & + n_0 adw_1^6 w_2^2 w_4 + n_0 adw_1^4 w_2^2 w_6 + acw_2^4 w_3^2 + n_0(1 + ar_0 + b)\, w_2^4 w_6 + \\ & + n_0 aw_1^2 w_2^2 w_8 + n_0 aw_3^2 w_8 + n_0 aw_2^2 w_{10} + n_0(1 + a)\, w_1^2 w_3^2 w_6 + \\ & + n_0 aw_1^2 w_4^3 + n_0 aw_4 w_5^2 + n_0 aw_4^2 w_6 + aw_1^2 w_6^2 + aw_7^2 + n_0 w_{14}\,; \end{aligned}$$

$$\begin{aligned} w_{16}(G_{n,r}) = {} & ((an_1 + r_1)\, b(e + r_1) + er_2 + f)\, w_1^{16} + n_0(1 + a)\, b(e + r_1)\, w_1^{14} w_2 + \\ & + n_0(an_1 + b)(br_1 + e)\, w_1^{12} w_4 + (1 + a)(1 + bn_2 + e)\, w_1^{10} w_3^2 + \\ & + n_0(1 + a)(br_1 + e)\, w_1^{10} w_6 + (1 + a(1 + er_0) + \end{aligned}$$

$$
\begin{aligned}
&+ b(c + n_2r_0 + n_1r_0r_1) + (1 + b)\,e)\,w_1^8w_2^4 + n_0(1 + a(e + n_1) + \\
&+ ab(n_1 + r_1) + b)\,w_1^8w_2w_3^2 + (1 + a(e + n_1r_1) + b + \\
&+ (cr_0 + n_1r_1)\,n_2)\,w_1^8w_4^2 + n_0((an_1 + r_1)\,b + e)\,w_1^8w_8 + \\
&+ n_0(1 + a)(1 + b)\,w_1^6w_2^5 + n_0(1 + a)(1 + b)\,w_1^4w_2^4w_4 + \\
&+ n_0(1 + a)(1 + b)\,w_1^6w_3^2w_4 + (1 + a)\,bw_1^6w_5^2 + n_0(1 + a)\,bw_1^6w_{10} + \\
&+ (1 + (1 + a)(1 + b))\,w_1^4w_3^4 + n_0(an_1 + b)\,w_1^4w_{12} + (a(ed + r_0) + \\
&+ b(n_1n_2r_0 + n_1r_1 + r_0))\,w_1^{12}w_2^2 + n_0(a(e + r_0) + \\
&+ b(n_1r_0 + n_2r_0 + n_1r_1))\,w_1^{10}w_2^3 + n_0acw_1^6w_2w_4^2 + acw_1^4w_2^6 + \\
&+ n_0acw_1^2w_2^7 + n_0acw_1^4w_2w_5^2 + n_0b(1 + a)\,w_1^2w_2w_3^4 + n_0aw_1^2w_2w_6^2 + \\
&+ n_0(a(e + r_0) + b(n_1r_1 + n_1r_0 + n_2r_0))\,w_1^8w_2^2w_4 + n_0adw_1^6w_2^2w_6 + \\
&+ (1 + a)\,bw_1^2w_2^4w_3^2 + n_0(1 + a)(1 + b)\,w_1^2w_2^4w_6 + n_0adw_1^4w_2^2w_8 + \\
&+ n_0(1 + a)\,w_1^2w_3^2w_8 + n_0aw_1^2w_2^2w_{10} + n_0(1 + ac)\,w_1^4w_3^2w_6 + \\
&+ n_0(1 + ad)\,w_1^4w_4^3 + n_0(1 + a)\,w_1^2w_4w_5^2 + n_0aw_1^2w_4^2w_6 + adw_1^4w_6^2 + \\
&+ (1 + a)\,w_1^2w_7^2 + n_0(1 + a)\,w_1^2w_{14} + (1 + a)(1 + b)\,w_1^6w_2^2w_3^2 + \\
&+ aw_1^4w_2^2w_4^2 + ((ar_0 + n_2)\,b + e)\,w_2^8 + n_0acw_2^6w_4 + \\
&+ n_0(1 + ar_0 + b)\,w_2^4w_8 + n_0(1 + a)\,w_1^2w_2^2w_3^2w_4 + aw_1^2w_2^2w_5^2 + acw_2^2w_3^4 + \\
&+ n_0aw_2^2w_{12} + n_0acw_2^5w_3^2 + n_0(1 + ar_0 + b)\,w_3^4w_4 + aw_1^2w_3^2w_4^2 + w_3^2w_5^2 + \\
&+ n_0aw_3^2w_{10} + acw_2^4w_4^2 + (ar_0 + b)\,w_4^4 + n_0aw_4^2w_8 + n_0aw_5^2w_6 + \\
&+ n_0aw_4w_6^2 + n_0aw_2w_7^2 + aw_8^2 + n_0w_{16}\,.
\end{aligned}
$$

Since the formulae are very long, we have omitted those for $w_k(G_{n,r})$ with odd values of k. Nevertheless, one can close the gaps easily, if required.

Namely, by the classical formula of Wu (e.g. Borel [4, 7.1]), if $i \leqq j$, then

$$
Sq^i(w_j(\xi)) = \sum_{k=0}^{i} \binom{j - i + k - 1}{k} w_{i-k}(\xi)\, w_{j+k}(\xi) \tag{1.5}
$$

for an arbitrary vector bundle ξ (Sq^i being the i-th Steenrod square,

$$
\binom{u}{v} = u!/(u - v)!\, v!)\,.
$$

Therefore we have

$$
w_k(G_{n,r}) = w_1(G_{n,r})\, w_{k-1}(G_{n,r}) + Sq^1(w_{k-1}(G_{n,r}))
$$

whenever k is odd. Hence, keeping in mind that ([3])

$$
w_1(G_{n,r}) = n_0 w_1\,,
$$

one can compute the omitted formulae without difficulties.

Moreover, using elementary facts about the binomial coefficients, including

$$
\binom{u}{v} \equiv \prod_i \binom{u_i}{v_i} \bmod 2\,, \tag{1.6}
$$

it is easy to verify the following assertion:

(1.7) *For a positive even integer x there exist even integers $m > n > 0$ such that $m + n = x$ and* $\binom{m-1}{n} \equiv 1 \bmod 2$ *iff x is not a power of* 2.

This implies (cf. (1.5)) that for an arbitrary $w_k(G_{n,r})$ with $2^i < k < 2^{i+1}$, the formula can be derived from those for $w_j(G_{n,r})$ with $j \leqq 2^i$.

In particular, the formulae for $w_k(G_{n,r})$, $k = 10, 12, 14$ could be computed using only Steenrod squares and the knowledge already contained in [3]. Nevertheless, we have prefered to give them here not only for reader's convenience but also for future references.

More specifically speaking, one can observe, for instance, that w_i^2 has the coefficient $n_1 + r_0 \bmod 2$ in $w_{2i}(G_{n,r})$ $i = 5, 6, 7, 8$. One can suspect that this will be the case for $i = 9, 10$ etc., as well.

As a matter of fact, this property can be verified. A research of phenomena of this kind was initiated in [7], and we hope that the formulae for $w_k(G_{n,r})$, $k = 10, 12, 14$, will be useful in its development.

Finally, supposing $n \geqq 2r$ (which is not restrictive because $G_{n,r}$ and $G_{n,n-r}$ are diffeomorphic) we obtain from Theorem 1.4 the following

(1.8) **Corollary.** *If $n \geqq 10$ is even and $r \geqq 3$ is odd then*

$$\text{span } G_{n,r} \leqq r(n - r) - 16$$

with the exception of

$$\text{span } G_{10,3} \leqq 7 .$$

Recall that $G_{n,1}$ is the projective space (therefore span $G_{n,1}$ = span S^{n-1}; cf. [1] for its values) and that span $G_{n,r} = 0$ if dim $G_{n,r} = r(n - r)$ is even. Moreover, span $G_{6,3} \leqq 7$ and span $G_{8,3} = 7$ (cf. [3]).

We also remark that the minimal from the two upper bounds should be always taken: one given by 1.8, the other by [7], when estimating the span of some grassmannian. For lower bounds we refer to [8] or [6].

Concluding this section we observe that although our method has produced the best known upper bounds for the span of grassmannians, its disadvantage is considerable. Namely, briefly speaking, the higher we go, the longer and more complicated all the computations involved become.

Hence, in this way we can enrich our knowledge to some extent, but must remember that we stay still very far from achieving the final, general solution of the vector fields problem on $G_{n,r}$'s (if such is possible at all), unless some new, intensive approach appears.

We postpone the proof of Theorem 1.4, proving first its corollary.

(2.1) Proof of (1.8). As we have mentioned already, $J^{(k)}_{n,r} = 0$ for $k \leqq n - r$, in (1.2). From 1.4 we read that the coefficients of $w_1^4 w_3^4$ and $w_1^6 w_2^2 w_3^2$ in $w_{16}(G_{n,r})$ are always different mod 2. This yields that $w_{16}(G_{n,r}) \neq 0$ and therefore span $G_{n,r} \leqq$ $\leqq r(n - r) - 16$, when $16 < n - r$.

To make the proof complete, we are left with 26 cases where $n - r < 16$.

Thanks to (1.3), we can find all generators of $J^{(16)}_{n,r}$ and also decide whether $w_{16}(G_{n,r}) \in J^{(16)}_{n,r}$ or not.

To facilitate this task, it is useful to recall that if

$$i_1 : G_{n,r} \hookrightarrow G_{n+1,r}$$

$$i_2 : G_{n+1,r} \hookrightarrow G_{n+2,r+1}$$

are the usual inclusions, then

$$i^*(w_k(G_{n+2,r+1})) = w_k(G_{n,r})$$

for $i = i_2 \circ i_1$. Hence, for example,

$$w_k(G_{n,r}) \neq 0 \quad \text{implies} \quad w_k(G_{n+4,r+2}) \neq 0 .$$

It turns out that from all the cases, only for $G_{10,3}$ the 16-th class vanishes. Fortunately, in a similar way one checks that $w_{14}(G_{10,3}) \neq 0$.

Besides some other facts, we shall need the following two lemmas for the proof of Theorem 1.4.

(2.2) **Lemma.** *Let η be an r-plane bundle over a paracompact space M. Let us abbreviate the Stiefel-Whitney class $w_i(\eta) \in H^i(M; Z_2)$ to w_i. Then:*

$$\begin{aligned} w_{10}(\eta \otimes \eta) = {} & (1 + r_0)(1 + r_1 + r_2)\, w_1^{10} + (1 + r_0 r_1)\, w_1^4 w_3^2 + \\ & + (1 + r_0)(1 + r_1)\, w_1^2 w_2^4 + r_0(1 + r_1)\, w_1^6 w_2^2 + r_0 w_1^2 w_4^2 + r_0 w_5^2 ; \end{aligned}$$

$$\begin{aligned} w_{12}(\eta \otimes \eta) = {} & ((1 + r_0)(1 + r_1) + (1 + r_0 + r_1) r_2)\, w_1^{12} + r_1(1 + r_0)\, w_1^6 w_3^2 + \\ & + r_0(1 + r_1)\, w_1^4 w_2^4 + r_0(1 + r_1 + r_2)\, w_1^8 w_2^2 + \\ & + (1 + r_0(1 + r_1))\, w_1^4 w_4^2 + r_0(1 + r_1)\, w_2^6 + (1 + r_0)\, w_1^2 w_2^2 w_3^2 + \\ & + (1 + r_0)\, w_1^2 w_5^2 + (1 + r_1)\, w_3^4 + r_0 w_6^2 ; \end{aligned}$$

$$\begin{aligned} w_{14}(\eta \otimes \eta) = {} & (1 + r_0)(1 + r_1)(1 + r_2)\, w_1^{14} + (r_0(1 + r_2) + r_1)\, w_1^8 w_3^2 + \\ & + r_0 r_1 w_1^6 w_4^2 + r_0 r_1 w_1^4 w_5^2 + r_1(1 + r_0)\, w_1^2 w_3^4 + \\ & + r_0(1 + r_1 + r_2)\, w_1^{10} w_2^2 + r_0(1 + r_1)\, w_1^2 w_2^6 + r_0(1 + r_1)\, w_2^4 w_3^2 + \\ & + r_0 w_1^2 w_6^2 + r_0 w_7^2 ; \end{aligned}$$

$$\begin{aligned} w_{16}(\eta \otimes \eta) = {} & ((1 + r_0)(1 + r_1)(1 + r_2) + r_3)\, w_1^{16} + \\ & + r_0(1 + r_1)(1 + r_2)\, w_1^{12} w_2^2 + (1 + r_1)(1 + r_2)\, w_1^8 w_2^4 + \\ & + (1 + r_0)(1 + r_2)\, w_1^{10} w_3^2 + r_1(1 + r_0)\, w_1^6 w_2^2 w_3^2 + (r_0(1 + r_2) + \end{aligned}$$

$$+ r_1(1 + r_0)) w_1^8 w_4^2 + (1 + r_1 + r_2) w_2^8 + r_1(1 + r_0) w_1^2 w_2^4 w_3^2 +$$
$$+ r_0(1 + r_1) w_1^4 w_3^4 + r_0 w_1^4 w_2^2 w_4^2 + (1 + r_0)(1 + r_1) w_1^6 w_5^2 +$$
$$+ r_0(1 + r_1) w_2^2 w_3^4 + r_0(1 + r_1) w_2^4 w_4^2 + r_0 w_1^2 w_3^2 w_4^2 + r_0 w_1^2 w_2^2 w_5^2 +$$
$$+ r_0(1 + r_1) w_1^4 w_6^2 + r_1 w_4^4 + w_3^2 w_5^2 + (1 + r_0) w_1^2 w_7^2 + r_0 w_8^2 .$$

(2.3) **Lemma.** *With the notation of (2.2) we have for the n-fold Whitney sum $n\eta$:*

$$\begin{aligned} w_{16}(n\eta) = {} & n_0 w_{16} + n_0 n_1 w_1^2 w_{14} + n_0 n_1 w_2^2 w_{12} + n_0 n_2 w_1^4 w_{12} + n_0 n_1 w_3^2 w_{10} + \\ & + n_0 n_1 n_2 w_1^6 w_{10} + n_1 w_8^2 + n_0 n_1 w_4^2 w_8 + n_0 n_2 w_2^4 w_8 + n_0 n_1 n_2 w_1^4 w_2^2 w_8 + \\ & + n_0 n_3 w_1^8 w_8 + n_1 n_2 w_1^4 w_6^2 + n_0 n_1 w_5^2 w_6 + n_0 n_1 n_2 w_1^4 w_3^2 w_6 + \\ & + n_0 n_1 n_2 w_1^2 w_2^4 w_6 + n_0 n_1 n_3 w_1^{10} w_6 + n_0 n_1 w_4 w_6^2 + n_2 w_4^4 + \\ & + n_0 n_1 n_2 w_1^4 w_4^3 + n_1 n_2 w_2^4 w_4^2 + n_1 n_3 w_1^8 w_4^2 + n_0 n_2 w_3^4 w_4 + \\ & + n_0 n_1 n_2 w_2^6 w_4 + n_0 n_1 n_3 w_1^8 w_2^2 w_4 + n_0 n_2 n_3 w_1^{12} w_4 + n_0 n_1 w_2 w_7^2 + \\ & + n_0 n_1 n_2 w_1^4 w_2 w_5^2 + n_0 n_1 n_2 w_2^5 w_3^2 + n_0 n_1 n_3 w_1^8 w_2 w_3^2 + n_1 n_2 w_2^2 w_3^4 + \\ & + n_3 w_2^8 + n_2 n_3 w_1^8 w_2^4 + n_1 n_2 n_3 w_1^{12} w_2^2 + n_0 n_1 n_2 w_1^2 w_2 w_3^4 + \\ & + n_0 n_1 n_2 n_3 w_1^{14} w_2 + n_4 w_1^{16} . \end{aligned}$$

Assuming (2.2) and (2.3) we are able to prove (1.4).

(2.4) Proof of Theorem 1.4. By (1.5), we have the following Wu's formulae for an arbitrary vector bundle ξ:

$$w_{10}(\xi) = w_2(\xi)\, w_8(\xi) + Sq^2(w_8(\xi)), \tag{2.5}$$

$$w_{12}(\xi) = w_4(\xi)\, w_8(\xi) + Sq^4(w_8(\xi)), \tag{2.6}$$

$$w_{14}(\xi) = w_2(\xi)\, w_{12}(\xi) + Sq^2(w_{12}(\xi)). \tag{2.7}$$

By [3, 1.1], we know $w_k(G_{n,r})$ for $k = 2, 4, 8$. Hence, we can compute the formulae for $w_k(G_{n,r})$, $k = 10, 12, 14$, putting $\xi = TG_{n,r}$ and using only elementary properties of Steenrod squares.

Having done this, we can compute $w_{16}(G_{n,r})$ as well (of course, in another way: cf. (1.7)).

Namely, recalling that odd-dimensional Stiefel-Whitney classes of $\gamma_{n,r} \otimes \gamma_{n,r}$ vanish (cf. [3, 2.1]), we obtain from Hsiang and Szczarba's relation (1.1):

$$w_{16}(G_{n,r}) = \sum_{i=1}^{8} w_{16-2i}(G_{n,r})\, w_{2i}(\gamma_{n,r} \otimes \gamma_{n,r}) + w_{16}(n\gamma_{n,r}) .$$

Clearly [3, 11.1, 2.1], Lemma 2.2 and Lemma 2.3 now provide all the information needed.

As the proof of Lemma 2.3 is very easy (therefore omitted), all that now remains is to prove (2.2).

(2.8) Proof of Lemma 2.2. Let $w(\eta \otimes \eta)$ denote the total Stiefel-Whitney class of the tensor square $\eta \otimes \eta$, and $\sigma_1, \ldots, \sigma_r$ the elementary symmetric functions in variables $x_1, \ldots, x_r$. Then (cf. [10])

$$w(\eta \otimes \eta) = \Phi_r(w_1, \ldots, w_r), \tag{2.9}$$

where Φ_r is the only element in $Z_2[x_1, \ldots, x_r]$ such that

$$\Phi_r(\sigma_1, \ldots, \sigma_r) = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}} (1 + x_i + x_j). \tag{2.10}$$

This makes it clear that our ability to express Stiefel-Whitney classes of $\eta \otimes \eta$ in terms of $w_1, \ldots, w_r$ is determined by our knowledge of the polynomial Φ_r.

However, (2.10) obviously yields

$$\Phi_r(\sigma_1, \ldots, \sigma_r) = 1 + \bar{\sigma}_1^2 + \ldots + \bar{\sigma}_{\binom{r}{2}}^2, \tag{2.11}$$

where $\bar{\sigma}_k$ denotes the k-th elementary symmetric function in variables $x_i + x_j$, $i < j$.

Since each $\bar{\sigma}_k$ can be expressed in a unique way as a polynomial in $\sigma_1, \ldots, \sigma_k$, our strategy is straightforward.

As a matter of fact, an improved (as compared with [3]) alrogorithm for computing $\bar{\sigma}_k$ in terms of $\sigma_1, \ldots, \sigma_k$ mod 2 now follows.

(2.12) Algorithm for computing $\bar{\sigma}_p$ provided that we have already computed $\bar{\sigma}_k$ for $k < p$.

The procedure will have three steps.

(2.13) Step 1. Let

$$\mathcal{M}_p = \{\sigma_1^{i(1)} \ldots \sigma_p^{i(p)};\ i(1) + 2\,i(2) + \ldots + p\,i(p) = p\}.$$

We define the *leading monomial of* $\sigma_1^{i(1)} \ldots \sigma_p^{i(p)} \in \mathcal{M}_p$ to be

$$\operatorname{lm}(\sigma_1^{i(1)} \ldots \sigma_p^{i(p)}) := x_1^{i(1)+\ldots+i(p)} x_2^{i(2)+\ldots+i(p)} \ldots x_p^{i(p)}.$$

Further, we order the set

$$\{\operatorname{lm}(\sigma_1^{i(1)} \ldots \sigma_p^{i(p)});\ \sigma_1^{i(1)} \ldots \sigma_p^{i(p)} \in \mathcal{M}_p\}$$

by the rule

$$x_1^{s(1)} \ldots x_p^{s(p)} < x_1^{t(1)} \ldots x_p^{t(p)} \quad \text{if} \tag{R_p}$$

$$(s(1), \ldots, s(p)) < (t(1), \ldots, t(p)) \quad \text{in} \quad \text{NLO},$$

where NLO is an abbreviation for the natural lexicographical order.

Finally, we order $\mathcal{M}_p$ as follows:

$$\sigma_1^{i(1)} \ldots \sigma_p^{i(p)} < \sigma_1^{j(1)} \ldots \sigma_p^{j(p)} \quad \text{if}$$

$$\operatorname{lm}(\sigma_1^{j(1)} \ldots \sigma_p^{j(p)}) < \operatorname{lm}(\sigma_1^{i(1)} \ldots \sigma_p^{i(p)}).$$

This completes the first step of the procedure. The set $(\mathcal{M}_p, <)$ has the following very important property:

if we interpret $\sigma_1^{i(1)} \ldots \sigma_p^{i(p)} \in \mathcal{M}_p$ as an element of $Z_2[x_1, \ldots, x_r]$, then the coefficient of

$$\operatorname{lm}(\sigma_1^{i(1)} \dots \sigma_p^{i(p)}) \quad \text{in} \quad \sigma_1^{j(1)} \dots \sigma_p^{j(p)},$$
$$\operatorname{lm}(\sigma_1^{i(1)} \dots \sigma_p^{i(p)}) \dashv \sigma_1^{j(1)} \dots \sigma_p^{j(p)}$$

for short, can be 1 only when

$$\sigma_1^{j(1)} \dots \sigma_p^{j(p)} \leqq \sigma_1^{i(1)} \dots \sigma_p^{i(p)} \tag{2.14}$$

in the set $(\mathcal{M}_p, <)$. This is easily implied by the definition of the order in $\mathcal{M}_p$ and by the simple fact that $\operatorname{lm}(\sigma_1^{i(1)} \dots \sigma_p^{i(p)})$ is the greatest element in the set

$$\{x_1^{s(1)} \dots x_r^{s(r)};\ (x_1^{s(1)} \dots x_r^{s(r)} \dashv \sigma_1^{i(1)} \dots \sigma_p^{i(p)}) = 1\,,$$

ordered by the rule (R_r).

(2.15) Examples.

$p = 1$: $\mathcal{M}_1 = \{\sigma_1\}$.

$p = 2$: $\mathcal{M}_2 = \{\sigma_1^2, \sigma_2\}$.

Since $\operatorname{lm}(\sigma_1^2) = x_1^2$ and $\operatorname{lm}(\sigma_2) = x_1x_2$, we have $\operatorname{lm}(\sigma_2) < \operatorname{lm}(\sigma_1^2)$ and therefore

$$(\mathcal{M}_2, <) = \{\sigma_1^2 < \sigma_2\}\,.$$

$p = 3$: $\mathcal{M}_3 = \{\sigma_1^3, \sigma_1\sigma_2, \sigma_3\}$.

Now, $\operatorname{lm}(\sigma_1^3) = x_1^3$, $\operatorname{lm}(\sigma_1\sigma_2) = x_1^2x_2$ and $\operatorname{lm}(\sigma_3) = x_1x_2x_3$. Hence

$$(\mathcal{M}_3, <) = \{\sigma_1^3 < \sigma_1\sigma_2 < \sigma_3\}\,.$$

We observe that the number of all elements in $\mathcal{M}_p$ is part (p), the number of partitions of p. For instance (cf. [2]):

p	1	2	3	4	5	6	7	8	9	10	11	...
part (p)	1	2	3	5	7	11	15	22	30	42	55	...

(2.16) Step 2. Let us search for the number

$$x_1^{s(1)} \dots x_h^{s(h)} \dashv \bar{\sigma}_p \tag{2.17}$$

with $s(1) \geqq \dots \geqq s(h) > 0$, $\sum_{i=1}^{h} s(i) = p$.

To begin with, let us denote

$$\mathcal{S}_r = \{x_i + x_j;\ 1 \leqq i < j \leqq r\}\,,$$
$$\mathcal{L}_h = \{x_1 + x_j;\ 2 \leqq j \leqq h\}\,,$$

both considered as naturally lexicographically ordered, and

$$\mathcal{G}_h = \{x_1^{g(1)} \dots x_h^{g(h)};\ (x_1^{g(1)} \dots x_h^{g(h)} \dashv \prod_{j=2}^{h} (1 + x_1 + x_j)) = 1 \bmod 2\,,$$

and $s(i) - g(i) \geqq 0$ for $i = 1, \dots, h\}$.

By definition, $\bar{\sigma}_p$ is the sum of all products of the form

$$\prod_{k=1}^{p} (x_{i(k)} + x_{j(k)}),$$

where $\{x_{i(k)} + x_{j(k)}\}_{k=1}^{p}$ is an increasing sequence in $\mathscr{S}_r$.

This fact, a little thinking of the "list"

$$\underbrace{x_1 + x_2, \ldots, x_1 + x_h}_{\mathscr{L}_h}, x_1 + x_{h+1}, \ldots, x_1 + x_r \mid x_2 + x_3, \ldots, x_2 + x_r, \ldots, x_{r-1} + x_r$$

with realizing how the elements of the set $\mathscr{G}_h$ are related with $\mathscr{L}_h$, clarify the following result:

$$(2.18) \qquad (x_1^p \dashv \bar{\sigma}_p) = \binom{r-1}{p} \bmod 2, \quad \text{and}$$

$$(x_1^{s(1)} \ldots x_h^{s(h)} \dashv \bar{\sigma}_p) = \sum \binom{r-h}{s(1) - g(1)} N_{s(2)-g(2),\ldots,s(h)-g(h)} \bmod 2$$

if $h \geqq 2$.

Here the sum is taken over all h-tuples $(g(1), \ldots, g(h))$ such that $x_1^{g(1)} \ldots x_h^{g(h)} \in \mathscr{G}_h$, and $N_{s(2)-g(2),\ldots,s(h)-g(h)} \in Z_2$ is

$$x_2^{s(2)-g(2)} \ldots x_h^{s(h)-g(h)} \dashv \bar{\sigma}_t(x_2 + x_3, \ldots, x_2 + x_r; \ldots; x_{r-1} + x_r)$$

with $t = \sum_{i=2}^{h} (s(i) - g(i))$.

It is clear, however, that $N_{s(2)-g(2),\ldots,s(h)-g(h)}$ coincides with

$$x_1^{s(2)-g(2)} \ldots x_{h-1}^{s(h)-g(h)} \dashv \bar{\sigma}_t(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1}).$$

So an induction can come in, finally. Indeed, since $t < p$ and therefore we have

$$\bar{\sigma}_t(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1})$$

in terms of $\sigma_i(x_1, \ldots, x_{r-1})$, $i = 1, \ldots, t$, already computed (cf. 2.12), we are able to find all the numbers $N_{s(2)-g(2),\ldots,s(h)-g(h)}$.

Taking successively all the leading monomials $\mathrm{lm}(\sigma_1^{i(1)} \ldots \sigma_p^{i(p)})$, $\sigma_1^{i(1)} \ldots \sigma_p^{i(p)} \in \bar{\mathscr{M}}_p$ for $x_1^{s(1)} \ldots x_h^{s(h)}$ in (2.17), we accomplish the second step of our algorithm.

We just note that the binomial coefficients in (2.18) are easily expressible in terms of dyadic coefficients, using (1.6), and that it is very useful to remember that

$$\prod_{j=2}^{h} (1 + x_1 + x_j) = \sum_{i=1}^{h} (1 + x_1)^{h-i} \sigma_{i-1}(x_2, \ldots, x_h),$$

when forming the set $\mathscr{G}_h$.

(2.19) Step 3. For a while let us denote by A_k the k-th element of the ordered set $(\bar{\mathscr{M}}_p, <)$. Then, of course,

$$(2.20) \qquad \bar{\sigma}_p = a(1) A_1 + a(2) A_2 + \ldots + a(\text{part}(p)) A_{\text{part}(p)}$$

for some $a(k) \in Z_2$, and our final aim is to find all these $a(k)$. We are ready to do this.

Namely, for any fixed A_k it is easy to find all A_j with the property

$$(\mathrm{lm}\,(A_k) \dashv A_j) = 1\,. \tag{2.21}$$

Recall that $(\mathrm{lm}\,(A_k) \dashv A_k) = 1$, and that all candidates for (2.21) have to satisfy $A_j \leqq A_k$ (cf. (2.14)). Hence, from (2.20) we get a linear equation over Z_2, where the left-hand side is the sum of $a(k)$ and some $a(i)$'s, $i < k$, while the right-hand side is the number $\mathrm{lm}\,(A_k) \dashv \bar{\sigma}_p$, computed in Step 2.

So, finding such an equation for every $k = 1, 2, \ldots$, part (p), we obtain a very simple system of linear equations over Z_2 giving us all $a(k)$ as desired.

This completes the last step of our algorithm.

(2.22) Example. Say, we have

$$\bar{\sigma}_1 = (1 + r_0)\,\sigma_1\,, \tag{2.23}$$

$$\bar{\sigma}_2 = (1 + r_0 + r_1)\,\sigma_1^2 + r_0\sigma_2\,, \tag{2.24}$$

and we wish to compute $\bar{\sigma}_3$.

The first step was made in 2.15. Recall its result:

$$(\mathscr{M}_3, <) = \{\sigma_1^3 < \sigma_1\sigma_2 < \sigma_3\}\,.$$

Step 2.

a) $(x_1^3 \dashv \bar{\sigma}_3) = \binom{r-1}{3} = 1 + r_0 + r_1 + r_0 r_1 \mod 2.$

b) For the leading monomial $x_1^2 x_2$ of $\sigma_1\sigma_2$ we have

$$\mathscr{G}_2 = \{1, x_1, x_2\}\,.$$

So (2.18) gives

$$(x_1^2 x_2 \dashv \bar{\sigma}_3) = \binom{r-2}{2} N_1 + \binom{r-2}{1} N_1 + \binom{r-2}{2} N_0\,.$$

We find

$$N_0 = (1 \dashv \bar{\sigma}_0(x_1 + x_2, \ldots, x_{r-2} + x_{r-1})) = 1\,, \quad \text{and}$$

(cf. (2.23))

$$N_1 = (x_1 \dashv \bar{\sigma}_1(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1})) = r_0\,.$$

Therefore

$$(x_1^2 x_2 \dashv \bar{\sigma}_3) = 1 + r_1 + r_0 r_1 \mod 2\,.$$

c) For the leading monomial $x_1 x_2 x_3$ of σ_3 we get

$$\mathscr{G}_3 = \{1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3\}\,.$$

Now (2.18) reads

$$(x_1x_2x_3 \dashv \bar{\sigma}_3) = \binom{r-3}{1} N_{1,1} + \binom{r-3}{1} N_{0,1} + \binom{r-3}{1} N_{1,0} + \binom{r-3}{0} N_{0,1} +$$
$$+ \binom{r-3}{0} N_{1,0} + \binom{r-3}{1} N_{0,0}\,.$$

We find

$$N_{0,0} = (1 \dashv \bar{\sigma}_0(x_1 + x_2, \ldots, x_{r-2} + x_{r-1})) = 1\,,$$

$$N_{0,1} = N_{1,0} = (x_1 \dashv \bar{\sigma}_1(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1}) = r_0\,,$$

and (cf. (2.24))

$$N_{1,1} = (x_1 x_2 \dashv \bar{\sigma}_2(x_1 + x_2, \ldots, x_1 + x_{r-1}; \ldots; x_{r-2} + x_{r-1}) = 1 + r_0\,.$$

Hence we obtain

$$(x_1 x_2 x_3 \dashv \bar{\sigma}_3) = 0 \bmod 2\,.$$

Step 3. Writing

$$\bar{\sigma}_3 = a(1)\,\sigma_1^3 + a(2)\,\sigma_1\sigma_2 + a(3)\,\sigma_3$$

we get the system

$$a(1) = 1 + r_0 + r_1 + r_0 r_1\,,$$

$$a(1) + a(2) = 1 + r_1 + r_0 r_1\,,$$

$$a(2) + a(3) = 0\,.$$

Clearly, $\bar{\sigma}_3 = (1 + r_0)(1 + r_1)\,\sigma_1^3 + r_0\sigma_1\sigma_2 + r_0\sigma_3$.

Continuing these computations for $\bar{\sigma}_k$, $k = 4, 5, 6, 7, 8$, one is able to check Lemma 2.2.

We observe that Wu's formula (1.5) is also true for elementary symmetric functions (in arbitrary variables).

Hence, when k is not a power of 2, also the Steenrod squares techniques can be used in order to compute $\bar{\sigma}_k$.

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