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TOLERANCE DISTRIBUTIVE AND TOLERANCE MODULAR VARIETIES OF COMMUTATIVE SEMIGROUPS

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By a tolerance on a semigroup S we mean a reflexive and symmetric subsemigroup of the direct product $S \times S$. The set $LT(S)$ of all tolerances on S forms a complete algebraic lattice with respect to set inclusion (see [1] and [2]). A variety V of semigroups is called *tolerance distributive* (*modular*) if each S from V has distributive (modular) $LT(S)$ (see [3]).

In this paper we shall describe all varieties of commutative semigroups which are tolerance distributive or tolerance modular. Non defined terminology and notation can be found in [4] and [5].

Let S be a commutative semigroup. The notation S^1 stands for S if S has an identity, otherwise for S with an identity adjoined. By $\vee$ and $\wedge$ we denote the join and the meet in the lattice $LT(S)$ respectively.

Let $A, B \in LT(S)$. Clearly we have $A \wedge B = A \cap B$. It is easy to show that

$$(1) \qquad (x, y) \in A \vee B \quad \text{if and only if}$$

either $(x, y) \in A \cup B$ or $(x, y) = (x_1, y_1)(x_2, y_2)$, where $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$.

For $a, b \in S$ we denote by $T(a, b)$ the least tolerance on S containing (a, b), i.e. $T(a, b)$ is the principal tolerance on S generated by (a, b). If $a \neq b$, then for $x, y \in S$, $x \neq y$, we have

$$(2) \qquad (x, y) \in T(a, b) \quad \text{if and only if}$$

there exist $z \in S^1$ and a positive integer m such that either $(x, y) = (a, b)^m (z, z)$ or $(x, y) = (b, a)^m (z, z)$.

By $W(i_1 = i_2)$ we denote the variety of all commutative semigroups satisfying the identity $i_1 = i_2$.

Theorem 1. *A variety V of commutative semigroups is tolerance modular if and only if V is a subvariety of $W(xy = xyz^n)$ for a positive integer n.*

First, we shall prove the following lemmas:

Lemma 1. *For any positive integer n the variety $W(xy = xyz^n)$ is tolerance modular.*

Proof. Suppose that S is a semigroup from $W(xy = xyz^n)$ which is not tolerance modular. Then there exist $A, B, C \in LT(S)$ such that $A \subseteq C$ and $(A \vee B) \wedge C \neq A \vee (B \wedge C)$. Since $A \vee (B \wedge C) \subseteq (A \vee B) \wedge C$, there exists $(u, v) \in (A \vee B) \wedge C$ such that $(u, v) \notin A \vee (B \wedge C)$. By (1) we have $(u, v) = (p, q)(r, s)$, where $(p, q) \in A$ and $(r, s) \in B$. We have $S \in W(xy = xyz^n)$ and so S is periodic having exactly one idempotent (say e) in which S^2 is a maximal subgroup. Hence we obtain $w^{2n} = e$ for every $w \in S$. Using (2) we have $(er, es) = (p^{2n}r, q^{2n}s) = (p^{2n-1}u, q^{2n-1}v) = (p, q)^{2n-1}(u, v) \in C$ and $(er, es) = (e, e)(r, s) \in B$. It follows from (1) that $(u, v) = (p, q)(er, es) \in A \vee (B \wedge C)$, which is a contradiction.

Lemma 2. *Let $P = \{a, b, c, p, q, r, 0\}$ be a semigroup with the multiplication table*

	a	b	c
a	p	p	0
b	p	q	r
c	0	r	r

and $xy = 0 = yx$ for $x \in P$ and $y \in \{p, q, r, 0\}$. Then the lattice $LT(P)$ is not modular.

Proof. Clearly we have $xy = yx$ and $(xy)z = x(yz)$ for all $x, y, z \in P$. Put $A = T(a, b)$, $B = T(b, c)$ and $C = A \vee T(p, r)$. We have $A \subseteq C$ and so, by (1), $(p, r) = (a, b)(b, c) \in (A \vee B) \wedge C$. According to (1) and (2), it can be shown that $(p, r) \notin A \vee (B \wedge C)$. Therefore $LT(P)$ is not modular.

Lemma 3. *Let $Q = \{a, b, c, p, r, 0\}$ be a semigroup with the multiplication table*

	a	b	c
a	0	p	0
b	p	0	r
c	0	r	0

and $xy = 0 = yx$ for $x \in Q$ and $y \in \{p, r, 0\}$. Then the lattice $LT(Q)$ is not modular.

Proof. This can be proved by an argument analogous to that in the proof of Lemma 2.

Lemma 4. *Let V be a tolerance modular variety of commutative semigroups. If S is a semilattice from V, then S is trivial.*

Proof. This follows from Example 4 of [3].

Proof of Theorem 1. Let V be a tolerance modular variety of commutative semigroups. According to Lemma 1 it suffices to show that V is a subvariety of $W(xy = xyz^n)$ for a positive integer n.

1. Every semigroup from V is periodic.

Suppose that there exists a non-periodic element u in a semigroup S from V. By U we denote the subsemigroup of S generated by u. Clearly $U \in V$ and so the lattice $LT(U)$ is modular. Since U is cancellative, we have by Corollary 3 of [6] that U is a group, which is a contradiction.

2. There exists a positive integer n such that $V \subseteq W(x^n x^n = x^n)$.

Suppose that for any positive integer m there exists an element u_m in a semigroup S_m from V such that u_m^i is not idempotent for $i = 1, 2, \ldots, m$. Then the direct product $S = \mathsf{X}_{m=1}^{\infty} S_m$ is not periodic, but $S \in V$, a contradiction.

3. We have $V \subseteq W(x^n = y^n)$.

Let S be a semigroup from V. By $E(S)$ we denote the semilattice of all idempotents of S. Clearly $E(S) \in V$. Lemma 4 implies that card $E(S) = 1$.

4. We have $V \subseteq W(x^2 = x^2 x^n)$.

Suppose that there exists an element u belonging to a semigroup S from V such that $u^2 \neq u^2 u^n$. By U denote the subsemigroup of S generated by u and put $I = u^2 U$. We shall show that the Rees quotient $R = S/I$ has exactly three elements u, u^2 and 0. Indeed, if $u^2 \in I$, then $u^2 = u^2 u^m$ for a positive integer m and so $u^2 = u^2(u^m)^n = u^2(u^n)^m = u^2 u^n$, a contradiction. Therefore we have $u^2 \notin I$ and $u^2 \neq u \notin I$.

Now, we shall define a mapping $\varphi\colon P \to R \times R$, where P is the semigroup from Lemma 2. Let us put $\varphi(a) = (u, u^2)$, $\varphi(b) = (u, u)$, $\varphi(c) = (u^2, u)$, $\varphi(p) = (u^2, 0)$, $\varphi(q) = (u^2, u^2)$, $\varphi(r) = (0, u^2)$ and $\varphi(0) = (0, 0)$. It is easy to show that φ is an isomorphism. Since $R \times R \in V$, we have $P \in V$, which is a contradiction (see Lemma 2).

5. We have $V \subseteq W(xy = xyz^n)$.

Suppose that there exist elements u, v and w belonging to a semigroup S from V such that $uv \neq uvw^n$. By U we denote the subsemigroup of S generated by u and v. Let us put $I = eU$, where e is an idempotent of S. It follows from 2, 3 and 4 that $e \in U$, $u \neq v$ and $u^2, v^2 \in I$. We shall show that the Rees quotient $R = U/I$ has exactly four elements u, v, uv and 0. Indeed, if $uv \in I$, then $uv = es$ for some $s \in U$ and so $uv = euv = uvw^n$, a contradiction. Therefore we have $uv \notin I$ and $u, v \notin I$. If $u = uv$, then $u = uv^n = ue$, a contradiction. Consequently, we have $u \neq uv \neq v$.

Let us define a mapping $\varphi\colon Q \to R \times R$, where Q is the semigroup from Lemma 3. We put $\varphi(a) = (u, u)$, $\varphi(b) = (v, v)$, $\varphi(c) = (u, 0)$, $\varphi(p) = (uv, uv)$, $\varphi(r) = (uv, 0)$ and $\varphi(0) = (0, 0)$. Evidently φ is an isomorphism. We have $R \times R \in V$. This implies that $Q \in V$, which contradicts Lemma 3.

Theorem 2. *A non-trivial variety V of commutative semigroups is tolerance distributive if and only if V is the variety of all zero-semigroups.*

Proof. It is easy to show that the variety of all zero-semigroups is $W(xy = xyz) = W(x_1 y_1 = x_2 y_2)$. Evidently the lattice $LT(S)$ is distributive, whenever S is a zero-semigroup.

Let V be a non-trivial tolerance distributive variety of commutative semigroups. Suppose that $V \neq W(xy = xyz)$. It is well known that the variety of all zero-semigroups is minimal and so V is no subvariety of $W(xy = xyz)$. This and Theorem 1 imply

(3) $$V \subseteq W(xy = xyz^n)$$

for a positive integer $n \geqq 2$. It is easy to show that

(4) $$V \subseteq W(x^n x^n = x^n) \cap W(x^n = y^n) .$$

Since $V \nsubseteq W(xy = xyz)$, there are elements u, v and w belonging to a semigroup S from V such that $a = uv \neq uvw = b$. It follows from (4) that the semigroup S has exactly one idempotent (say e). Therefore we have either $a \neq e$ or $b \neq e$. Suppose that $a \neq e$ (without loss of generality). Let U denote the subsemigroup of S generated by a. According to (3) and (4), we have $a = a^{n+1}$. This means that U is a cyclic non-trivial finite subgroup of S. Therefore S contains a cyclic subgroup R of a primer order. Clearly $R \times R \in V$ and so the lattice $LT(R \times R)$ is distributive. It is well known (see [7]) that every tolerance on a commutative group is a congruence and thus, by Ore's Theorem [8], the group $R \times R$ is locally cyclic. Since $R \times R$ is finite, we obtain that $R \times R$ is cylic, which is a contradiction.

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