Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 3, 376–392

Persistent URL: http://dml.cz/dmlcz/102100

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

ENDOMORPHISMS OF PARTIAL MONOUNARY ALGEBRAS

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(Received November 5, 1984)

This paper deals with systems of partial monounary algebras which have the same underlying set and the same set of endomorphisms. It is proved that the result of [1] concerning (complete) mounounary algebras ([1], Thm. 5.1) can be generalized to partial monounary algebras. The main result of the present paper is Thm. 4.11. As a corollary we obtain a Ramsey-type theorem for systems of partial monounary algebras (Thm. 4.12).

Homomorphisms and endomorphisms of (complete) monounary algebras were investigated in [1], [4]–[7]; for the case of partial monounary algebras, cf. [3], [2].

1. PRELIMINARIES

Let $A \neq \emptyset$ be a set, $F = F(A)$ the system of all partial mappings of A into A, and let $F_0 = F_0(A)$ be the system of all mappings of the set A into A. If $f \in F$, then (A, f) is said to be a *partial monounary algebra*. For $f \in F$ we shall denote by D_f the set of all $x \in A$ such that $f(x)$ does not exist. If $f \in F_0$, then $D_f = \emptyset$ and (A, f) is called a *(complete) monounary algebra*.

Let N be the set of all positive integers. Suppose that (A, f) is a partial monounary algebra. For each $x \in A$ we put $f^0(x) = x$. Let $n \in N$. If $f^m(x)$ is defined for each $m \in N \cup \{0\}$, $m < n$, and if $f^{n-1}(x) \notin D_f$, then we put $f^n(x) = f(f^{n-1}(x))$. Further, denote $f^{-n}(x) = \{y \in A : f^n(y) = x\}$ for each $n \in N$. For $x, y \in A$ we shall write $x \equiv_f y$, if there exist $m, n \in N \cup \{0\}$ such that $f^n(x) = f^m(y)$. The relation $\equiv_f$ is an equivalence relation on A and the elements of the set $P_f = A/\equiv_f$ are called *connected components* of the partial monounary algebra (A, f). If $A/\equiv_f$ has one element, then we shall say that (A, f) is connected. The connected component containing the element $x \in A$ will be denoted by $K_f(x)$.

A mapping $H: A \to A$ is called an *endomorphism* of the partial monounary algebra (A, f) (cf. [3]), if the following relation is valid:

$$(\forall x \in A - D_f)\,(H(x) \in A - D_f \;\&\; H(f(x)) = f(H(x)))\,.$$

The system of all endomorphisms of (A, f) will be denoted by the symbol End (A, f).

A partial monounary algebra (A, f) is said to be of *type* τ or π, if there is $x \in A$ such that $f(y) = x$ for each $y \in A$, or $f(x) = x$ and $A - \{x\} = D_f$, respectively.

Let (A, f) be a partial monounary algebra. Put

$$Eq(f) = \{g \in F(A): \operatorname{End}(A, f) = \operatorname{End}(A, g)\},$$
$$Eq_0(f) = \{g \in F_0(A): \operatorname{End}(A, f) = \operatorname{End}(A, g)\}.$$

The set $Eq_0(f)$ for the case of the (complete) monounary algebra (A, f) was investigated in [1] and it was shown that the following assertions hold (Lemma 4, Thm. 5.1):

(A1) *If (A, f) is a (complete) monounary algebra and $g \in Eq_0(f)$, then $P_f = P_g$.*

(A2) *If (A, f) is a (complete) monounary algebra, then*

$$\operatorname{card} Eq_0(f) \leqq c$$

(independently of the cardinality of the set A) and this estimate is the best possible.

In the paper [2] the set $Eq(f)$ was studied and it was proved that

(B1) *if (A, f) is a partial monounary algebra which is neither of type τ nor of type π, then $P_f = P_g$ for each $g \in Eq(f)$.*

In the present paper the following theorem will be established (using (A2) and (B1)):

(B2) *If (A, f) is a partial monounary algebra, then*

$$\operatorname{card} Eq(f) \leqq c$$

(independently of the cardinality of the set A) and this estimate is the best possible.

The notion of degree $s_f(x)$ for $x \in A$ was introduced in [3]. We remark that s_f is a mapping of A into the union of the set $\{\infty_1, \infty_2\}$ with the class of all ordinals and $\alpha < \infty_1 < \infty_2$ for each ordinal α (for the thorough definition of s_f cf. [3]).

For a partial monounary algebra (A, f) we denote

$$A_2^f = \{x \in A: (\exists z \in D_f)(\exists n \in N \cup \{0\})(f^n(x) = z)\},$$
$$A_1^f = A - A_2^f.$$

Let $x, y \in A_2^f$, $n \in N$.

(a) We put $x\, ur_0^f\, y$, if $x = y$.

(b) We put $x\, ur_n^f\, y$, if either $f^{n-1}(x) \neq f^{n-1}(y)$, $f^n(x) = f^n(y)$, or $K_f(x) \neq K_f(y)$ and $\{f^{n-1}(x), f^{n-1}(y)\} \subseteq D_f$.

If there is $i \in N \cup \{0\}$ such that $x\, ur_i^f\, y$, then we shall write also $x\, ur^f\, y$. It is obvious that the relations ur_i^f for $i \in N \cup \{0\}$ are equivalence relations on A_2^f.

1.1. Lemma. *If $x\, ur_i^f\, y$ and $x\, ur_j^f\, y$, $x, y \in A_2^f$, $i, j \in N \cup \{0\}$, then $i = j$.*

Proof. First, let $K_f(x) \neq K_f(y)$. It follows from the definition that $\{f^{i-1}(x), f^{i-1}(y)\} \subseteq D_f$ and $\{f^{j-1}(x), f^{j-1}(y)\} \subseteq D_f$, thus $i = j$. Now assume that $K_f(x) = K_f(y)$. If $x = y$, the assertion is obvious. If $x \neq y$, then $f^{i-1}(x) \neq f^{i-1}(y)$ and $f^i(x) = f^i(y)$, and analogously for j, which yields that $i = j$.

1.2. Lemma. *The relation ur^f is an equivalence relation on A_2^f.*

Proof. The fact that ur^f is reflexive and symmetric is obvious. Now let $x\, ur^f\, y$, $y\, ur^f\, z$. There are $i, j \in N \cup \{0\}$ such that $x\, ur_i^f\, y$, $y\, ur_j^f\, z$. We shall assume that x, y, z are distinct elements (in the opposite case we evidently have $x\, ur^f\, z$). If $K_f(x) \neq K_f(y) \neq K_f(z)$, then the definition implies that $i = j$ and hence $x\, ur^f\, y$. Therefore it suffices to investigate two cases: (i) $K_f(x) = K_f(y) \neq K_f(z)$, (ii) $K_f(x) = K_f(y) = K_f(z)$. Assume that (i) is valid. Then $f^i(x) = f^i(y)$, $\{f^j(y), f^j(z)\} \subseteq D_f$, thus $f^j(x) \in D_f$. Hence $\{f^j(x), f^j(z)\} \subseteq D_f$ and $x\, ur_j^f z$, i.e. $x\, ur^f\, z$. Now suppose that (ii) holds and let $i \leqq j$. Then $f^{i-1}(x) \neq f^{i-1}(y), f^i(x) = f^i(y), f^{j-1}(y) \neq f^{j-1}(z)$, $f^j(y) = f^j(z)$. If $i < j$, then $f^{j-1}(x) = f^{j-1}(y) \neq f^{j-1}(z)$, $f^j(x) = f^j(y) = f^j(z)$, i.e. $x\, ur_j^f\, z$. If $i = j$, then there is $0 < k \leqq i$ such that $f^{k-1}(x) \neq f^{k-1}(z)$, $f^k(x) = f^k(z)$, i.e. $x\, ur_k^f\, z$. Hence ur^f is transitive.

Instead of writing $x\, ur^f\, y$ we shall write also $y \in x\, ur^f$, and analogously for ur_n^f.

It is obvious that if $x \in f^{-k}(z)$, $z \in D_f$, $k \in N \cup \{0\}$, then $x\, ur^f = f^{-k}(D_f)$.

If $M \subseteq A_2^f$, put

$$M\, ur^f = \{y \in A_2^f \colon (\exists x \in M)\, (x\, ur^f\, y)\}\,,$$

and similarly for $M\, ur_n^f$.

Through the whole paper we suppose that A is a nonempty set, f is a *fixed* element of $F(A)$ and g is an *arbitrary* element of the set $Eq(f)$.

Moreover, in Sections 2 and 3 we assume that (A, f) is a partial monounary algebra which is neither of type τ nor of type π. We shall use the assertion (B1) without quotation.

2. RELATION ur^f

The following propositions (T1) and (T2) which will be often used in the sequel are immediate consequences of 3.3 and 4.8 [3].

(T1) *Let (A, h) be a partial monounary algebra, $x \in D_h$, $y \in A - K_h(x)$. Then $s_h(x) \leqq s_h(y)$ if and only if there exists $H \in \mathrm{End}\,(A, h)$ such that the following conditions are fulfilled*:

(i) $H(x) = y$,

(ii) $H(h^{-n}(x)) \subseteq h^{-n}(y)$ *for each* $n \in N$,

(iii) $H(t) = t$ *for each* $t \in A - \bigcup_{n \in N \cup \{0\}} h^{-n}(x)$.

(T2) *Let (A, h) be a partial monounary algebra, $x, y \in A$, $x \neq h(x) = h(y)$. Then $s_h(x) \leqq s_h(y)$ if and only if there exists $H \in \mathrm{End}\,(A, h)$ such that* (i), (ii) *and* (iii) *from* (T1) *are valid.*

2.1. Lemma. *Let $x, y \in A_2^f$.*

(i) *If $x\, ur^f\, y$, then there are $z \in K_f(x) \cup K_f(y)$ and $H \in \mathrm{End}\,(A, f)$ such that $H(x) = H(y) = H(z) = z$ and $H(t) = t$ for each $t \in \bigcup_{k \in N} f^{-k}(z)$.*

(ii) *If $n \in N \cup \{0\}$, $y \in K_f(x) \cap x\, ur_n^f$, then there are $z \in K_f(x)$ and $H \in \mathrm{End}\,(A, f)$ such that $H(x) = H(y) = H(z) = z$, $H(t) = t$ for each $t \in \bigcup_{k \in N} f^{-k}(z)$ and $H(f^n(x)) = f^n(x)$.*

Proof. We shall prove both the assertions by induction. Let $n \in N \cup \{0\}$ be such that $x\ ur_n^f\ y$ is valid. If $n = 0$, i.e. $x = y$, then we can put $z = x$, $H = \mathrm{id}_A$. Let $n = 1$. Then $x \neq y$ and either $\{x, y\} \subseteq D_f$ or $f(x) = f(y)$ (where $x \neq f(x)$, $y \neq f(y)$, since $\{x, y\} \subseteq A_2^f$). Without loss of generality we can consider the case $s_f(x) \leqq s_f(y)$. From (T1) or (T2) it follows that there is $H \in \mathrm{End}(A, f)$ such that

(1) $H(x) = H(y) = y$, $H(t) = t$ for each $t \in \bigcup_{k \in N} f^{-k}(y)$, and if $x \notin D_f$, then $H(f(x)) = f(x)$.

Hence we can put $z = y$; obviously $z \in K_f(x) \cup K_f(y)$, and the assertion for $n = 1$ is valid.

Now let $n > 1$ and suppose that for each $m < n$ and for each $x', y' \in A_2^f$ such that $x'\ ur_m^f\ y'$ the assertion holds. Denote $x' = f(x)$, $y' = f(y)$. Then $x'\ ur_{n-1}^f\ y'$, thus

(2) there are $z' \in K_f(x') \cup K_f(y')$ and $H' \in \mathrm{End}(A, f)$ such that $H'(x') = H'(y') = H'(z') = z'$, $H'(t) = t$ for each $t \in \bigcup_{k \in N} f^{-k}(z')$, and if $y' \in K_f(x')$ (i.e. $f^{n-2}(x') \notin D_f$), then $H(f^{n-1}(x')) = f^{n-1}(x')$.

Put $H'(x) = x_1$, $H'(y) = y_1$. This implies

(3) $f(x_1) = f(H'(x)) = H'(f(x)) = H'(x') = z' = H'(y') = H'(f(y)) = f(H'(y)) = f(y_1)$,

therefore either $x_1\ ur_0^f\ y_1$ or $x_1\ ur_1^f\ y_1$. By using the relations proved above (and taking x_1, y_1 instead of x, y) we obtain

(4) there are $z \in \{x_1, y_1\}$ and $H_1 \in \mathrm{End}(A, f)$ such that $H_1(x_1) = H_1(y_1) = H_1(z) = z$, $H_1(t) = t$ for each $t \in \bigcup_{k \in N} f^{-k}(z)$, and if $x_1 \notin D_f$, then $H_1(f(x_1)) = f(x_1)$.

From (3) it follows that

(5) $z \in \{x_1, y_1\} \subseteq f^{-1}(z')$,

and then (2) implies

(6) $H'(z) = z$.

Denote $H = H' \circ H_1$. According to (6) and (4) we get

(7) $$\begin{aligned} H(z) &= H_1(H'(z)) = H_1(z) = z, \\ H(x) &= H_1(H'(x)) = H_1(x_1) = z, \\ H(y) &= H_1(H'(y)) = H_1(y_1) = z. \end{aligned}$$

Now let $t \in \bigcup_{k \in N} f^{-k}(z)$. Then $t \in \bigcup_{j \in N, j > 1} f^{-j}(z')$ with respect to (5), and hence according to (4) and (2) we infer that $H_1(t) = t$ and $H'(t) = t$; therefore

(8) $H(t) = H_1(H'(t)) = H_1(t) = t$.

Further, let $y \in K_f(x)$, thus $f^{n-1}(x) \notin D_f$. Then $f^{n-2}(x') = f^{n-1}(x) \notin D_f$ and (2) yields

(9) $H'(f^n(x)) = H'(f^{n-1}(x')) = f^{n-1}(x') = f^n(x)$.

According to (3) and (5) we obtain

$$f^n(z) = f^n(x_1) = f^n(y_1),$$

hence

(10) $f^n(x) = H'(f^n(x)) = f^n(H'(x)) = f^n(x_1) = f^n(z)$.

Therefore (10) and (8) yield

(11) $H(f^n(x)) = H(f^n(z)) = f^n(H(z)) = f^n(z) = f^n(x)$,

completing the proof.

For $x \in A$ denote

$$V_f(x) = \{v \in A\colon (\exists z \in K_f(x) \cup K_f(v)) (\exists H \in \operatorname{End}(A, f)) \\ (H(x) = H(v) = H(z) = z)\} .$$

Let us notice that if $g \in Eq(f)$, then $V_f(x) = V_g(x)$ for each $x \in A$.

2.2. Corollary. *If $x, y \in A_2^f$, $y \in x\ ur^f$, then $y \in V_f(x)$.*

2.3. Lemma. *Let $x \in A_2^f$, $y \in K_f(x)$. Then $y \in x\ ur^f$ if and only if $y \in V_f(x)$.*

Proof. 2.2 implies that $x\ ur^f \subseteq V_f(x)$. Now let $y \in V_f(x)$. Since $y \in K_f(x)$, there are $z \in K_f(x)$ and $H \in \operatorname{End}(A, f)$ such that $H(x) = H(y) = H(z) = z$. Assume that $y \notin x\ ur^f$. Then there exists $m \in N$ such that either $x\ ur^f f^m(y)$ or $y\ ur^f f^m(x)$. Since the situation is symmetric with respect to y and x, it suffices to investigate the case when $x\ ur^f f^m(y)$. Now let $n \in N \cup \{0\}$ be such that $x\ ur_n^f f^m(y)$. Denote $w = f^n(x) = f^n(f^m(y))$. Then

$$f^n(z) = f^n(H(x)) = H(f^n(x)) = H(w) = H(f^{n+m}(y)) = \\ = f^{n+m}(H(y)) = f^{n+m}(z) ,$$

which is a contradiction.

2.4. Corollary. *If $x \in A_2^f \cap A_2^g$, then $x\ ur^f \cap K_f(x) = x\ ur^g \cap K_g(x)$.*

2.5. Lemma. *Let $x, y \in A_2^f - D_g$, $x\ ur^f\ y$. Then $g(x)\ ur^f\ g(y)$.*

Proof. Since $x\ ur^f\ y$, by 2.1 there are $z \in K_f(x) \cup K_f(y)$ and $H \in \operatorname{End}(A, f)$ such that $H(x) = H(y) = H(z) = z$. Without loss of generality we can assume that $z \in K_f(x)$. We have $H \in \operatorname{End}(A, g)$, thus

(1) $H(g(x)) = g(H(x)) = g(z) = g(H(y)) = H(g(y))$.

Further, $g(z) \in K_f(x)$, hence there is $m \in N \cup \{0\}$ such that either

(2.1) $g(x)\ ur_n^f f^m(g(z))$ for some $n \in N \cup \{0\}$,

or

(2.2) $g(z)\ ur_n^f f^m(g(x))$ for some $n \in N \cup \{0\}$.

Then either

(3.1) $f^n(g(z)) = f^n(g(H(x))) = H(f^n(g(x))) = H(f^n(f^m(g(z)))) = \\ = f^{n+m}(g(H(z))) = f^{n+m}(g(z))$,

or

(3.2) $f^n(g(z)) = f^n(g(H(z))) = H(f^n(g(z))) = H(f^n(f^m(g(x)))) = \\ = f^{n+m}(g(H(x))) = f^{n+m}(g(z))$,

which in both cases implies $m = 0$. Therefore

(4) $g(x)\ ur^f\ g(z)$.

If, moreover, $z \in K_f(y) = K_f(x)$, then analogously to (4) we obtain

(4′) $g(y)\, ur^f\, g(z)$,

and hence (4) and (4′) imply that $g(x)\, ur^f\, g(y)$.

Now let $y \notin K_f(x)$. There exist $k, j \in N \cup \{0\}$ such that

(5) $f^k(g(z)) \in D_f$, $f^j(g(y)) \in D_f$.

Further, there exists $i \in N \cup \{0\}$ such that either

(6.1) $g(z)\, ur^f\, f^i(g(y))$,

or

(6.2) $g(y)\, ur^f\, f^i(g(z))$.

In the first case we obtain $i + k = j$ and

(7.1) $H(f^j(g(y)) = f^{i+k}(g(H(y))) = f^{i+k}(g(z))$,

thus $f^{i+k-1}(g(z)) \notin D_f$, which (with respect to (5)) implies

(8.1) $k > i + k - 1$,

$1 > i$,

$i = 0$,

i.e.

(9.1) $g(z)\, ur^f\, g(y)$.

In the second case $i + j = k$ is valid and then

(7.2) $H(f^j(g(y)) = f^j(g(H(y))) = f^{k-i}(g(z))$.

Since $z \in V_f(x) \cap K_f(x)$, 2. 3 yields

(8.2) $z\, ur^f\, x$,

which implies

(9.2) $z\, ur^f\, y$, i.e. $z\, ur_l^f\, y$ for some $l \in N$.

Then $\{f^l(z), f^l(y)\} \subseteq D_f$ and we have

(10.2) $H(f^l(y)) = f^l(H(y)) = f^l(z)$.

Since $f^j(g(y)) \in D_f$ by (5), the relation $f^j(g(y)) = f^l(y)$ holds and according to (7.2) we get

(11.2) $f^{k-i}(g(z)) = H(f^j(g(y)) = H(f^l(y)) = f^l(z)$,

hence

$$f^j(g(z)) = f^{k-i}(g(z)) \in D_f\,.$$

Then (5) implies

$$\{f^j(g(z), f^j(g(y))\} \subseteq D_f\,,$$

thus

(12.2) $g(z)\, ur^f\, g(y)$.

Therefore, in both cases the following relation is valid (by (9.1) and (12.2)):

(13) $g(z)\, ur^f\, g(y)$.

From (4) and (13) we obtain

(14) $g(x)\, ur^f\, g(y)$.

2.6. Lemma. *There is no $x \in A_2^f - D_g$ such that the relation $g(x)\, ur^f\, x$ holds.*

Proof. Assume that there is $x \in A_2^f - D_g$ with $g(x)\, ur^f\, x$. According to 2.1 there are $z \in K_f(x)$ and $H \in \mathrm{End}\,(A, f)$ such that $H(x) = H(g(x)) = H(z) = z$, from which we obtain

$$z = H(g(x)) = g(H(x)) = g(z).$$

Then 2.0 [2] implies that $f(z) = z$, a contradiction with the fact that $x \in A_2^f$.

2.7. Lemma. *Let $x, y \in A_2^f \cap A_2^g - D_g$, $g(x)\, ur^f\, g(y)$. Then $x\, ur^f\, y$.*

Proof. First assume that $y \in K_f(x)$. Since $K_f(x) = K_g(x)$, there exist $m, n \in N \cup \cup \{0\}$ such that $g^m(x) = g^n(y)$. We can suppose that $m \leqq n$. By 2.5 the following relation is valid:

(1) $g^2(x)\, ur^f\, g^2(y), \ldots, g^m(x)\, ur^f\, g^m(y)$.

Further, according to 2.4, (1) implies

(2) $g^m(x)\, ur^g\, g^m(y)$.

We get

(3) $g^n(y)\, ur^g\, g^m(y)$,

hence $m = n$ and $x\, ur^g\, y$. By using 2.4 again, we obtain that $x\, ur^f\, y$.

Now assume that $y \notin K_f(x)$. Then there are $i, j \in N \cup \{0\}$ such that

(4) $\{f^i(x), f^j(y)\} \subseteq D_f$;

we can suppose that $i \leqq j$. Further, let $k \in N \cup \{0\}$ be such that

(5) $\{f^k(g(x)), f^k(g(y))\} \subseteq D_f$.

Denote $v = f^{j-i}(y)$. Since

(6) $f^i(v) = f^{i+j-i}(y) = f^j(y)$

holds, (4) and (6) yield that $\{f^i(x), f^i(v)\} \subseteq D_f$, thus

(7) $v\, ur^f\, x$.

Then 2.5 implies

(8) $g(v)\, ur^f\, g(x)$

and according to the assumption of the lemma we obtain

(9) $g(v)\, ur^f\, g(y)$.

Since v and y belong to the same component, we obtain from the first part of the proof and from (9) that

(10) $v\, ur^f\, y$.

This and (7) imply that $y\, ur^f\, x$.

2.8.1. Corollary. *Let $x, y \in A_2^f \cap A_2^g - D_g$, $g(x) = g(y)$. Then $x\, ur^f\, y$.*

2.8.2. Corollary. *Let $x, y \in A_2^f \cap A_2^g - D_g$. Then $x\, ur^f\, y$ if and only if $g(x)\, ur^f\, g(y)$.*

Proof. The assertion follows from 2.5 and 2.7.

3. AUXILIARY RESULTS

For $i \in N$ let us introduce the following notations:

$U_i^f = \{x \in A_2^f \cap A_2^g - D_g\colon g(x)\, ur^f\, f^i(x)\}$,

$L_i^f = \{x \in A_2^f \cap A_2^g - D_g\colon g(x) \in f^{-i}(x)\, ur^f\}$,

$U^f = \bigcup_{i\in N} U_i^f$,
$L^f = \bigcup_{i\in N} L_i^f$.
It follows from 2.6 that $K_f(x) = (U^f \cup L^f \cup D_g) \cap K_f(x)$ for each $x \in A_2^f \cap A_2^g$.

3.1. Lemma. *If $i \in N$, $x \in U_i^f$, then $g(x) = f^i(x)$.*

Proof. Let the assumption of the lemma be satisfied. Denote $f^i(x) = z$, $g(x) = y$ and suppose that $z \neq y$. Obviously $z\, ur^f\, y$; let $n \in N$ be such that $z\, ur_n^f\, y$ holds. Put $f^{n-1}(z) = x'$, $f^{n-1}(y) = y'$. It is obvious that $x' \neq y'$ and then $x' \neq f(x') = f(y') \neq y'$ (since all elements considered belong to the same component $K_f(x) \subseteq A_2^f$). First, assume that $s_f(x') \leqq s_f(y')$. Then (T2) implies that there is $H \in \operatorname{End}(A, f)$ such that

(1) $H(x') = H(y') = y'$, $H(y) = y$, $H(x) = u \in f^{-i-n+1}(y')$.

Thus $H \in \operatorname{End}(A, g)$ and we get

(2) $g(x) = y = H(y) = H(g(x)) = g(H(x)) = g(u)$.

If $g(x) = x$ or $g(u) = u$, then 2.0 [2] implies that $f(x) = x$ of $f(u) = u$, which is a contradiction, since $K_f(x) \subseteq A_2^f$. If $s_g(x) \geqq s_g(u)$, then there is $H_1 \in \operatorname{End}(A, g)$ such that

(3) $H_1(u) = H_1(x) = x$, $H_1(y) = y$,

according to (T2). Hence we obtain

$$x' = f^{n-1+i}(x) = f^{n-1+i}(H_1(u)) = H_1(f^{n-1+i}(u)) = H_1(y') =$$
$$= H_1(f^{n-1}(y)) = f^{n-1}(H_1(y)) = f^{n-1}(y) = y',$$

a contradiction. If $s_g(x) < s_g(u)$, then (T2) implies that there exists $H_2 \in \operatorname{End}(A, g)$ such that

(4) $H_2(x) = H_2(u) = u$, $H_2^{-1}(t) = \{t\}$ for each $t \in y\, ur^g \cap K_g(y)$.

We have $y \in A_2^f \cap A_2^g$, thus $y\, ur^g \cap K_g(y) = y\, ur^f \cap K_f(y)$ by 2.4, and therefore

(5) $H_2^{-1}(t) = \{t\}$ for each $t \in y\, ur^f \cap K_f(y)$.

Then

$$z = H_2(z) = H_2(f^i(x)) = f^i(H_2(x)) = f^i(u) \in f^{-n+1}(y'),$$

which is a contradiction.

Hence the relation $s_f(x') > s_f(y')$ is valid. According to (T2) there exists $H_3 \in \operatorname{End}(A, f)$ such that $H_3(y') = H_3(x') = x'$, $H_3(x) = x$, $H_3(y) \neq y$. Then we have

$$y = g(x) = g(H_3(x)) = H_3(g(x)) = H_3(y) \neq y,$$

a contradiction. Therefore $z = y$.

3.2.0. Notation. Let $z \in D_f$, $x \in K_f(z)$, $n \in N \cup \{0\}$. If $f^{-n}(z) \neq \emptyset$ and $f^{-n-1}(z) = \emptyset$, then we shall write

$$x \in M_0^n(z).$$

Further, we put

$$M_0^n = \bigcup_{t\in D_f} M_0^n(t),$$
$$M_0 = \bigcup_{n\in N\cup\{0\}} M_0^n.$$

3.2.1. Lemma. *Let $n \in N \cup \{0\}$, $x, x' \in M_0^n \cap D_g$. Then $x\, ur^f\, x'$.*

Proof. The assertion is obvious if $x = x'$. Suppose that $x \neq x'$. Then $x' \notin K_f(x)$ and there are $z, z' \in D_f$ with $x \in K_f(z)$, $x' \in K_f(z')$. Since $x, x' \in M_0^n$, we have

(1) $s_f(z) = n = s_f(z')$.

According to (T1) there exists $H \in \operatorname{End}(A, f)$ such that

(2) $H(t) \in f^{-k}(z')$ for each $t \in f^{-k}(z)$, $k \in \{0, 1, \ldots, n\}$. It follows from 2.4 that $x\, ur_f \cap K_f(x) = \{x\}$ and $x'\, ur_f \cap K_f(x') = \{x'\}$. Take an arbitrary $k \in \{0, \ldots, n\}$ with $f^{-k}(z) \neq \{x\}$. Let $t \in f^{-k}(z)$. Then $t \notin D_g$, thus $H(t) \notin D_g$, since $H \in \operatorname{End}(A, g)$. Therefore $H(t) \notin x'\, ur^f$. This and (2) yield

(3) $f^{-k}(z') \neq \{x'\}$.

The above conditions concerning k imply that $x\, ur^f\, x'$.

3.2. Lemma. *Let $x, x' \in (A_2^f - M_0) \cap D_g$. Then $x\, ur^f\, x'$.*

Proof. In the case $x = x'$ the assertion is obvious. Let $x \neq x'$. Then $x' \notin K_f(x)$ and there are $z, z' \in D_f$, $m, n \in N \cup \{0\}$ with

(1) $f^m(x) = z$, $f^n(x') = z'$.

We can assume that $s_f(z) \leqq s_f(z')$. According to (T1) there exists $H \in \operatorname{End}(A, f)$ such that

(2) $H(t) \in f^{-k}(z')$ for each $t \in f^{-k}(z)$, $k \in N \cup \{0\}$.

Let $k \neq m$ be an arbitrary number from $N \cup \{0\}$, $t \in f^{-k}(z)$. Lemma 2.4 yields

(3) $x\, ur^f \cap K_f(x) = \{x\}$, $x'\, ur^f \cap K_f(x') = \{x'\}$.

Then we have $t \notin D_g$, $H(t) \notin D_g$ (since $H \in \operatorname{End}(A, g)$) and according to (2) and (3) we get

(4) $k \neq n$.

From the assumption $x \notin M_0$ we obtain that $f^{-i}(z) \neq \emptyset$ for each $i \in N$. Therefore the number k can run over the set $N \cup \{0\} - \{m\}$, thus (4) yields that $m = n$, i.e. $x\, ur^f\, x'$.

3.3. Lemma. *Let $i \in N$, $x \in L_i^f$. Then there is $x' \in x\, ur^f \cap K_f(x)$ such that $g(x') \in f^{-i}(x')$.*

Proof. Let us denote $g(x) = u$, $f^i(u) = y$. Then $x\, ur^f\, y$ and by 2.1 there are $x' \in K_f(x)$ and $H \in \operatorname{End}(A, f)$ such that $H(x) = H(y) = H(x') = x'$. We have

$$x' = H(y) = H(f^i(u)) = f^i(H(u)),$$

thus $H(u) = t$ for some $t \in f^{-i}(x')$. Hence

$$t = H(u) = H(g(x)) = g(H(x)) = g(x').$$

Let us introduce the following notation. For $x, u \in A_2^f \cap A_2^g$ such that $u \in x\, ur^f \cap K_f(x)$ put

$$M_f(x, u) = \{z \in K_f(x) \colon (\exists H, G_1, G_2 \in \operatorname{End}(A, f)) (H(x) = H(u) = H(z) = z \;\&\; G_1(x) = G_1(z) = z \;\&\; G_1(u) = u \;\&\; G_2(u) = G_2(z) = z \;\&\; G_2(x) = x)\},$$

$$T_f(x) = \{v \in K_f(x) \cap x\, ur^f \colon M_f(x, v) = \{v\}\}.$$

3.4. Lemma. *If $x, u \in A_2^f \cap A_2^g$, $u \in x\, ur^f \cap K_f(x)$, then $M_f(x, u) \neq \emptyset$, $M_f(x, u) = M_g(x, u)$ and $T_f(x) = T_g(x)$.*

Proof. Let the assumption of the lemma be satisfied. It follows from 2.1 and 2.4 that there are $z \in K_f(x)$ and $H \in \operatorname{End}(A, f)$ such that $H(x) = H(u) = H(z) = z$. If $z \in f^{-i}(D_f)$, $x \in f^{-j}(D_f)$ for some $i, j \in N \cup \{0\}$, then

$$f^j(H(x)) = H(f^j(x)) = H(f^i(z)) = f^i(H(z)) = f^i(z) = f^j(x),$$

thus $H(x)\, ur^f\, x$; analogously $H(u)\, ur^f\, u$. Then there exist $m, n \in N \cup \{0\}$ with $x\, ur_n^f\, z$, $z\, ur_m^f\, u$. Put

$$G_1(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k \in N \cup \{0\}} f^{-k}(f^{n-1}(u)), \\ H(t) & \text{otherwise}, \end{cases}$$

$$G_2(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k \in N \cup \{0\}} f^{-k}(f^{n-1}(x)), \\ H(t) & \text{otherwise}. \end{cases}$$

We have $G_1, G_2 \in \operatorname{End}(A, f)$ and

$$G_1(x) = G_1(z) = z, \quad G_1(u) = u,$$
$$G_2(u) = G_2(z) = z, \quad G_2(x) = x.$$

Thus $M_f(x, u) \neq \emptyset$. The relations $M_f(x, u) = M_g(x, u)$ and $T_f(x) = T_g(x)$ follow from 2.4 and from the fact that $\operatorname{End}(A, f) = \operatorname{End}(A, g)$.

3.5. Lemma. *Let $i \in N$, $x \in L_i^f$. If $g(x) \in f^{-i}(u)$ for $u \neq x$, then $u \in T_f(x)$.*

Proof. Denote $g(x) = y$. Then $f^i(y) = u$, $u \in x\, ur^f \cap K_f(x)$. We want to prove that $M_f(x, u) = \{u\}$. Let $n \in N$ be such that $x\, ur_n^f\, u$. Further, assume that there exists $z \in M_f(x, u) - \{x, u\}$. Hence there are $H, G_1, G_2 \in \operatorname{End}(A, f)$ such that

(1) $H(x) = H(u) = H(z) = z$,

(2) $G_1(x) = G_1(z) = z$, $G_1(u) = u$,

(3) $G_2(u) = G_2(z) = z$, $G_2(x) = x$.

Since $H(z) = z$, we have $H(t)\, ur^f\, t$ for each $t \in K_f(z)$. Namely, if $t \in K_f(z)$, $f^j(t) \in D_f$, $f^i(z) \in D_f$ for some $i, j \in N \cup \{0\}$, then

$$f^j(H(t)) = H(f^j(t)) = H(f^i(z)) = f^i(H(z)) = f^i(z) = f^j(t),$$

thus $H(t)\, ur^f\, t$. Consequently,

(4) $u\, ur_m^f\, z$ for some $m \in N$.

Further, (2) implies

$$f^n(z) = f^n(G_1(x)) = G_1(f^n(x)) = G_1(f^n(u)) = f^n(G_1(u)) = f^n(u),$$

which yields the relation $m \leqq n$. Put

$$H_2(t) = \begin{cases} G_2(t) & \text{if} \quad t \in f^{-i}(f^j(u)), \quad i \in N \cup \{0\}, \quad j \in N, \quad j < m, \\ t & \text{otherwise}. \end{cases}$$

Obviously, $H_2 \in \operatorname{End}(A, f)$. Analogously to (4) we obtain

(4′) $x\, ur_k^f\, z$ for some $k \in N$, and (3) yields

$$f^n(z) = f^n(x)\,,$$

thus $k \leqq n$. Then we put

$$H_1(t) = \begin{cases} G_1(t) & \text{if } t \in f^{-i}(f^j(x))\,, \quad i \in N \cup \{0\}\,, \quad j \in N\,, \quad j < k\,, \\ t & \text{otherwise}\,. \end{cases}$$

We have $H_1 \in \operatorname{End}(A, f)$ and

(5) $H_1(y) = y$,

(6) $H_2^{-1}(y) = \emptyset$.

Hence

(7) $g(x) = y = H_1(y) = H_1(g(x)) = g(H_1(x)) = g(z)$.

Since $u \in x\, ur^f \cap K_f(x) = x\, ur^g \cap K_g(x)$ (by 2.4) and $x \notin D_g$, then $u \notin D_g$ and according to (7) we get

(8) $y = g(z) = g(H_2(u)) = H_2(g(u))$,

which is a contradiction with (6).

Therefore $M_f(x, u) \subseteq \{x, u\}$. Suppose that $x \in M_f(x, u)$. Then there is $G \in \operatorname{End}(A, f)$ such that $G(x) = G(u) = x$, which implies

$$y = g(x) = g(G(x)) = G(g(x)) = G(y)\,,$$
$$x = G(u) = G(f^i(y)) = f^i(G(y)) = f^i(y) = u\,,$$

a contradiction. Since $M_f(x, u) \neq \emptyset$ with respect to 3.4 and since we have already proved that $M_f(x, u) \subseteq \{u\}$, the relation $M_f(x, u) = \{u\}$ holds.

3.6. Lemma. *Let $i \in N$, $x \in L_i^f$. If $g(x) \in f^{-i}(u)$ for some $u \neq x$, then $g(u) = g(x)$.*

Proof. It follows from 3.5 that $u \in T_f(x)$, i.e. that $M_f(x, u) = \{u\}$. Then there is $H \in \operatorname{End}(A, f)$ such that $H(x) = H(u) = u$ and then there exists $n \in N$ with $x\, ur_n^f\, u$. Put

$$G(t) = \begin{cases} H(t) & \text{if } t \in f^{-k}(f^{n-1}(x))\,, \quad k \in N \cup \{0\}\,, \\ t & \text{otherwise}\,. \end{cases}$$

We have $G \in \operatorname{End}(A, f)$ and $G(g(x)) = g(x)$, since $g(x) \notin \bigcup_{k \in N \cup \{0\}} f^{-k}(f^{n-1}(x))$. Hence

$$g(x) = G(g(x)) = g(G(x)) = g(H(x)) = g(u)\,.$$

3.7. Lemma. *Let $i \in N$, $x \in L_i^f$. If $f^{-i}(x) \neq \emptyset$, then $g(x) \in f^{-i}(x)$.*

Proof. Suppose that the assertion fails to hold, i.e. that $y = g(x) \in f^{-i}(u)$, $u \neq x$. According to 3.6, $g(u) = g(x)$. First, let $s_g(u) \leqq s_g(x)$. Then (T2) implies that there is $H \in \operatorname{End}(A, g)$ such that $H(u) = H(x) = x$, from which we obtain

$$u = f^i(y) = f^i(g(x)) = f^i(g(H(x))) = f^i(H(g(x))) =$$
$$= f^i(H(y)) = H(f^i(y)) = H(u) = x\,,$$

a contradiction. Hence $s_g(x) < s_g(u)$. According to (T2) there exists $G \in \operatorname{End}(A, g)$ such that $G(x) = G(u) = u$, $G^{-1}(y') = \{y'\}$ for each $y' \in y\, ur^g \cap K_g(x)$. Since $G \in \operatorname{End}(A, f)$, we obtain

(1) $G(f^{-i}(x)) \subseteq f^{-i}(u)$.

Assume that $t \in f^{-i}(x) \neq \emptyset$ and put $y' = G(t)$. Then $y' \in y\, ur^f \cap K_f(x) = y\, \mathrm{ur}^g \cap \cap K_g(x)$ (with respect to 2.4). Further,

(2) $G^{-1}(y') \supseteq \{y', t\}$,

which is a contradiction.

3.8. Lemma. *Let $i \in N$, $x \in L_i^f$, $y = g(x) \in f^{-i}(x)$. Then y is the unique element belonging to $f^{-i}(x)$ such that $H(x) = x$ implies $H(y) = y$ for each $H \in \mathrm{End}\,(A, f)$.*

Proof. Let the assumption of the lemma be satisfied. Further, let $H \in \mathrm{End}\,(A, f)$ be such that $H(x) = x$. Then $H \in \mathrm{End}\,(A, g)$, which implies

$$H(y) = H(g(x)) = g(H(x)) = g(x) = y\,.$$

Suppose that $y \neq y' \in f^{-i}(x)$. Thrn $y'\, ur_n^f\, y$ for some $n \in N$, $n \leqq i$, thus 2.1 yields that there are $z \in K_f(x)$ and $G \in \mathrm{End}\,(A, f)$ such that

$$G(y) = G(y') = G(z) = z\,, \quad G(f^n(y)) = f^n(y)\,.$$

Thus $G(f^i(y)) = f^i(y)$, since $n \leqq i$, i.e. $G(x) = x$, and from this we obtain that $G(y) = y$. Then

$$G(y') = G(y) = y \neq y'\,,$$

and therefore y' does not possess the property considered.

3.8.1. Remark. The element y from Lemma 3.8 will be denoted by the symbol $y_0^f(x, i)$.

3.9. Lemma. *Let $i \in N$, $x \in L_i^f$, $f^{-i}(x) = \emptyset$. There exists the least $m \in N$ such that $f^{-i-m}(f^m(x)) \neq \emptyset$. Then $f^{-i-m}(f^m(x))$ contains a unique element v such that $H(x) = x$ implies $H(v) = v$ for each $H \in \mathrm{End}\,(A, f)$, and we have $g(x) = v$.*

Proof. Since $x \in L_i^f$, $f^{-i}(x) = \emptyset$, it is obvious that a positive integer m (and also the least positive integer m) with the required property $f^{-i-m}(f^m(x)) \neq \emptyset$ exists. Denote

(1) $L = L(x, i) = \{v \in f^{-i-m}(f^m(x))\colon (\forall H \in \mathrm{End}\,(A, f))$
$(H(x) = x \Rightarrow H(v) = v)\}$.

Further, let $g(x) = y$, $f^i(y) = u$. We have $f^{-i-m}(f^m(x)) \neq \emptyset$; let $y_1 \in f^{-i-m}(f^m(x))$ and put $u_1 = f^i(y_1)$, $y_1' = g(u_1)$. Then

(2) $y\, ur^f\, y_1$,

(3) $u\, ur^f\, u_1\, ur^f\, x$.

Since $f^{-i}(u) \neq \emptyset, f^{-i}(u_1) \neq \emptyset$, according to 3.6, 3.7, 3.8 and 3.8.1 we obtain

(4) $y = g(u) = y_0^f(u, i) \in f^{-i}(u)$,
$y_1' = g(u_1) = y_0^f(u_1, i) \in f^{-i}(u_1)$.

Let $n \in N$ be such that $x\, ur_n^f\, u$. From the fact that m is the least positive integer such that $f^{-i-m}(f^m(x)) \neq \emptyset$ it follows that $n \geqq m$. Put

$$H(t) = \begin{cases} f^{i+k_1-k_2}(y_1') & \text{if} \quad t \in f^{-k_2}(f^{k_1}(x))\,, \quad 0 \leqq k_1 < m\,, \quad 0 \leqq k_2 < i + m\,, \\ t & \text{otherwise}\,. \end{cases}$$

Then $H \in \mathrm{End}(A, f)$, thus $H \in \mathrm{End}(A, g)$, and since $n \geqq m$, we have $H(y) = y$. Further, we have

(5) $y'_1 = g(u_1) = g(H(x)) = H(g'x)) = H(y) = y$.

Then $y \in f^{-i-m}(f^m(x))$ and obviously $y \in L$. According to (4) and (5) the following identities are valid:

(6) $u = f^i(y) = f^i(y'_1) = u_1$.

Since $y_1 \in f^{-i}(u_1) = f^{-i}(u)$ and $y \in f^{-i}(u)$, there exists $k \in N$, $k \leqq i$, such that $y\, ur^f_k\, y_1$. It follows from 2.1 that there are $z \in K_f(y)$ and $G \in \mathrm{End}(A, f)$ such that

(7) $G(y) = G(y_1) = G(z) = z$, $G(f^k(y)) = f^k(y)$.

Put

$$G'(t) = \begin{cases} G(t) & \text{if} \quad t \in f^{-j}(f^k(y)), \quad j \in N \cup \{0\}, \\ t & \text{otherwise}. \end{cases}$$

Then $G' \in \mathrm{End}(A, f)$ and

(8) $G'(y) = G'(y_1) = G'(z) = z$, $G'(x) = x$.

If we assume that $y_1 \in L$, then (8) implies

(9) $y_1 = z$,

and since $y \in L$, according to (8) we get

(10) $y = z$.

Thus $y = y_1$ and the proof is complete.

3.9.1. Remark. The element v from Lemma 3.9 will be denoted by the symbol $v^f_0(x, i)$.

4. THE ESTIMATE OF THE CARDINALITY OF THE SET $Eq(f)$

We start by introducing the following notations.

Let (B, h_1) and (B, h_2) be partial monounary algebras such that

(i) $B = \{x_i: i \in N\} \cup \bigcup_{i \in N, i > 1} B_i$, where x_i, $i \in N$, are distinct elements, B_i, $i \in N$, $i > 1$, are disjoint sets and $x_i \notin B_j$ for $i, j \in N$, $j > 1$;

(ii) $h_1(b_{i+1}) = x_{i+2}$ for each $b_i \in B_i \cup \{x_i\}$, $i \in N$, and $h_1(x_1) = x_2$;

(iii) $h_2(b_{i+1}) = x_i$ for each $b_{i+1} \in B_{i+1} \cup \{x_{i+1}\}$, $i \in N$, and $x_1 \in D_{h_2}$.

The algebra (B, h_1) or (B, h_2) is said to be of *type* σ or ϱ, respectively. If (B, h_1) or (B, h_2) is of type σ or ϱ, respectively, and there are x_i (for each $i \in N$) and B_i (for each $i \in N$, $i > 1$) fulfilling (i), (ii) or (i), (iii), then we write $(B, h_1) \in \sigma$, $(B, h_2) \in \varrho$, or more explicitly,

(1) $(B, h_1) = \sigma(x_1, x_2, \ldots, B_2, B_3, \ldots)$,

(2) $(B, h_2) = \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots)$.

If (1) and (2) are valid, then we shall write

$$(B, h_1) = (B, h_2)^\sigma, \quad (B, h_2) = (B, h_1)^\varrho.$$

Further, we need the following two results (cf. 3.2 and 4.7 [2]):

4.1. Lemma. (i) *Let* $f = \mathrm{id}_A$. *If* $g \mp f$, *then* $A = D_g$.

(ii) *Let* $A = D_f$. *If* $g \mp f$, *then* $g = \mathrm{id}_A$.

4.2. Lemma. *Suppose that* (A, f) *is a partial monounary algebra which is neither of type* τ *nor of type* π, $A \neq D_f$, $f \neq \mathrm{id}_A$. *Further, let* $x \in A$ *and let* $g \mid K_f(x) \neq f \mid K_f(x)$.

(i) *If* $x \in A_1^f \cap A_1^g$, *then* $(K_f(x), f \mid K_f(x)) \notin \sigma$.

(ii) *If* $x \in A_1^f \cap A_2^g$, *then* $(K_f(x), f \mid K_f(x)) \in \sigma$ *and* $(K_f(x), g \mid K_f(x)) = (K_f(x), f \mid K_f(x))^\varrho$.

(iii) *If* $x \in A_2^f \cap A_2^g$, *then* $(K_f(x), f \mid K_f(x)) \notin \varrho$.

4.3. Corollary. (i) *Let* $f = \mathrm{id}_A$. *Then* card $Eq(f) = 2$.

(ii) *Let* $A = D_f$. *Then* card $Eq(f) = 2$.

Proof. The assertion follows from 4.1 and from the fact that the element $g \neq f$ considered in 4.1 obviously belongs to $Eq(f)$.

In the following Lemmas 4.4–4.9.1 we shall assume that (A, f) is a partial monounary algebra which is neither of type τ nor of type π, and for which $A \neq D_f$ and $f \neq \mathrm{id}_A$ hold. Let us denote

$M_1 = \{x \in A_1^f : (K_f(x), f \mid K_f(x)) \notin \sigma\}$,

$M_2 = A_1^f - M_1$,

$M_3 = \{x \in A_2^f : (K_f(x), f \mid K_f(x)) \in \varrho\}$,

$M_4 = A_2^f - M_3$,

$M_4' = M_4 - M_0$

(M_0 was introduced in 3.2.0).

4.4. Lemma. *If* $M_1 \neq \emptyset$, *then* card $Eq(f \mid M_1) \leqq c$.

Proof. Since $(K_f(x), f \mid K_f(x)) \notin \sigma$ for each $x \in M_1$, according to 4.2 (ii) we have $x \notin A_2^g$, i.e. $x \in A_1^g$. Hence f and g considered on the set M_1 are complete unary operations. Then

(1) card $Eq(f \mid M_1) =$ card $Eq_0(f \mid M_1)$,

and (A2) yields

(2) card $Eq_0(f \mid M_1) \leqq c$.

Thus we obtain the assertion of the lemma from (1) and (2).

4.5. Lemma. *If* $M_2 \neq \emptyset$, *then* card $Eq(f \mid M_2) \leqq 2$.

Proof. Let $x \in M_2$. Then

(1) $(K_f(x), f \mid K_f(x)) = \sigma(x_1, x_2, \ldots, B_2, B_3, \ldots)$ for some $x_1, x_2, \ldots, B_2, B_3, \ldots$.

According to 4.2 (i), (ii) either

(2.1) $g \mid K_f(x) = f \mid K_f(x)$

or

(2.2) $(K_f(x), g \mid K_f(x)) = (K_f(x), f \mid K_f(x))^\varrho = \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots)$

holds. Thus there are only two possibilities for the operation g considered on $K_f(x)$. We shall prove that under each of them, the operation g on $K_f(x')$ for $x' \in M_2$ is uniquely determined, and the proof will be complete.

Let $x' \in M_2$. We shall denote by (1'), (2.1'), (2.2') the conditions analogous to (1), (2.1) and (2.2) (for the element x' instead of x). Then we obtain that (1') is valid and either (2.1′) or (2.2′) is valid. Further, there exists $H \in \operatorname{End}(A, f)$ such that $H(x_1) = x_1'$, $H(x_2) = x_2'$, $H(x_3) = x_3'$. First assume that (2.1) holds. Then

(3.1) $g(x_1') = g(H(x_1)) = H(g(x_1)) = H(f(x_1)) = H(x_2) = x_2' = f(x_1')$,

thus (2.2′) fails to hold, i.e. (2.1′) is valid. Now let (2.2) hold. Then

(3.2) $g(x_2') = g(H(x_2)) = H(g(x_2)) = H(x_1) = x_1' \neq f(x_2')$,

and therefore (2.2′) is satisfied.

4.6. Lemma. *If* $M_3 \neq \emptyset$, *then* card $Eq(f \mid M_3) \leqq 2$.

Proof. Let $x \in M_3$. Then

(1) $(K_f(x), f \mid K_f(x)) = \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots)$ for some $x_1, x_2, \ldots, B_2, B_3, \ldots$.

Hence $x \in A_1^g$ according to 4.2 (iii), thus $x \in A_1^g \cap A_2^f$, and from 4.2 (ii) (with f and g interchanged) and from the definition of σ and ϱ we obtain that either

(2.1) $g \mid K_f(x) = f \mid K_f(x)$,

or

(2.2) $(K_f(x), g \mid K_f(x)) = (K_f(x), f \mid K_f(x))^\sigma = \sigma(x_1, x_2, \ldots, B_2, B_3, \ldots)$.

Analogously as in the proof of 4.5 we shall show that if $x' \in M_3$, then for each of these two possibilities for g on $K_f(x)$, the operation g on $K_f(x')$ is uniquely determined.

Let $x' \in M_3$. Then (1′) is valid and either (2.1′) or (2.2′) ((1′), (2.1′), (2.2′) are analogous to (1), (2.1) and (2.2), for x' instead of x). Further, there exists $H \in \operatorname{End}(A, f)$ such that $H(x_1) = x_1'$, $H(x_2) = x_2'$, $H(x_3) = x_3'$. First assume that (2.1) is valid. Then

(3.1) $f(x_2') = x_1' = H(x_1) = H(f(x_2)) = H(g(x_2)) = g(H(x_2)) = g(x_2')$,

and hence (2.1′) holds. Now let (2.2) be valid. We obtain

(3.2) $x_2' = H(x_2) = H(g(x_1)) = g(H(x_1)) = g(x_1')$,

thus (2.2′) holds.

4.7. Lemma. *Let* $x \in M_4$. *Then* $x \in A_2^f \cap A_2^g$ *and one of the following conditions is satisfied*:

(4.7.1) $x \in D_g$,

(4.7.2) *there is* $i \in N$ *such that* $g(x) = f^i(x)$,

(4.7.3) *there is* $i \in N$ *such that* $x \in L_i^f$, *and then*

a) *if* $f^{-i}(x) \neq \emptyset$, *then* $g(x) = y_0^f(x, i)$,

b) *if* $f^{-i}(x) = \emptyset$, *then* $g(x) = v_0^f(x, i)$.

Proof. Since $(K_f(x), f \mid K_f(x)) \notin \varrho$ for $x \in M_4$, according to 4.2 (ii) (with f and g interchanged) we obtain $x \in A_2^f \cap A_2^g$. Further, $x \in D_g \cup U^f \cup L^f$. Assume that $x \notin D_g$. Then 2.6 implies that there is $i \in N$ such that $x \in U_i^f \cup L_i^f$. It follows from 3.1 that if $x \in U_i^f$, then $g(x) = f^i(x)$. Now let $x \in L_i^f$. If $f^{-i}(x) \neq \emptyset$, then $g(x) \in f^{-i}(x)$ according to 3.7 and $g(x) = y_0^f(x, i)$ with respect to 3.8 and 3.8.1. If $f^{-i}(x) = \emptyset$, we infer from 3.9 and 3.9.1 that $g(x) = v_0^f(x, i)$.

4.7.1. Remark. Lemma 4.7 implies the following result: *If* $x \in M_4 \cap U_i^f$ *for* $i \in N$ *or if* $x \in M_4 \cap L_i^f$ *for* $i \in N$, *then* $g(x)$ *is uniquely determined by the number* i.

4.8. Lemma. *Let $n \in N \cup \{0\}$, $x, x' \in M'_4 \cup M^n_0$, $x\, ur^f\, x'$.*

(i) *If $x \in D_g$, then $x' \in D_g$.*

(ii) *If $x \notin D_g$, then $x' \notin D_g$ and $g(x')$ is uniquely determined by $g(x)$.*

Proof. According to 4.7, $M_4 \subseteq A^f_2 \cap A^g_2$, thus ur^f and ur^g are defined on M_4.

(i) Let $x \in D_g$, $z' \in K_f(x') \cap D_g$. 3.2 or 3.2.1 imply that $x\, ur^f\, z'$, thus

(1) $x' \in z'\, ur^f \cap K_f(z')$.

Further, we have

(2) $z'\, ur^g \cap K_g(z') = \{z'\}$

and (with respect to 2.4)

(3) $z'\, ur_g \cap K_g(z') = z'\, ur^f \cap K_f(z')$.

Hence (1), (2) and (3) yield that $x' = z'$, thus $x' \in D_g$.

(ii) Let $x \notin D_g$. Then $x' \notin D_g$ with respect to (i) (if we interchange x and x'), and 2.5 yields

(4) $g(x')\, ur^f\, g(x)$.

Thus if $i \in N$, $x \in U^f_i$, then $x' \in U^f_i$, and hence by 4.7 and 4.7.1 we obtain that $g(x')$ is uniquely determined. Similarly, if $i \in N$, $x \in L^f_i$, then $x' \in L^f_i$, and 4.7 and 4.7.1 imply that $g(x')$ is uniquely determined.

4.9. Lemma. *If $M'_4 \neq \emptyset$, then* card $Eq(f \mid M'_4) \leqq c$.

Proof. From 4.7 we obtain that $M'_4 \subseteq A^f_2 \cap A^g_2$. Then there exist $N_1 \subseteq N$ and distinct elements $x_j \in M'_4$ for $j \in N_1$ such that

(1) for each $x' \in M'_4$ there is $j \in N_1$ with $x'\, ur^f\, x_j$,

(2) $x_k \notin x_j\, ur^f$ for $k, j \in N_1$, $k \neq j$.

Let $j \in N_1$. Consider the number of possibilities for $g(x_j)$. Since by 4.7 and 4.7.1 either $x_j \in D_g$ or $g(x_j)$ is uniquely determined by a number $i \in N$ in the cases when $x_j \in U^f_i$ or $x_j \in L^f_i$, the number of possibilities for $g(x_j)$ is at most $\aleph_0$. Further, card $N_1 \leqq \aleph_0$, and then (1) and 4.8 imply that the number of possibilities for $g \mid M'_4$ is at most $\aleph_0^{\aleph_0}$, i.e.,

$$\text{card } Eq(f \mid M'_4) \leqq c\,.$$

4.9.1. Lemma. *If $n \in N \cup \{0\}$ and $M^n_0 \neq \emptyset$, then* card $Eq(f \mid M^n_0) \leqq c$.

The proof is analogous to that of 4.9.

In the following lemma we suppose only that (A, f) is a partial monounary algebra.

4.10. Lemma (3.2 [2]). *If (A, f) is of type τ or of type π, then* card $Eq(f) = 2$.

Now we shall prove the main result of this paper; we shall repeat all assumptions.

4.11. Theorem. *Let (A, f) be a partial monounary algebra. Then*

$$\text{card } Eq(f) \leqq c$$

(independently of the cardinality of the set A), and this estimate is the best possible.

Proof. First suppose that (A, f) satisfies some of the assumptions of the assertions

4.3 (i), 4.3 (ii), 4.10. Then we obtain

$$\operatorname{card} Eq(f) = 2,$$

according to 4.3 (i), 4.3 (ii) and 4.10.

Now let none of these assumptions be satisfied. Then 4.4, 4.5, 4.6, 4.9 and 4.9.1 imply (since $A = M_1 \cup M_2 \cup M_3 \cup M'_4 \cup \bigcup_{n \in N \cup \{0\}} M_0^n$)

$$\operatorname{card} Eq(f) \leqq c \,.\, 2 \,.\, 2 \,.\, c \,.\, c^{\aleph_0} = c \,.$$

The fact that the estimate is the best possible follows from (A2).

Let us suppose that A is an infinite set. Let us consider the graph $G(A)$ whose set of vertices coincides with $F(A)$ and whose set E of edges is defined as follows: for $f_1, f_2 \in F(A)$ we put $(f_1, f_2) \in E$ if and only if $f_2 \notin Eq(f_1)$. For $f \in F(A)$ let $K(f)$ be the component of the graph $(F(A), E)$ which contains f.

From Thm. 4.11 we obtain as a corollary the following Ramsey-type result:

4.12. Theorem. *Let A be an infinite set. The graph $(F(A), E)$ has the following property: if $f \in F(A)$, then* $\operatorname{card} K(f) = \operatorname{card} F(A)$.

Let us remark that in 4.12 the symbol $F(A)$ can be replaced by $F_0(A)$ (cf. (A2) in Section 1).

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