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ON THE OSCILLATORY BEHAVIOR OF SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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INTRODUCTION

In this paper we are interested in obtaining results on the oscillatory behavior of solutions of a broad class of second order nonlinear differential equations. During the last twenty-five years there has been a great deal of work on the asymptotic behavior of solutions of equations of the type

$$(E_1) \qquad (a(t)\,x')' + q(t)\,f(x) = 0\,, \quad q(t) \geqq 0\,.$$

Considerably less is known about the behavior of solutions of (E_1) when $q(t)$ is allowed to change signs,

$$(E_2) \qquad (a(t)\,x')' + q(t)\,f(x) = 0\,, \quad q(t) \text{ oscillatory}\,,$$

and as recent contributions to this study we cite the papers of Chen [1, 2], Grace and Lalli [4], Graef and Spikes [5], Grammatikopoulos, et al. [6, 9], Kamenev [8], Kura [10], Kusano, et al. [11], Kwong and Wong [12, 13], Mahfoud and Rankin [14], Philos [15, 16], Staikos and Sficas [17], and Yeh [19–21]. There are a number of papers which discuss the behavior of just the bounded solutions of (E_2), but since we are interested in obtaining results for all solutions, we chose not to list those papers here. Recently there has been interest in obtaining results on the asymptotic behavior of solutions of nonlinear equations of the form

$$(E_3) \qquad (a(t)\,\psi(x)\,x')' + q(t)\,f(x) = 0$$

where $\psi(x) > 0$ for $x \neq 0$. Such results can be found in the papers of Domshlak [3] and Jannelli [7] as well as the references [4], [9], [14], [17], and [21] mentioned above.

Here we consider the equation

$$(E) \qquad (a(t)\,\psi(x)\,x')' + q(t)\,f(x) = r(t)$$

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where $q(t)$ is allowed to change signs and we give sufficient conditions for any solution $x(t)$ of (E) to be either oscillatory or satisfy $\liminf_{t\to\infty} |x(t)| = 0$. Two other results give sufficient conditions for all solutions of (E) to be oscillatory in the case when $r(t) \equiv 0$. The results presented here differ in several respects from those of other authors. Due first to the fact that q may change signs and secondly to the presence of the function ψ our results differ from those previously obtained for equations (E_1) and (E_2) respectively. Moreover, our results will cover all solutions not just the bounded ones (for example, see some of the results in [3] and [11]). In addition, we do not require $\int^\infty q(s)\, ds = \infty$ as many authors do (see some of the results in [3], [11], or [17]), and in that respect even when $\psi(x) \equiv 1$, our results differ from some of those previously known for equation (E_2). Some comparisons between our theorems and those of other authors are indicated, and some examples illustrating our results are also included.

OSCILLATORY AND ASYMPTOTIC BEHAVIOR

Consider the equation

$$(a(t)\, \psi(x)\, x')' + q(t) f(x) = r(t) \tag{1}$$

where $a, q, r\colon [t_0, \infty) \to R$ and $f, \psi\colon R \to R$ are continuous, $a(t) > 0$, and $\psi(x) > 0$ for $x \neq 0$. The results in this paper pertain only to the continuable solutions of (1). Such a solution $x(t)$ is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* otherwise. The following conditions will be utilized as they are needed:

$$\int_{t_0}^{\infty} [1/a(s)]\, ds = \infty\,; \tag{2}$$

$$x f(x) > 0 \quad \text{for all} \quad x \neq 0\,; \tag{3}$$

and

$$\int_{t_0}^{\infty} |r(s)|\, ds < \infty\,. \tag{4}$$

Also, to simplify notation we let $W(t) = a(t)\, \psi(x(t))\, x'(t)/f(x(t))$ for any nonoscillatory solution $x(t)$ of (1).

We first extend a result of Kwong and Wong [12, Lemma 2]. For the proof we need the following lemma which can be found in [18, p. 14].

Lemma 1. *Let $k(t, s, z)$ be a real-valued function of t and s in $[T, C)$ and z in $[T_1, C_1]$ such that for fixed t and s, k is a nondecreasing function of z. Let $g(t)$ be a given function on $[T, C)$, and let u and v be functions on $[T, C)$ satisfying $u(s)$ and $v(s)$ are in $[T_1, C_1]$ for all s in $[T, C)$, $k(t, s, v(s))$ and $k(t, s, u(s))$ are locally integrable in s for fixed t, and for all t in $[T, C)$*

$$v(t) = g(t) + \int_T^t k(t, s, v(s))\, ds\,,$$

and

$$u(t) \geqq g(t) + \int_T^t k(t, s, u(s))\, ds .$$

Then $v(t) \leqq u(t)$ *for all* t *in* $[T, C)$.

Lemma 2. *Suppose that* (3) *holds and that*

$$f'(x) \geqq 0 \quad for \quad x \neq 0 . \tag{5}$$

Let $x(t)$ *be a positive (negative) solution of* (1) *on* $[T_1, C)$ *for some positive* T_1 *such that* $t_0 \leqq T_1 < C \leqq \infty$. *If there exist* T *in* $[T_1, C)$ *and a positive constant* A_1 *such that*

(6)
$$-W(T_1) + \int_{T_1}^t [q(s) - r(s)/f(x(s))]\, ds + \int_{T_1}^T [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds \geqq A_1$$

for all t *in* $[T, C)$, *then* $a(t)\, \psi(x(t))\, x'(t) \leqq -A_1 f(x(T))$ $(a(t)\, \psi(x(t))\, x'(t) \geqq -A_1 f(x(T)))$ *for all* t *in* $[T, C)$.

Proof. Let $x(t)$ be a solution of (1) satisfying the hypotheses of the lemma. Since

$$W'(t) + f'(x(t))\, W^2(t)/a(t)\, \psi(x(t)) = r(t)/f(x(t)) - q(t) ,$$

we have

$$-W(t) = -W(T_1) + \int_{T_1}^t [q(s) - r(s)/f(x(s))]\, ds + \int_{T_1}^t [f'(x(s))\, W^2(s)/a(s)\, \psi(s))]\, ds$$

for $T_1 \leqq t < C$, and thus from (6) we see that

$$-W(t) \geqq A_1 + \int_T^t [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds \tag{7}$$

for $T \leqq t < C$. Since the integral in (7) is nonnegative, we have $x(t)\, x'(t) < 0$ on $[T, C)$.

If $x(t)$ is positive, let $u(t) = -a(t)\, \psi(x(t))\, x'(t)$. Then (7) becomes

$$u(t) \geqq A_1 f(x(t)) + \int_T^t [f(x(t)) f'(x(s)) (-x'(s))\, u(s)/f^2(x(s))]\, ds .$$

Define $k(t, s, z) = f(x(t)) f'(x(s)) (-x'(s))\, z/f^2(x(s))$ for t and s in $[T, C)$ and z in $[0, \infty)$. Notice that, in its domain of definition, $k(t, s, z)$ is nondecreasing in z for fixed t and s. Hence, Lemma 1 applies, with $g(t) = A_1 f(x(t))$, and we have $u(t) \geqq v(t)$, where $v(t)$ satisfies the equation

$$v(t) = A_1 f(x(t)) + \int_T^t [f(x(t)) f'(x(s)) (-x'(s))\, v(s)/f^2(x(s))]\, ds$$

provided $v(s)$ is in $[0, \infty)$ for each s in $[T, C)$. Multiplying the last equation by $1/f(x(t))$ and then differentiating, we obtain $v'(t)/f(x(t)) \equiv 0$ so that $v(t) \equiv v(T) =$

$= A_1 f(x(T))$ for all t in $[T, C)$. Thus, by Lemma 1, $a(t)\,\psi(x(t))\,x'(t) \leqq -A_1 f(x(T))$ for $T \leqq t < C$.

The proof for $x(t)$ negative follows by a similar argument by taking $u(t) = a(t)\,\psi(x(t))\,x'(t)$ and $g(t) = -A_1 f(x(t))$.

Lemma 3. *Suppose that* (2)–(5) *hold and that*

$$\int_{t_0}^{\infty} q(s)\,\mathrm{d}s \text{ converges} \tag{8}$$

and

$$f(x) \to \pm\infty \quad as \quad x \to \pm\infty\,. \tag{9}$$

If $x(t)$ *is a solution of* (1) *such that* $\liminf_{t\to\infty} |x(t)| > 0$, *then*

$$\int^{\infty} [f'(x(s))\,W^2(s)/a(s)\,\psi(x(s))]\,\mathrm{d}s < \infty\,, \tag{10}$$

$$W(t) \to 0 \text{ as } t \to \infty\,, \tag{11}$$

and

$$W(t) = \int_t^{\infty} [f'(x(s))\,W^2(s)/a(s)\,\psi(x(s))]\,\mathrm{d}s + \int_t^{\infty} [q(s) - r(s)/f(x(s))]\,\mathrm{d}s \tag{12}$$

for all sufficiently large t.

Proof. Let $x(t)$ be a nonoscillatory solution of (1) satisfying $\liminf_{t\to\infty} |x(t)| > 0$. Then there exist $m > 0$, $M > 0$, and $t_1 > t_0$ such that $|x(t)| \geqq m$ and $|f(x(t))| \geqq M$ for $t \geqq t_1$. This, together with (4), implies that

$$\left|\int_{t_1}^{t} [r(s)/f(x(s))]\,\mathrm{d}s\right| \leqq \int_{t_1}^{t} |r(s)/f(x(s))|\,\mathrm{d}s \leqq N_1 \tag{13}$$

for some $N_1 > 0$ and all $t \geqq t_1$.

Now suppose that (10) does not hold. Then, in view of (8), there exist $A_1 > 0$ and $t_2 > t_1$ such that (6) holds for all $t \geqq t_2$. If $x(t) > 0$ on $[t_1, \infty)$, it follows from Lemma 2 and its proof that $x'(t) < 0$ and $a(t)\,\psi(x(t))\,x'(t) \leqq -A_1 f(x(t_2))$ for $t \geqq t_2$. Since $x(t)$ is positive and decreasing on $[t_2, \infty)$, $0 < \psi(x(t)) \leqq A_2$ on $[t_2, \infty)$ for some positive constant A_2. Thus $x'(t) \leqq -A_1 f(x(t_2))/A_2 a(t)$ and integrating we have $x(t) \leqq x(t_2) - (A_1 f(x(t_2))/A_2) \int_{t_2}^{t} [1/a(s)]\,\mathrm{d}s$ which, in view of (2), contradicts the assumption that $x(t)$ is positive on $[t_1, \infty)$. A similar argument handles the case when $x(t) < 0$ on $[t_1, \infty)$.

Since

$$W'(t) + f'(x(t))\,W^2(t)/a(t)\,\psi(x(t)) = r(t)/f(x(t)) - q(t)\,,$$

we have

$$W(z) + \int_t^{z} [f'(x(s))\,W^2(s)/a(s)\,\psi(x(s))]\,\mathrm{d}s = W(t) + \int_t^{z} [r(s)/f(x(s)) - q(s)]\,\mathrm{d}s\,. \tag{14}$$

From (8), (10), (13), and (14), we see that $\lim_{z\to\infty} W(z)$ exists, say $W(z) \to A_3$ as $z \to \infty$, so that from (14) we have

$$(15)\quad W(t) = A_3 + \int_t^\infty [q(s) - r(s)/f(x(s))]\, ds + \int_t^\infty [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds$$

for $t \geqq t_1$.

To show that (11) and (12) hold we must show that $A_3 = 0$. Suppose first that $x(t) > 0$ on $[t_1, \infty)$. If $A_3 < 0$, then from (8), (10), and (13) there exists $T_1 > t_1$ such that $|\int_{T_1}^t q(s)\, ds| \leqq -A_3/8$ and

$$\left|\int_{T_1}^t [r(s)/f(x(s))]\, ds\right| \leqq -A_3/8 \quad \text{for} \quad t \geqq T_1\,, \quad \text{and}$$

$$\int_{T_1}^\infty [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds < -A_3/8\,.$$

From (15) we see that (6) holds on $[T_1, \infty)$ with $T = T_1$. But then, as argued above, Lemma 2 and its proof contradict the assumption that $x(t)$ is positive on $[t_1, \infty)$. If $A_3 > 0$, it follows from (8), (10), (13), and (15) that $W(t) \to A_3$ as $t \to \infty$ so there exists $T_2 > t_1$ such that $a(t)\, \psi(x(t))\, x'(t)/f(x(t)) \geqq A_3/2$ for $t \geqq T_2$. Thus

$$\int_{T_2}^t [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds = \int_{T_2}^t [f'(x(s))\, (x'(s))^2\, a(s)\, \psi(x(s))/f^2(x(s))]\, ds \geqq$$

$$\geqq (A_3/2) \int_{T_2}^t [f'(x(s))\, x'(s)/f(x(s))]\, ds = (A_3/2) \ln \frac{f(x(t))}{f(x(T_2))}\,.$$

But this, together with (9) and (10), implies that $x(t)$ is bounded above and, hence, $0 < \psi(x(t)) \leqq A_4$ for some positive constant A_4. Since $a(t)\, \psi(x(t))\, x'(t) \geqq$ $\geqq (A_3/2) f(x(t))$, then $x'(t) > 0$ for $t \geqq T_2$ which, together with (5), implies that $f(x(t)) \geqq f(x(T_2))$ for $t \geqq T_2$. Therefore $x'(t) \geqq A_3 f(x(T_2))/2A_4 a(t)$ on $[T_2, \infty)$ and integrating we have

$$x(t) \geqq x(T_2) + (A_3 f(x(T_2))/2A_4) \int_{T_2}^t [1/a(s)]\, ds$$

for $t \geqq T_2$ which, in view of (2), contradicts the boundedness of $x(t)$. This completes the proof that $A_3 = 0$ for case the $x(t) > 0$ on $[t_1, \infty)$. The proof that $A_3 = 0$ when $x(t) < 0$ on $[t_1, \infty)$ is similar and will be omitted.

Remark. Lemma 3 generalizes Lemma 1 in [5].

Before stating our first theorem we observe that if (4) and (8) hold, then

$$h_0(t) = \int_t^\infty [q(s) - P|r(s)|]\, ds/a^{1/2}(t)$$

is a well-defined function on $[t_0, \infty)$ for every positive constant P in the sense that the improper integral converges. As long as the improper integrals involved converge, we can define

$$h_1(t) = \int_t^\infty h_0^2(s)\, ds$$

and

$$h_{n+1}(t) = \int_t^\infty [h_0(s) + Lh_n(s)/a^{1/2}(s)]^2\, ds$$

for $n = 1, 2, 3, \ldots$, where L is any positive constant.

In the next two theorems we will need the condition that for every constant $L > 0$ there exists a positive integer N such that

$$h_n \text{ exists for } \quad n = 0, 1, 2, \ldots, N-1 \quad \text{and } h_N \text{ does not exist}. \tag{16}$$

Theorem 4. *Suppose that* (2)–(5), (8), *and* (16) *hold, and for any* $\lambda_1 > 0$ *there exists* $\lambda_2 > 0$ *such that*

$$f'(x)/\psi(x) \geqq \lambda_2 \quad \text{for all} \quad |x| \geqq \lambda_1 . \tag{17}$$

Suppose, furthermore, that for any $P > 0$

$$h_0(t) \geqq 0 \tag{18}$$

for all sufficiently large t. Then any solution $x(t)$ *of* (1) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Assume that the conclusion of the theorem is false. Then there is a solution $x(t)$ of (1) such that $\liminf_{t\to\infty} |x(t)| > 0$. It then follows from (5) that $|f(x(t))| \geqq M$ for some $M > 0$ and all $t \geqq t_1$ for some $t_1 \geqq t_0$. From (12) and (17) we then have

$$W(t) \geqq a^{1/2}(t)\, h_0(t) + L \int_t^\infty [W^2(s)/a(s)]\, ds \tag{19}$$

for $t \geqq t_1$ and some $L > 0$. Now (10) implies that

$$\int_{t_1}^\infty [W^2(s)/a(s)]\, ds < \infty \tag{20}$$

and (18) and (19) imply that $W(t) \geqq a^{1/2}(t)\, h_0(t) \geqq 0$ so

$$W^2(t)/a(t) \geqq h_0^2(t) . \tag{21}$$

If $N = 1$, then (20) and (21) imply that $h_1(t) = \int_t^\infty h_0^2(s)\, ds < \infty$ which contradicts the nonexistence of $h_N(t) = h_1(t)$. If $N = 2$, then (19) and (21) yield

$$W(t) \geqq a^{1/2}(t)\, h_0(t) + L \int_t^\infty h_0^2(s)\, ds = a^{1/2}(t)\, h_0(t) + Lh_1(t) \tag{22}$$

so

$$W^2(t)/a(t) \geqq [h_0(t) + Lh_1(t)/a^{1/2}(t)]^2 .$$

In view of (20), an integration of the above inequality would give a contradiction to the nonexistence of $h_N(t) = h_2(t)$. A similar argument leads to a contradiction for any integer $N > 2$.

As an example of an equation satisfying the hypotheses of Theorem 4 consider

$$(x^2x')' + [(2 + \sin t - 2t \cos t)/2t^{3/2}]\, x^3 =$$
$$= 4/t^5 + (2 + \sin t - 2t \cos t)/2t^{9/2}\,, \quad t \geqq 1$$

which has the nonoscillatory solution $x(t) = 1/t$. Here $f'(x)/\psi(x) \equiv 3$ and

$$\int_t^\infty q(s)\, ds = (2 + \sin t)/t^{1/2} \geqq 1/t^{1/2}\,.$$

Now $|r(t)| \leqq 7/t^3$ so

$$\int_t^\infty |r(s)|\, ds \leqq 7/2t^2$$

and, hence, $h_0(t) \geqq 0$ for all sufficiently large t. Since

$$\int_t^\infty h_0^2(s)\, ds \geqq \int_t^\infty [1/s^{1/2} - 7P/2s^2]^2\, ds = \infty$$

we have that $N = 1$.

Although Staikos and Sficas [17] obtained results similar to Theorem 4 for equation (1), none of the results in [17] apply to the preceding example. Also, Theorem 4 generalizes Theorem 2 in [5].

Our next two theorems are oscillation results for the case when $r(t) \equiv 0$. Observe that in this case equation (1) becomes

$$(a(t)\, \psi(x)\, x')' + q(t) f(x) = 0 \tag{23}$$

and

$$h_0(t) = \int_t^\infty q(s)\, ds/a^{1/2}(t)\,.$$

Theorem 5. *Suppose that conditions* (2)–(3), (8), (9), (16), *and* (18) *hold and there exists* $\lambda > 0$ *such that*

$$f'(x)/\psi(x) \geqq \lambda \quad \text{for all} \quad x \neq 0\,. \tag{24}$$

Then all solutions of (23) *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (23). Then there exists $t_1 \geqq t_0$ such that $|x(t)| > 0$ for $t \geqq t_1$, and since (24) implies that $f'(x) \geqq 0$ for $x \neq 0$, we have $|f(x(t))| > 0$ for $t \geqq t_1$. It is easy to see that Lemma 2 is valid for equation (23) with condition (6) replaced by

$$-W(T_1) + \int_{T_1}^t q(s)\, ds + \int_{T_1}^T [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds \geqq A_1\,. \tag{6'}$$

Proceeding as in the proof of Lemma 3, we again obtain (10), i.e.,

$$\int^{\infty} [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds < \infty \tag{10}$$

since (13) obviously holds. Using (14) with $r(t) \equiv 0$ and continuing as in the proof of Lemma 3, we again obtain

$$W(t) \to 0 \quad \text{as} \quad t \to \infty \tag{11}$$

and (12) with $r(t) \equiv 0$, i.e.,

$$W(t) = \int_t^{\infty} [f'(x(s))\, W^2(s)/a(s)\, \psi(x(s))]\, ds + \int_t^{\infty} q(s)\, ds \tag{12'}$$

for all sufficiently large t. The remainder of the proof of this theorem is similar to the proof of Theorem 4 and will be omitted.

Notice that obtaining (10), (11), and (12′) in the proof of Theorem 5 extends the lemmas in [2] and [8] and Corollary 4 and Theorem 1 in [12]. Also, Theorem 5 reduces to Theorem 1 in [2] when $\psi(x) \equiv 1$ and to Theorem 1 in [8] when $\psi(x) \equiv 1$ and $a(t) \equiv 1$.

Also notice that for the case $\psi(x) \equiv a(t) \equiv 1$ that (24) becomes

$$f'(x) \geqq \lambda > 0\,, \quad x \neq 0\,. \tag{24'}$$

For this case Kwong and Wong [12, Th. 3] obtained the conclusion of Theorem 5 under the hypotheses (3), (8), (24′),

$$\int_{\pm 1}^{\pm\infty} [1/f(x)]\, dx < \infty\,, \tag{25}$$

and $\limsup_{T\to\infty} \int_{t_0}^T (h_0(t) + \int_t^\infty H_+^2(s)\, ds)\, dt = \infty$ where $H_+(s) = \max\{h_0(s), 0\}$.

Observe that Theorem 5 implies that the equation

$$x'' + [(2 + \sin t - 2t\cos t)/2t^{3/2}]\,(x + x^{1/3}) = 0\,, \quad t > 0$$

is oscillatory whereas this result cannot be deduced from [12, Th. 3] since (25) fails.

Theorem 5 is also related to numerous other results contained in and referred to in our list of references. However, none of these appear to contain Theorem 5. We illustrate the independence of Theorem 5 and a number of these results by considering the examples

$$x'' + [(2 + \sin t - 2t\cos t)/2t^{3/2}]\, x = 0\,, \quad t > 0 \tag{26}$$

$$x'' + [(2 + \sin t - 2t\cos t)/2t^{3/2}]\,(x + x^3) = 0\,, \quad t > 0 \tag{27}$$

$$(x^2x')' + [(2 + \sin t - 2t\cos t)/2t^{3/2}]\, x^3 = 0\,, \quad t > 0 \tag{28}$$

and

$$x'' + (x + x^3)/t \ln^2 t = 0\,, \quad t > 1\,. \tag{29}$$

Theorem 5 implies that each of (26)–(29) is oscillatory. However, Theorem 2.1 in [4], Theorem 6 in [9], Corollary 1 in [13], the Corollary and Theorem 2 in [19], and Corollary 1 in [20] do not apply to either (26) or (27); none of the results in [16] apply to (27); Theorem 2.3 in [4], Theorems 4 and 5 in [14], and none of the results in [1] or [17] apply to (28); Corollary 3 in [12] and Corollary 2 in [20] do not apply to (29).

Theorem 6. *Suppose that conditions* (2)–(3), (8), (18), *and* (24) *hold,*

$$\int_1^{\infty} [\psi(s)/f(s)]\, ds < \infty \quad \text{and} \quad \int_{-1}^{-\infty} [\psi(s)/f(s)]\, ds < \infty \tag{30}$$

and for some positive integer N the functions h_n exist for $n = 0, 1, 2, \ldots, N$. If for every $B > 0$

$$\int^{\infty} \{[h_0(s) + B h_N(s)/a^{1/2}(s)]/a^{1/2}(s)\}\, ds = \infty\,,$$

then all solutions of (23) *are oscillatory.*

Proof. Letting $x(t)$ be a nonoscillatory solution of (23) and proceeding as in the proof of Theorem 5 (also see (22) in the proof of Theorem 4), we eventually obtain

$$W(t) \geqq a^{1/2}(t)\, h_0(t) + B h_N(t)$$

for $t \geqq t_1$ for some $t_1 \geqq t_0$ and $B > 0$. Hence

$$\psi(x(t))\, x'(t)/f(x(t)) \geqq [h_0(t) + B h_N(t)/a^{1/2}(t)]/a^{1/2}(t)$$

and integrating we obtain

$$\int_{x(t_1)}^{x(t)} [\psi(s)/f(s)]\, ds \geqq \int_{t_1}^{t} \{[h_0(s) + B h_N(s)/a^{1/2}(s)]/a^{1/2}(s)\}\, ds \to \infty$$

as $t \to \infty$ which contradicts (30).

Theorem 6 extends Theorem 2 in [2] and Theorem 2 in [8] and applies to equations to which Theorem 2 in [17] is not applicable.

It is interesting to observe that, in view of (18), Theorem 6 reduces to one case of Theorem 3 in [12] when $\psi(x) \equiv a(t) \equiv 1$ and $N = 1$. Notice also that because of (30), Theorem 6 is not applicable when $f(x) = x$ whereas Theorem 5 is.

References

[1] *L. Chen:* Some remarks on oscillation of second order differential equations, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *67* (1979), 45–52.

[2] *L. Chen* and *C. Yeh:* Oscillation theorems for second order nonlinear differential equations with an "integrally small" coefficient, J. Math. Anal. Appl. *78* (1980), 49–57.

[3] *Y. I. Domshlak:* Oscillation of solutions of second-order differential equations, Differential'nye Uravnenija *7* (1971), 205–214 (Russian).

[4] *S. R. Grace* and *B. S. Lalli:* Oscillation theorems for certain second order perturbed nonlinear differential equations, J. Math. Anal. Appl. *77* (1980), 205—214.

[5] *J. R. Graef* and *P. W. Spikes:* Asymptotic behavior of the nonoscillatory solutions of differential equations with integrable coefficients, Publ. Math. Debrecen, to appear.

[6] *M. K. Grammatikopoulos:* Oscillation theorems for second order ordinary differential inequalities and equations with alternating coefficients, An. Stiint. Univ. "Al. I. Cuza" Iasi Sect. Ia Mat. *26* (1980), 67—76.

[7] *V. L. Jannelli:* Sull'esistenza ed il comportamento asintotico di una classe di soluzioni monotone dell'equazione $(p(x)f(y)\,y')' = q(x)\,g(y)$, Boll. Un. Mat. Ital. C (6) *2* (1983), 307—316.

[8] *I. V. Kamenev:* Oscillation of solutions of a second-order differential equation with an "integrally small" coefficient, Differential'nye Uravnenija *13* (1977), 2141—2148 (Russian).

[9] *M. R. Kulenovic* and *M. K. Grammatikopoulos:* On the asymptotic behavior of second order differential inequalities with alternating coefficients, Math. Nachr. *98* (1980), 317—327.

[10] *T. Kura:* Oscillation theorems for a second order sublinear ordinary differential equation, Proc. Amer. Math. Soc. *84* (1982), 535—538.

[11] *T. Kusano, H. Onose* and *H. Tobe:* On the oscillation of second order nonlinear ordinary differential equations, Hiroshima Math. J. *4* (1974), 491—499.

[12] *M. K. Kwong* and *J. S. W. Wong:* An application of integral inequality to second order nonlinear oscillation, J. Differential Equations *46* (1982), 63—77.

[13] *M. K. Kwong* and *J. S. W. Wong:* Linearization of second-order nonlinear oscillation theorems, Trans. Amer. Math. Soc. *279* (1983), 705—722.

[14] *W. E. Mahfoud* and *S. M. Rankin:* Some properties of solutions of $(r(t)\,\psi(x)\,x')' + a(t)f(x) = 0$, SIAM J. Math. Anal. *10* (1979), 49—54.

[15] *Ch. G. Philos:* Oscillation of sublinear differential equations of second order, Nonlinear Anal. *7* (1983), 1071—1080.

[16] *Ch. G. Philos:* A second order superlinear oscillation criterion, Canad. Math. Bull. *27* (1984), 102—112.

[17] *V. A. Staikos* and *Y. G. Sficas:* Oscillations for forced second order nonlinear differential equations, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *60* (1973), 25—30.

[18] *W. Walter:* Differential and Integral Inequalities, Springer-Verlag, Berlin/New York, 1970.

[19] *C. C. Yeh:* An oscillation criterion for second order nonlinear differential equations with functional arguments, J. Math. Anal. Appl. *76* (1980), 72—76.

[20] *C. C. Yeh:* Oscillation theorems for nonlinear second order differential equations with damped term, Proc. Amer. Math. Soc. *84* (1982), 397—402.

[21] *C. C. Yeh:* Comparison theorems for second order nonlinear differential equations, to appear.

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