Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 2, 259–267

Persistent URL: http://dml.cz/dmlcz/102089

ASYMPTOTIC BEHAVIOR OF A CERTAIN FUNCTIONAL EQUATION VIA LIMITING EQUATIONS

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(Received September 21, 1984)

1. INTRODUCTION

The main aim of this work is to deal with the asymptotic character of the bounded solutions of the scalar functional equation with delays

$$x(t) = F(t, x(\alpha_1(t)), \ldots, x(\alpha_k(t)), \int_0^t a(t-s)\, g(s, x(s))\, ds), \quad t \geqq 0 \tag{1.1}$$

which is a more general version of the Volterra integral equation

$$x(t) = f(t) + \int_0^t a(t-s)\, h(x(s))\, ds, \quad t \geqq 0. \tag{1.2}$$

Studying the convergence of bounded solutions of (1.2), Londen (1974) showed that if f is continuous and $f(t) \to f_0$ as $t \to +\infty$, h is continuous, $a \in L^1([0, +\infty))$ and nonincreasing, then any bounded solution x of (1.2) is slowly varying in the usual sense, i.e., for each $r > 0$,

$$\lim_{t \to +\infty} |\sup_{t \leqq s \leqq t+r} x(s) - \inf_{t \leqq s \leqq t+r} x(s)| = 0,$$

and it approximates, as the time increases, the set of the asymptotic equilibrium states of (1.2), i.e., the set $\{\xi \in \mathbb{R}: \xi = f_0 + h(\xi) \int_0^\infty a(s)\, ds\}$. By this result he improved some results of Miller-Sell (1970), Londen (1973), Levin-Shea (1972) etc. Later on, such a behavior of the bounded solutions has been discussed for (1.1) with one delay (i.e. $k = 1$) by Karakostas (1981) for the scalar case and (1982) for the n-dimensional case. A common point in these works is that the function $g(t, x)$, $t \geqq 0$, $x \in \mathbb{R}$, converges as $t \to +\infty$, to a function $h(x)$ and that (1.1) behaves asymptotically like (1.2).

Before describing what is going to be examined in this paper we present the following example:

Example. Consider the Volterra integral equation

$$x(t) = 2 \cos \sqrt{t} + \int_0^t e^{-(t-s)} g(s, x(s))\, ds, \quad t > 0 \tag{1.3}$$

where

$$g(t, x) = -2 + 5x - 4x \cos \frac{\sqrt{t}}{2} + \frac{3}{4\sqrt{t}} \sin \frac{\sqrt{t}}{2}, \quad t > 0, \quad x \in \mathbb{R}.$$

Observe that neither $g(t, x)$ nor $2 \cos \sqrt{t}$ converge as $t \to +\infty$, but they are slowly varying functions. Furthermore, the (unique) solution of (1.3) $x(t) = 1 + \cos(\sqrt{t}/2)$, $t \geqq 0$ is slowly varying.

This example (and many others) motivates us to look for slowly varying solutions of (1.1) in case where $F(t, z_1, z_2, \ldots, z_k, y)$ and $g(t, z)$ do not necessarily converge as $t \to +\infty$. In particular, we show that if these functions are slowly varying then all bounded solutions of (1.1) are so. Also we are able to prespecify the set which contains all limiting points of all such solutions. These results improve (partially or totally) and extend related results of Miller and Sell (1970), Londen (1974), Karakostas (1982), Grippenberg (1978).

Moreover, another concept is introduced here: the concept of almost slowly varying functions. Then, the results described above still hold if we replace the phrase "slowly" by "almost slowly".

It is, however, of interest that in our present analysis we use topological dynamics considerations applied to causal operator equations and especially the theory of limiting equations along solutions as is developed in [3].

2. THE MAIN RESULTS

First we present our basic assumptions and then the main results.

Assume that

(i) $F: [0, +\infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ *is continuous and uniformly continuous in the first variable and such that for each sequence* (t_n) *in* $[0, +\infty)$ *with* $t_n \to \infty$ *there are a subsequence* (t_m) *and a function* $F_0: \mathbb{R} \to \mathbb{R}$ *such that*

$$F(t_m + t, z_1, z_2, \ldots, z_k, y) \to F_0(y) \quad as \quad m \to +\infty,$$

uniformly for all $(t, z_1, z_2, \ldots, z_k, x)$ *in bounded subsets of* $[0, +\infty) \times \mathbb{R}^{k+1}$.

(ii) *For each* $i = 1, 2, \ldots, k$ *the delay* $\alpha_i(t)$ *is continuous and stays near to* t, *i.e. the function* $t - \alpha_i(t)$ *is bounded.*

(iii) $g: (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is continuous and uniformly continuous in the first variable and such that for each sequence* (t_n) *in* $(0, +\infty)$ *with* $t_n \to +\infty$ *there are a subsequence* (t_m) *and a function* $g_0: \mathbb{R} \to \mathbb{R}$ *such that*

$$g(t_m + t, y) \to g_0(y) \quad as \quad m \to +\infty,$$

uniformly for all (t, y) *in bounded subsets of* $[0, +\infty) \times \mathbb{R}$.

(iv) *The kernel* a *is in* $L^1([0, +\infty))$.

Remark 2.1. It is not hard to see that the convergence of F and g in (i), (iii) respectively could be expressed in other words as: $F(\cdot, z_1, \ldots, z_k, y)$ and $g(\cdot, y)$ are slowly varying functions uniformly for $(z_1, z_2, \ldots, z_k, y)$ and y in bounded subsets of $\mathbb{R}^{k+1}$ and $\mathbb{R}$, respectively. Moreover, the limiting functions F_0 and g_0 are continuous functions. This follows from the almost uniform convergence on bounded sets.

Remark 2.2. On the other hand, by the classical Ascoli's theorem, (i) and (iii) imply that $F(t, z_1, \ldots, z_k, y)$ and $g(t, y)$ are uniformly continuous and bounded functions on sets of the form $(0, +\infty) \times W$ and V, respectively, where $W \subseteq \mathbb{R}^{k+1}$ and $V \subseteq \mathbb{R}$ are bounded sets.

In the sequel we shall denote by $L(F)$ and $L(g)$ the sets of all functions of the form F_0 and g_0, respectively (appearing in (i), (iii)) for all sequences (t_n).

Theorem A. *Assume that* (i)–(iv) *hold and in addition to* (iv) *assume that a is nonincreasing on $[0, +\infty)$. If for each $F_0 \in L(F)$ and $g_0 \in L(g)$ the set $\{\xi \in \mathbb{R}: \xi = (g_0 \circ F_0)(\xi) \int_0^\infty a(s)\, ds\}$ is connected, then each bounded continuous solution x of* (1.1) *is slowly varying and satisfies*

$$[\liminf_{t \to +\infty} x(t), \limsup_{t \to +\infty} x(t)] \subseteq B\,, \tag{2.1}$$

where $B = \{\eta \in \mathbb{R}: \eta = F_0(g_0(\eta) \int_0^\infty a(s)\, ds),\ F_0 \in L(F)$ and $g_0 \in L(g)\}$.

Two significant implications of Theorem A are the following:

Corollary 1. *Under the conditions of Theorem* A, *if the set B consists of isolated points then any bounded continuous solution $x(t)$, $t \geqq 0$ of* (1.1) *converges as $t \to +\infty$ to a point of B.*

This follows by the continuity of the solution, since in this case the set of all limit points of $x(t)$ (as $t \to +\infty$) is a continuum of $\mathbb{R}$.

Corollary 2. *Under the conditions of Theorem* A, *if the set B is empty then there exist no bounded solutions of* (1.1).

Before presenting other results concerning (1.1) we introduce the following notion:

Definition. We shall say that a function $x: [0, +\infty) \to \mathbb{R}$ is *almost slowly varying* if for each sequence (t_n) in $[0, +\infty)$ with $t_n \to +\infty$ there exists a subsequence (t_m) such that for each $r > 0$,

$$\lim_{t \to +\infty} \sup_{0 \leqq s_1, s_2 \leqq r} \lim_{m \to \infty} |x(t_m + t + s_1) - x(t_m + t + s_2)| = 0\,.$$

It is not hard to see that if x is slowly varying function then it is almost slowly varying. The inverse is not true; indeed, the function

$$x(t) = \begin{cases} \sin t, & t \in [2^m\pi + \pi/2\,,\ (2^m + 1)\pi + \pi/2)\,, \quad m = 0, 1, 2, \ldots \\ 1\,, & \text{otherwise} \end{cases}$$

is almost slowly varying but not slowly varying.

Assume now the following:

(v) $F: [0, +\infty) \times \mathbb{R}^{k+1} \to \mathbb{R}$ *is continuous and uniformly continuous in the first variable and for each sequence* (t_n) *with* $t_n \to \infty$ *there exist a subsequence* (t_m) *and a function* $\bar{F}: [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *such that*

$$F(t_m + t, z_1, \ldots, z_k, y) \to \bar{F}(t, y) \quad \text{as} \quad m \to +\infty ,$$

uniformly for all $(t, z_1, \ldots, z_k, y)$ *in bounded subsets of* $[0, +\infty) \times \mathbb{R}^{k+1}$;

(vi) $g: [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ *is continuous and uniformly continuous in the first variable and such that for each sequence* (t_n) *with* $t_n \to +\infty$ *there exist a subsequence* (t_m) *and a function* $\bar{g}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that*

$$g(t_m + t, y) \to \bar{g}(t, y) \quad \text{as} \quad m \to +\infty ,$$

uniformly for all (t, y) in bounded subsets of $\mathbb{R} \times \mathbb{R}$.

We shall denote by $O(F)$ and $O(g)$ the sets of all functions of the form $\bar{F}$ and $\bar{g}$, respectively. Observe here that if F satisfies (i) (which is stronger than (v)) then $O(F) = L(F)$. Similarly for g.

Our next result on (1.1) states now as follows:

Theorem B. *Assume that* (ii), (iv), (v), (vi) *hold and, moreover, a is nonincreasing on* $[0, +\infty)$. *Assume also that each* $\bar{F} \in O(F)$ *satisfies* (i) *and each* $\bar{g} \in O(g)$ *satisfies* (iii) (*and so the sets* $L(\bar{F})$ *and* $L(\bar{g})$ *are nonempty*). *If each pair of functions* $F_0 \in L(O(F))$ *and* $g_0 \in L(O(g))$ *satisfy the conditions of Theorem* A, *then any bounded and uniformly continuous solution x of* (1.1) *is almost slowly varying and satisfies*

$$\lim_{t \to +\infty} \operatorname{dist}(x(t), C) = 0 \tag{2.2}$$

where $C = \{\eta \in \mathbb{R}: \eta = F_0(g_0(\eta) \int_0^\infty a(s)\, ds),\ F_0 \in L(O(F)),\ g_0 \in L(O(g))\}$.

The proofs of our results rely heavily on the theory of limiting equations which is found in Karakostas (1982), and will be given in Sections 4, 5 .

3. SOME AUXILIARY RESULTS

We state here some facts about shifting semi-flow and some auxiliary results needed for the proofs of the theorems.

Let C denote the set of all continuous functions $x: \mathbb{R}^+ \to \mathbb{R}$ endowed with the topology of the uniform convergence on compact sets. This is a metrizable topology but we shall not use any metric.

Let $x \in C$; for any $t \geqq 0$ the symbol $x_t(s) = x(t + s)$, $s \geqq 0$, defines a new function x_t which also belongs to C. Then $(t, x) \to x_t: \mathbb{R}^+ \times C \to C$ defines a semi-flow on C which is known as the *shifting semi-flow*, cf., e.g. [3]. It is easy to see that if x is a uniformly continuous and bounded function in C then the ω-limit set $\omega(x)$ with respect to the shifting semi-flow is nonempty, compact, connected and positively invariant. These properties are obviously guaranteed by Ascoli's Theorem and the uniform continuity of the function x. In this case the ω-limit set $\omega(x)$ of x is invariant, in the sense that the (original) semi-flow restricted to the set $\omega(x)$ can be extended to a flow in such a way that the full orbit of any function $y \in \omega(x)$ lies in $\omega(x)$. If $Q(y)$ is such a full orbit, then the function $Q(y)(t)$, $t \in \mathbb{R}$, say x^*, which is called a *full limiting function* of x, satisfies $x^*(t) = \lim x(t_k + t)$ for a certain sequence $\{t_k\}$ with $t_k \to \infty$ and uniformly for all t in compact sets. Notice that if y is a function on $\mathbb{R}$ to $\mathbb{R}$, then for any $t \in \mathbb{R}$ the symbol y_t will denote the function $y_t(s)$, $s \geqq 0$.

Another item we should notice is a characterization of a function as being slowly and almost slowly varying by the shifting semi-flow. This is given in the following lemma whose proof is evident.

Lemma 3.1. *Let x be a uniformly continuous and bounded function in C. Then*

(a) x is slowly varying if and only if each $x^* \in \omega^*(x)$ is a constant function.

(b) x is almost slowly varying if and only if each $x^* \in \omega^*(x)$ is a slowly varying function (and thus each $\bar{\bar{x}} \in \omega(\omega(x))$ is a constant function).

Now write (1.1) in the form

$$x = Tx \tag{3.1}$$

where $(Tx)(t) = F(t, x(\alpha_1(t)), \ldots, x(\alpha_k(t)), \int_0^t a(t-s)\, g(s, x(s))\, \mathrm{d}s)$, $t \geqq 0$, $x \in C$. Following Karakostas (1982) the translation of T along x by $\tau \geqq 0$ is the operator defined on $C(x(\tau)) =^{\text{def.}} \{\varphi \in C\colon \varphi(0) = x(\tau)\}$ by

$$(T_{\tau,x}\varphi)(t) = x(\tau) - (Tx)(\tau) + (T\mu_{\tau,x}\varphi)_\tau (t),$$

where $(\mu_{\tau,x}\varphi)(t) = x(t)$ if $t \leqq \tau$ and $= \varphi(t-\tau)$ if $t > \tau$.

A limiting equation of (1.1) along a solution x is any equation of the form

$$u = \bar{T}u \tag{3.2}$$

where $\bar{T} = \lim T_{t_n,x}$, for a sequence $t_n(\to +\infty)$, where the convergence is such that $\bar{T}x = \lim T_{t_n,x}x_n$ whenever $x_n(0) = x(t_n)$, and x_n converge to a certain function x in the topology of C. We say then that (3.2) is generated by (t_n).

The more general theory on the structure of the translations and on the skew semi-flow generated by the translations of operators and the shifting semi-flow involved are not needed here.

Observing now that the translation of T along a solution x by τ is given by

$$(T_{\tau,x}\varphi)(t) = F\Big(\tau + t, (\mu_{\tau,x}\varphi)_\tau(\alpha_1(\tau + t) - \tau), \ldots, (\mu_{\tau,x}\varphi)_\tau(\alpha_k(\tau + t) - \tau)$$
$$\int_{-\tau}^{0} a(t-s)\, g(\tau + s, x_\tau(s))\, \mathrm{d}s + \int_0^t a(t-s)\, g(\tau + s, \varphi(s))\, \mathrm{d}s\Big)$$

and following a computational routine we can show the following

Lemma 3.2. *Assume that* (ii), (iv), (v), (vi) *hold and, moreover, that $F(t, z_1, \ldots, z_k, y)$ and $g(t, y)$ are bounded on sets of the form $[0, +\infty) \times W$ and $(0, +\infty) \times V$, respectively, where $W \subseteq \mathbb{R}^{k+1}$ and $V \subseteq \mathbb{R}$ are bounded. Then the limiting equation* *of* (1.1) *along a uniformly continuous and bounded solution* x *is of the form*

$$y = \bar{F}\left(t, \int_{-\infty}^{t} a(t-s)\,\bar{g}(s, y(s))\,\mathrm{d}s\right), \quad t \in \mathbb{R}, \tag{3.3}$$

where $\bar{F} \in O(F)$ *and* $\bar{g} \in O(g)$.

Here let us recall the following result borrowed from Karakostas (1982).

Lemma 3.3. *Any limiting equation (of* (3.1)*) along a solution* x *is satisfied by at least one function of* $\omega^*(x)$.

It is also convenient to adapt Proposition 6.3 of [3] in our situation as follows:

Lemma 3.4. *Let* (t_n) *be a sequence such that* $t_n \to \infty$. *Assume also that the assumptions of Lemma* 3.2 *hold. If* x *is a uniformly continuous and bounded solution of* (1.1), *there is a subsequence* (t_m) *of* (t_n) *and a function* $\bar{x} \in \omega^*(x)$ *generated by* (t_m) *which satisfies a limiting equation, say* (3.3), *of* (1.1) *along* x *generated by the sequence* (t_m).

4. PROOF OF THEOREM A

Let $x(t)$, $t \geqq 0$ be a bounded solution of (1.1). Set

$$y(t) = \int_{0}^{t} a(t-s)\,g(s, x(s))\,\mathrm{d}s, \quad t \geqq 0$$

and observe that y is a uniformly continuous and bounded function. Then (1.1) and Remark 2.2 imply that also x is a uniformly continuous function.

Let x^* be a full limiting function of x. To show that x is slowly varying and satisfies (2.1) it is enough (by Lemma 3.1) to show that x^* is a constant function, say $x^* = p$, where $p \in B$.

By Lemmas 3.2, 3.3, 3.4 we get that x^* satisfies the equation

$$x^*(t) = F_0\left(\int_{-\infty}^{t} a(t-s)\,g_0(x^*(s))\,\mathrm{d}s\right), \quad t \in \mathbb{R} \tag{4.1}$$

for some $F_0 \in L(F)$ and $g_0 \in L(g)$ (see the lines before Theorem B).

Set

$$z(t) = \int_{-\infty}^{t} a(t-s)\,g_0(x^*(s))\,\mathrm{d}s, \quad t \in \mathbb{R} \tag{4.2}$$

and observe on the one hand that z is a full limiting function of y (i.e. $z \in \omega^*(y)$) and on the other hand that z satisfies

$$z(t) = \int_{-\infty}^{t} a(t-s)\,h(z(s))\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.3}$$

where $h = g_0 \circ F_0$.

Our next step is to show that the set $\alpha(z_0) \cup \omega(z_0)$ consists of constant functions.

Making some easy manipulations on (4.3) we obtain

$$(4.4) \qquad G(t; z) - G(\tau; z) = \frac{1}{2}\int_{\tau}^{t}\int_{-\infty}^{s} [h(z(s)) - h(z(r))]^2 \, da(s - r) \, ds$$

for all t, τ, where

$$G(t; z) = \int_0^{z(\tau)} h(u) \, du - \frac{1}{2}\int_{-\infty}^{t} a(t - s) \, h^2(z(s)) \, ds .$$

Let $v \in \alpha(z_0)$; then we can assume that v is defined on the whole $\mathbb{R}$ and $z(t_k + t) \to \to v(t)$, uniformly for t in compact sets, where (t_k) is a sequence such that $t_k \to -\infty$. Hence $G(t_k + t; v) \to G(t; v)$, uniformly for t in compact sets. Now by (4.4) we observe that the function $G(\cdot\,; v)$ is a constant and thus by (4.4) again, we get

$$\int_{\tau}^{t}\int_{-\infty}^{s} [h(v(s)) - h(v(r))]^2 \, da(s - r) \, ds = 0 , \quad t \geqq \tau$$

which implies that $h(v(s)) = h(v(r))$ for all s, r in $\mathbb{R}$. Since v satisfies also (4.3) we conclude that v is a constant function. The same reasoning applies also to $\omega(z_0)$ and so $\alpha(z_0) \cup \omega(z_0)$ contains only constant functions.

We claim that $\alpha(z_0) \cap \omega(z_0) \neq \emptyset$. Indeed, the sets $\alpha(z_0)$ and $\omega(z_0)$ are connected, compact and contain only constant functions. Thus we can write $\alpha(z_0) = [\xi_1, \xi_2]$ and $\omega(z_0) = [\eta_1, \eta_2]$, for some reals $\xi_1 \leqq \xi_2$ and $\eta_1 \leqq \eta_2$.

Assume that $\xi_2 < \eta_1$. By our assumption we have $A\,h(\xi) = \xi$ for all $\xi \in [\xi_1, \eta_2]$, where $A = \int_0^\infty a(s) \, ds$.

Let $p \in (\xi_2, \eta_1)$. Since h is continuous, we have $\alpha(h(z)_0) = h(\alpha(z_0))$ and therefore there is an $s_1 \in \mathbb{R}$ such that $z(s_1) = p$ and $h(z(s)) < h(z(s_1))$, $s < s_1$. Our purpose is to show that

$$(4.5) \qquad \frac{\xi_1}{A} = h(\xi_1) \leqq h(z(s)) \quad \text{for all} \quad s \leqq s_1 .$$

(recall that $\xi_1 = \liminf_{s \to -\infty} z(s)$).

Indeed, assume that (4.5) does not hold, so that there exists an $s_2 < s_1$ such that

$$(4.6) \qquad h(z(s)) < h(z(s_1)) , \quad s < s_1$$

and

$$(4.7) \qquad h(z(s_2)) < h(z(s)) , \quad s < s_2 .$$

From (4.3) and (4.6) we have

$$\dot{z}(s_1) = a(0) \, h(z(s_1)) - \int_{-\infty}^{s_1} h(z(s)) \, da(s_1 - s) > a(0) \, h(z(s_1)) - a(0) \, h(z(s_1)) = 0 ,$$

namely, z is strictly increasing in a neighborhood of s_1. This fact and (4.6) imply that $z(s) < z(s_1)$, $s < s_1$. By using (4.7) and following the same procedure with s_2 we obtain $z(s) \leqq z(s_2)$, $s \in [s_2, s_1]$. These two relations give $z(s_2) < z(s_1) < z(s_2)$, which is false. So, (4.5) is true.

Now, since $\zeta = A\, h(\zeta)$ for all $\zeta \in [\xi_1, \eta_2]$ we have the contradiction

$$z(s_1) = \int_{-\infty}^{s_1} a(s_1 - s)\, h(z(s))\, ds = \frac{1}{A} \int_{-\infty}^{s_1} a(s_1 - s)\, z(s)\, ds <$$
$$< \frac{1}{A} \int_{-\infty}^{s_1} a(s_1 - s)\, z(s_1)\, ds = z(s_1)\,.$$

So, there exists a $\xi \in \alpha(z_0) \cap \omega(z_0)$ and sequences $\{\tau_k\}$, $\{t_k\}$ such that $\tau_k \to -\infty$, $t_k \to \infty$, $z(\tau_k + \cdot) \to \xi$, $z(t_k + \cdot) \to \xi$. Then we see that $G(\tau_k; z)$ and $G(t_k; z)$ converge to $\int_0^\xi h(u)\, du - (1/2)\, h^2(\xi)\, A$. Thus, by (4.4) we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{s} [h(z(s)) - h(z(r))]^2\, da(s - r)\, ds = 0$$

which implies that $h(z(t))$, $t \in \mathbb{R}$ is constant. This in turn implies by (4.3) that z is a constant function, say ζ, which satisfies $\zeta = A\, h(\zeta)$. Then the full limiting function x^* of x, which by (4.1) is given by

$$x^* = F_0(\zeta)\,,$$

is a constant function, say $u \in \mathbb{R}^n$. Then, again by (4.1), we get

$$u = F_0(A\, g_0(u))\,,$$

i.e. $u \in B$, and the proof is complete.

5. PROOF OF THEOREM B

Let x be a uniformly continuous and bounded solution of (1.1). By Lemma 3.1 it is enough to show that each $x^* \in \omega^*(x)$ is slowly varying.

Indeed, let $x^* \in \omega^*(x)$. Then by (3.4) and Lemmas 3.2, 3.3 we conclude that x^* satisfies the limiting equation (3.3) for some $\bar{F} \in O(F)$ and $\bar{g} \in O(g)$.

Now observe that each $\bar{F}$ satisfies (i) and each $\bar{g}$ satisfies (iii). Thus, Theorem A is applicable to (3.3) since it can be written in the form

$$y(t) = F_1\left(t, \int_0^t a(t - s)\, \bar{g}(s, y(s))\, ds\right)$$

$$\text{where} \quad F_1(t, w) = \bar{F}\left(t, \int_{-\infty}^0 a(t - s)\, \bar{g}(s, x^*(s))\, ds + w\right).$$

Therefore

$$[\liminf_{t \to +\infty} x^*(t),\ \limsup_{t \to +\infty} x^*(t)] \subseteq B_1\,,$$

where $B_1 = \{\xi \in \mathbb{R}: \xi = F_0(g_0(\xi) \int_0^\infty a(s)\, ds),\ F_0 \in L(\bar{F}),\ g_0 \in L(\bar{g})\}$. Since $B_1 \subseteq C$, we get

$$\lim_{t \to +\infty} \operatorname{dist}(x^*(t), C) = 0\,,$$

which proves (2.2) because $\omega(\omega(x)) \subseteq \omega(x)$.

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