Czechoslovak Mathematical Journal

Pavol Marušiak

Oscillatory properties of solutions of nonlinear differential systems with deviating arguments

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 2, 223–231

Persistent URL: http://dml.cz/dmlcz/102086

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

OSCILLATORY PROPERTIES OF SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

PAVOL MARUŠIAK, Žilina

(Received July 2, 1984)

INTRODUCTION

In this paper we consider the nonlinear differential system with deviating arguments:

$$\text{(S)} \qquad \begin{aligned} y_i'(t) &= p_i(t)\, y_{i+1}(t)\,, \quad i = 1, 2, \ldots, n-2\,, \\ y_{n-1}'(t) &= p_{n-1}(t)\, f_{n-1}(y_n(h_n(t)))\,, \\ y_n'(t) &= -p_n(t)\, f_n(y_1(h_1(t)))\,. \end{aligned}$$

The following conditions are always assumed to be fulfilled:

(1) (a) $p_i\colon [0, \infty) \to [0, \infty)$, $i = 1, 2, \ldots, n$, are continuous functions and not identically zero on any subinterval of $[a, \infty) \subset [0, \infty)$; $\int^\infty p_i(t)\, dt = \infty$, $i = 1, 2, \ldots, n-1$.

(b) $h_i\colon [0, \infty) \to R$, $i = 1, n$, are continuous and $\lim_{t\to\infty} h_i(t) = \infty$;

(c) $f_i\colon R \to R$, $i = n-1, n$, $f_i(u) \,.\, u > 0$ fos $u \neq 0$, $f_i(u)$ are nondecreasing in u.

Definition 1. System (S) is called (α_{n-1}, α_n) *superlinear* if there are positive numbers α_{n-1}, α_n such that $\alpha_n \,.\, \alpha_{n-1} > 1$ and

$$\frac{|f_i(u)|}{|u|^{\alpha_i}} \geqq \frac{|f_i(v)|}{|v|^{\alpha_i}} \quad \text{for} \quad |u| > |v|\,, \quad u \,.\, v > 0\,, \quad i = n-1, n\,.$$

Denote by W the set of all solutions $y(t) = (y_1(t), \ldots, y_n(t))$ of the system (S) which exist on some ray $[T_y, \infty) \subset [0, \infty)$ and satisfy $\sup\{\sum_{i=1}^n |y_i(t)|\colon t \geqq T\} > 0$ for $T \geqq T_y$.

Definition 2. A solution $y \in W$ is called *oscillatory* if each of its components has arbitrarily large zeros. A solution $y \in W$ is called *nonoscillatory* (weakly nonoscillatory) if each of its components (at least one component, respectively) is eventually of a constant sign.

By Lemma 1 [4] it follows that every solution of (S) is either oscillatory or non-oscilatory.

Definition 3. We shall say that the system (S) has the property A, if every solution $y \in W$ is oscillatory for n even, while for n odd it is iether oscillatory or y_i ($i = 1, 2, \ldots, n$) tend monotonically to zero as $t \to \infty$.

The oscillation properties of two-dimensional nonlinear differential systems with deviating arguments were studied for example by Kitamura and Kusano [2, 3], Ševelo and Varech [5, 6, 7]. The oscillation results for n-dimensional systems were obtained by Foltynska and Werbowski, and by the present author [4].

In this paper we extend some results established in [7] to the system (S).

OSCILLATION THEOREMS

In what follows we shall use the following notations:

$$h_i^*(t) = \min \{h_i(t), t\}, \quad i = 1, n,$$

$$\gamma_i(t) = \sup \{s \geqq 0;\ t > h_i^*(s)\} \quad \text{for} \quad t \geqq 0, \quad i = 1, n,$$

$$\gamma(t) = \max \{\gamma_1(t), \gamma_n(t)\}.$$

Let $i_k \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, n - 1\}$, $t, s \in [a, \infty)$. We define: $I_0 = 1$,

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = \int_s^t p_{i_k}(x)\, I_{k-1}(x, s; p_{i_{k-1}}, \ldots, p_{i_1})\,\mathrm{d}x. \tag{2}$$

It is not difficult to verify that the following identities hold:

$$I_k(t, s; p_{i_k}, \ldots, p_{i_1}) = (-1)^k I_k(s, t; p_{i_1}, \ldots, p_{i_k}) = \tag{3}$$
$$= (-1)^k \int_t^s p_{i_1}(x)\, I_{k-1}(x, t; p_{i_2}, \ldots, p_{i_k})\,\mathrm{d}x, \quad k \in \{1, 2, \ldots, n - 1\}.$$

In the sequel we shall need the following lemmas [4; Lemma 2, Lemma 4].

Lemma 1. *Let* (1a)–(1c) *hold. Let* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S) *on the interval* $[a, \infty)$. *Then there exist an integer* $l \in \{1, 2, \ldots, n\}$, $n \equiv l \pmod 2$, *and a* $t_0 \geqq a$ *such that*

$$y_i(t)\, y_1(t) > 0 \quad on \quad [t_0, \infty) \quad for \quad i = 1, 2, \ldots, l, \tag{4}$$

$$(-1)^{n+i}\, y_i(t)\, y_1(t) > 0 \quad on \quad [t_0, \infty) \quad for \quad i = l + 1, \ldots, n. \tag{5}$$

Lemma 2. *Let* (1a)–(1c) *hold. Let* $y = (y_1, \ldots, y_n) \in W$ *be a solution on the interval* $[a, \infty)$. *Then the following relations hold*:

$$y_i(t) = \sum_{j=0}^{m} (-1)^j\, y_{i+j}(s)\, I_j(s, t; p_{i+j-1}, \ldots, p_i) + \tag{6}$$
$$+ (-1)^{m+1} \int_t^s y_{i+m+1}(x)\, p_{i+m}(x)\, I_m(x, t; p_{i+m-1}, \ldots, p_i)\,\mathrm{d}x$$

$$\text{for} \quad 0 \leqq m \leqq n - i - 2\,, \quad 1 \leqq i \leqq n - 2\,, \quad t, s \in [a, \infty)\,;$$

$$(7) \qquad y_i(s) = \sum_{j=0}^{n-i-1} (-1)^j\, y_{i+j}(t)\, I_j(t, s; p_{i+j-1}, \ldots, p_i) +$$

$$+ (-1)^{n-i} \int_s^t p_{n-1}(x)\, f_{n-1}(y_n(h_n(x)))\, I_{n-i-1}(x, s; p_{n-2}, \ldots, p_i)\, \mathrm{d}x$$

$$\text{for} \quad i = 1, 2, \ldots, n - 1\,, \quad t, s \in [a, \infty)\,.$$

The proofs of Lemma 1 and Lemma 2 are found in the paper [4].

Lemma 3. *Let* (1a)–(1c) *hold. Let* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S) *on the interval* $[a, \infty)$ *with* $y_1(t) > 0$ *for* $t \geqq a$.

Then there exist an integer $l \in \{1, 2, \ldots, n\}$, $l \equiv n \pmod 2$, *and a* $t_0 \geqq a$ *such that* (4), (5) *hold,*

$$(8) \qquad y_i(t) \geqq \int_{t_0}^t H_{i,l-1}(s, t_0)\, p_{n-1}(s)\, f_{n-1}(y_n(h_n(s)))\, \mathrm{d}s\,,$$

$$\text{for} \quad l \in \{2, 3, \ldots, n\}\,, \quad i = 1, 2, \ldots, l - 1\,, \quad t \geqq t_0\,,$$

where

$$(9) \qquad H_{i,l-1}(s, t_0) = \int_{t_0}^s I_{l-i-1}(t, x; p_i, \ldots, p_{l-2})\, p_{l-1}(x) \times$$

$$\times I_{n-l-1}(s, x; p_{n-2}, \ldots, p_l)\, \mathrm{d}x\,, \quad l \in \{2, 3, \ldots, n - 1\}\,, \quad s \geqq t_0\,,$$

$$(10) \qquad H_{i,n-1}(s, t_0) = I_{n-i-1}(t, s; p_i, \ldots, p_{n-2})\,, \quad l = n\,, \quad t_0 \leqq s \leqq t\,.$$

Proof. We put $m = l - i - 1$, $s = t_0$ in (6) and use (3), (4). Then we have

$$(11) \qquad y_i(t) = \sum_{j=0}^{l-i-1} y_{i+j}(t)\, I_j(t, t_0; p_i, \ldots, p_{i+j-1}) +$$

$$+ \int_{t_0}^t y_l(u)\, p_{l-1}(u)\, I_{l-i-1}(t, u; p_i, \ldots, p_{l-2})\, \mathrm{d}u \geqq$$

$$\geqq \int_{t_0}^t y_l(u)\, p_{l-1}(u)\, I_{l-i-1}(t, u; p_i, \ldots, p_{l-2})\, \mathrm{d}u \quad \text{for} \quad i = 1, 2, \ldots, l - 1\,, \quad t \geqq t_0\,.$$

On the other hand, we put $i = l$, $s = u$ in (7) and using (5) for $t \geqq u$ we then have

$$(12) \qquad y_l(u) = \sum_{j=0}^{n-l-1} (-1)^j\, y_{i+j}(t)\, I_j(t, u; p_{i+j-1}, \ldots, p_i) +$$

$$+ (-1)^{n-l} \int_u^t p_{n-1}(x)\, f_{n-1}(y_n(h_n(x)))\, I_{n-l-1}(x, u; p_{n-2}, \ldots, p_l)\, \mathrm{d}x \geqq$$

$$\geqq \int_u^t p_{n-1}(x)\, f_{n-1}(y_n(h_n(x)))\, I_{n-l-1}(x, u; p_{n-2}, \ldots, p_l)\, \mathrm{d}x\,.$$

Substituting (12) into (11), we get

$$y_i(t) \geqq \int_{t_0}^{t} (p_{l-1}(u)\, I_{l-i-1}(t, u; p_i, \ldots, p_{l-2}) \int_{u}^{t} p_{n-1}(x) f_{n-1}(y_n(h_n(x))) \times$$

$$\times I_{n-l-1}(x, u; p_{n-2}, \ldots, p_l)\, \mathrm{d}x)\, \mathrm{d}u =$$

$$= \int_{t_0}^{t} H_{i,l-1}(x, t_0)\, p_{n-1}(x) f_{n-1}(y_n(h_n(x)))\, \mathrm{d}x\,.$$

Let $l = n$. Put $t = t_0$, $s = t$ in (7) and use (3) and (4). We get

$$y_i(t) \geqq \int_{t_0}^{t} p_{n-1}(x)\, I_{n-i-1}(t, x; p_i, \ldots, p_{n-2}) f_{n-1}(y_n(h_n(x)))\, \mathrm{d}x \quad \text{for} \quad t \geqq t_0\,.$$

The proof of the lemma is complete.

Let us denote

$$\phi_n(t) = \int_{t}^{\infty} p_n(s)\, \mathrm{d}s\,,$$

$$J_{k,n}(t, t_0) = I_{n-1}(t, t_0;\ p_k, \ldots, p_{n-1})\,,$$

$$J_{k,l}(t, t_0) = \int_{t_0}^{t} H_{k,l-1}(s, t_0)\, p_{n-1}(s)\, \mathrm{d}s \quad \text{for} \quad l = 1, 2, \ldots, n-1\,.$$

Theorem 1. *Let there exist a continuous nondecreasing function g on $[a, \infty)$ such that*

(13) $$h_n(t) \leqq g(t)\,, \quad g(h_1(t)) \leqq t\,.$$

Let

(14) i) $f_n(u\,.\,v) \geqq K f_n(u) f_n(v)$ $(0 < K = \text{const.})$;

ii) $$\int_{0+}^{\alpha} \frac{\mathrm{d}x}{f_n(f_{n-1}(x))} < \infty\,, \quad \int_{0-}^{-\alpha} \frac{\mathrm{d}x}{f_n(f_{n-1}(x))} < \infty$$

for every constant $\alpha > 0$;

(15) $$\int_{\gamma(T)}^{\infty} p_n(t) f_n(J_{1,l}(h_1(t), T))\, \mathrm{d}t = \infty \quad \textit{for} \quad l = 2, 3, \ldots, n\,.$$

If n is odd, suppose in addition that for every constant $L > 0$,

(16) $$\int_{T}^{\infty} p_{n-1}(t)\, I_{n-2}(t) f_{n-1}(L \phi_n(h_n(t)))\, \mathrm{d}t = \infty\,.$$

Then the system (S) *has the property* A.

Proof. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S). Without loss of generality we may suppose that $y_1(t) > 0$, $y_1(h_1(t)) > 0$ for $t \geqq t_1 \geqq a$. Then the n-th equation of (S) implies that $y_n'(t) \leqq 0$ for $t \geqq t_1$ and it is not identically zero on any subinterval of $[t_1, \infty)$. Because $y_1(t) > 0$, $y_n'(t) \leqq 0$ for $t \geqq t_1$, then by Lemma 3, for $t \geqq t_2 \geqq t_1$ (4), (5) and (8) hold.

I. Let $l \in \{2, 3, \ldots, n\}$. For $i = 1$, $t_0 = t_2$, using the monotonicity of y_n, f_{n-1}, (13) and (3), we obtain from (8) that

$$(17) \qquad y_1(t) \geqq \int_{t_2}^{t} H_{1,l-1}(s, t_2)\, p_{n-1}(s) f_{n-1}(y_n(h_n(s)))\, ds \geqq$$
$$\geqq f_{n-1}(y_n(g(t)))\, J_{1,l}(t, t_2)\,, \quad t \geqq t_2\,.$$

Putting (17) into the n-th equation of (S) and then using (13), (14i), we get

$$(18) \qquad y_n'(t) = -p_n(t) f_n(y_1(h_1(t))) \leqq$$
$$\leqq -p_n(t) f_n(f_{n-1}(y_n(g(h_1(t))))\, J_{1,l}(h_1(t), t_2)) \leqq$$
$$\leqq -p_n(t) f_n(f_{n-1}(y_n(t))\, J_{1,l}(h_1(t), t_2)) \leqq$$
$$\leqq -K\, p_n(t) f_n(f_{n-1}(y_n(t))) f_n(J_{1,l}(h_1(t), t_2))$$

for $t \geqq t_3 = \gamma(t_2)$, $l = 2, 3, \ldots, n$.

Dividing (18) by $f_n(f_{n-1}(y_n(t)))$ and then integrating from t_3 to $u(\geqq t_3)$, we get

$$(19) \qquad \int_{t_3}^{u} \frac{y_n'(t)}{f_n(f_{n-1}(y_n(t)))}\, dt \leqq -K \int_{t_3}^{u} p_n(t) f_n(J_{1,l}(h_1(t), t_2))\, dt\,.$$

From (19) for $u \to \infty$ we obtain

$$K \int_{t_3}^{\infty} p_n(t) f_n(J_{1,l}(h_1(t), t_2))\, dt \leqq \int_0^{y_n(t_3)} \frac{dx}{f_n(f_{n-1}(x))} < \infty\,,$$

which contradicts (15).

Let $l = 1$ (n is odd). Then $y_1(t) \downarrow k$ as $t \uparrow \infty$, where $k \geqq 0$. We suppose that $k > 0$. If we put $i = 1$, $s = t_2$ in (7) and use (5), we have

$$(20) \quad y_1(t_2) \geqq \int_{t_2}^{t} p_{n-1}(x) f_{n-1}\, y_n(h_n(x))\, I_{n-2}(x, t_2; p_{n-2}, \ldots, p_1)\, dx \quad \text{for} \quad t \geqq t_2\,.$$

Integrating the n-th equation of (S) from t to ∞ and using $y_1(t) \geqq k$ for $t \geqq t_2$, we get

$$y_n(t) \geqq f_n(k) \int_t^{\infty} p_n(s)\, ds = L\phi_n(t)\,, \quad \text{where} \quad L = f_n(k) \neq 0\,.$$

Then in view of the monotonicity of y_n, f_{n-1} and (13), the inequaity (20) yields

$$y_1(t_2) \geqq \int_{t_2}^{t} p_{n-1}(x)\, I_{n-2}(x, t_2; p_{n-2}, \ldots, p_1) f_{n-1}(L\phi_n(h_n(x)))\, dx\,,$$

which contradicts (16) for $t \to \infty$.

Therefore $\lim_{t \to \infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$.

Remark. Theorem 1 extends the results of the author [4; Theorem 3], Kitamura and Kusano [3; Theorem 6], Ševelo and Varech [7; Theorem 1].

Theorem 2. *Suppose that* (14), (16) *hold and*

$$h_n(t) \leqq t\,, \quad h_1(t) \geqq t \quad on \quad [a, \infty)\,. \tag{21}$$

If

$$\int_T^\infty p_n(t)\, f_n(J_{1,l}(t, T))\, \mathrm{d}t = \infty \quad for \quad l = 2, 3, \ldots, n\,, \tag{22}$$

then the system (S) *has the property* A.

Proof. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S) such that $y_1(h_1(t)) > 0$ for $t \geqq t_1$. Proceeding in the same way as in the proof of Theorem 1 we get (4), (5), (7) and (8) for $t \geqq t_2 \geqq t_1$.

I. Let $l \in \{2, 3, \ldots, n\}$. For $i = 1$, $t_0 = t_2$, using (21) and the monotonicity of y_n, f_{n-1}, we obtain from (8) that

$$y_1(t) \geqq f_{n-1}(y_n(t))\, J_{1,l}(t, t_2)\,, \quad t \geqq t_2\,.$$

If we put the last inequality into the n-th equation, we get

$$y_n'(t) \leqq -p_n(t)\, f_n(y_1(t)) \leqq \tag{23}$$
$$\leqq -K\, p_n(t)\, f_n(f_{n-1}(y_n(t)))\, f_n(J_{1,l}(t, t_2)) \quad \text{for} \quad l = 2, 3, \ldots, n\,, \quad t \geqq t_2\,.$$

Dividing (23) by $f_n(f_{n-1}(y_n(t)))$ and then integrating from t_2 to $\tau \to \infty$ we get a contradiction to (22).

II. If $l = 1$ (n is odd) we proceed in the same way as in the case II of the proof of Theorem 1.

Theorem 3. *Let the system* (S) *be* (α_{n-1}, α_n) *superlinear. Let*

$$g_1(t) \leqq \min\{h_1(t), t\}\,, \quad h_n(t) \leqq t \quad on \quad [a, \infty)\,, \tag{24}$$

where g_1 *is an increasing function on* $[a, \infty)$ *and* $\lim_{t\to\infty} g_1(t) = \infty$.

Let

$$\int_a^\infty p_n(t)\, \mathrm{d}t < \infty\,, \tag{25}$$

$$\int_a^\infty J_{2,l}(g_1(t), a)\, p_1(g_1)(t))\, g_1'(t)\, f_{n-1}(K\, \phi_n(t))\, \mathrm{d}t = \infty \tag{26}$$
$$for\ any\ constant \quad K > 0\,, \quad l = 3, 4, \ldots, n\,.$$

In addition we suppose that a) *for n even,*

(27)
$$\int_a^\infty p_1(g_1(t))\, g_1'(t) \int_t^\infty p_{n-1}(x)\, f_{n-1}(K\, \phi_n(x))\, I_{n-3}(x, g_1(t); p_{n-2}, \ldots, p_2)\, \mathrm{d}x\, \mathrm{d}t = \infty$$
$$for\ any \quad K > 0\,;$$

b) *for n odd,* (16) *holds.*

Then the system (S) *has the property* A.

Proof. Let $y = (y_1, \ldots, y_n) \in W$ be a nonoscillatory solution of (S). Proceeding in the same way as in the proof of Theorem 1, we get (4), (5), (7) and (8). We suppose that $y_1(t) > 0$, $y_1(h_1(t)) > 0$ for $t \geqq T_1$. Integrating the n-th equation of (S) from $t(\geqq T_1)$ to τ, we get

$$y_n(\tau) - y_n(t) = -\int_t^\tau p_n(s)\, f_n(y_1(h_1(s)))\, ds\,,$$

and then for $\tau \to \infty$ we have

$$y_n(t) \geqq \int_t^\infty p_n(s) f_n(y_1(h_1(s)))\, ds\,, \quad t \geqq T_1\,. \tag{28}$$

I. Let $l \geqq 2$. Then y_1 is nondecreasing and therefore $y_1(h_1(t)) \geqq c$ for some $c > 0$ and $t \geqq T_2 \geqq T_1$. Using the fact that the system (S) is superlinear, we obtain

$$f_n(y_1(h_1(t))) \geqq \frac{f_n(c)}{c^{\alpha_n}} (y_1(h_1(t))) = c^{-\alpha_n} f_n(c)\, (y_1(h_1(t)))^{\alpha_n} \quad \text{for} \quad t \geqq T_3 \geqq T_2\,. \tag{29}$$

Combining (29) with (28) we get

$$y_n(t) \geqq c^{-\alpha_n} f_n(c) \int_t^\infty p_n(s)\, (y_1(h_1(s)))^{\alpha_n}\, ds\,, \quad t \geqq T_3\,. \tag{30}$$

Because $y_1(h_1(t)) \geqq c$ for $t \geqq T_2$, (28) implies

$$y_n(g_1(t)) \geqq f_n(c) \int_{g_1(t)}^\infty p_n(s)\, ds = M\, \phi_n(g_1(t))\,, \quad \text{where} \quad M = f_n(c)\,. \tag{31}$$

In view of (30), (24) and the monotonicity of y_n we have

$$y_n(g_1(t)) \geqq y_n(t) \geqq c^{-\alpha_n} M \int_t^\infty p_n(s)\, (y_1(h_1(s)))^{\alpha_n}\, ds\,. \tag{32}$$

Using the superlinearity of f_{n-1} and (31), we get

$$f_{n-1}(y_n(g_1(t))) \geqq \frac{f_{n-1}(M\, \phi_n(t))}{(M\, \phi_n(t))^{\alpha_{n-1}}} (y_n(g_1(t)))^{\alpha_{n-1}}\,. \tag{33}$$

a) Let $l \geqq 3$. We put $i = 2$, $T_3 = t_0$ in (8) and using the monotonicity of f_{n-1}, y_n and (24), we obtain

$$(34_l) \qquad y_2(t) \geqq \int_{T_3}^t H_{2,l-1}(s, T_3)\, p_{n-1}(s) f_{n-1}(y_n(h_n(s))) \geqq$$
$$\geqq f_{n-1}(y_n(t))\, J_{2,l}(t, T_3) \quad (l = 3, 4, \ldots, n-1)\,,$$

and

$$(34_n) \qquad y_2(t) \geqq \int_{T_3}^t I_{n-3}(t, s;\, p_2, \ldots, p_{n-2})\, p_{n-1}(s) f_{n-1}(y_n(h_n(s)))\, ds \geqq$$
$$\geqq f_{n-1}(y_n(t))\, J_{2,n}(t, T_3)\,.$$

Substituting (33) and (32) in (34), we get

$$y_2(g_1(t)) \geqq f_{n-1}(y_n(g_1(t)))\, J_{2,l}(g_1(t), T_3) \geqq$$

$$\geqq \frac{f_{n-1}(M\, \phi_n(t))}{(M\, \phi_n(t))^{\alpha_{n-1}}} \left(M\, c^{-\alpha_n} \int_t^\infty p_n(s)\, (y_1(g_1(s)))^{\alpha_n}\, \mathrm{d}s \right)^{\alpha_{n-1}} J_{2,l}(g_1(t), T_3) \geqq$$

$$\geqq f_{n-1}(M\, \phi_n(t))\, c^{-\alpha}(y_1(g_1(t)))^{\alpha}\, J_{2,l}(g_1(t), T_3)\,,$$

$$\text{where} \quad \alpha = \alpha_n \alpha_{n-1} > 1\,, \quad l = 3, 4, \ldots, n\,.$$

Multiplying the last inequality by $p_1(g_1(t))\,(y_1(g_1(t)))^{-\alpha}\, g_1'(t)$ and using the first equation of (S), we get

$$(35) \qquad \frac{y_1'(g_1(t))\, g_1'(t)}{(y_1(g_1(t)))^{\alpha}} \geqq c^{-\alpha} f_{n-1}(M\, \phi_n(t))\, J_{2,l}(g_1(t), T_3)\, p_1(g_1(t))\, g_1'(t)\,.$$

Integrating (35) from $T_4 = \gamma(T_3)$ to τ, we obtain

$$\frac{c^\alpha}{\alpha - 1} [y_1(g_1(T_3))]^{1-\alpha} \geqq \int_{T_4}^{\tau} J_{2,l}(g_1(t), T_3)\, p_1(g_1(t))\, g_1'(t)\, f_{n-1}(M\, \phi_n(t))\, \mathrm{d}t\,,$$

which contradicts (26) as $\tau \to \infty$.

Let $l = 2$. We put $i = 2$ in (7) and use (5), obtaining

$$(36) \quad y_2(t) \geqq \int_t^\tau p_{n-1}(x)\, f_{n-1}(y_n(h_n(x)))\, I_{n-3}(x, t;\, p_{n-2}, \ldots, p_2)\, \mathrm{d}x \quad \text{for} \quad \tau \geqq t\,.$$

Using the superlinearity of f_{n-1}, (24) and (30), we obtain

$$y_2(g_1(t)) \geqq \int_{g_1(t)}^{\tau} p_{n-1}(x) \frac{f_{n-1}(M\, \phi_n(x))}{(M\, \phi_n(x))^{\alpha_{n-1}}} (y_n(x))^{\alpha_{n-1}} \,.$$

$$.\, I_{n-3}(x, g_1(t);\, p_{n-2}, \ldots, p_2)\, \mathrm{d}x\,, \quad t \geqq T_3\,.$$

Multiplying the last inequality by $p_1(g_1(t))\, g_1'(t)$ and using the first equation of (S), (32) and (24), we get

$$(37) \quad y_1'(g_1(t))\, g_1'(t) \geqq p_1(g_1(t))\, g_1'(t) \int_t^\tau p_{n-1}(x)\, f_{n-1}(M\, \phi_n(x))\, c^{-\alpha}(y_1(g_1(x)))^{\alpha}\,.$$

$$.\, I_{n-3}(x, g_1(t);\, p_{n-2}, \ldots, p_2)\, \mathrm{d}x \geqq$$

$$\geqq c^{-\alpha}(y_1(g_1(t)))^{\alpha}\, p_1(g_1(t))\, g_1'(t) \int_t^\tau p_{n-1}(x)\, f_{n-1}(M\, \phi_n(x))\,.$$

$$.\, I_{n-3}(x, g_1(t);\, p_{n-2}, \ldots, p_2)\, \mathrm{d}x\,, \quad t \geqq T_3\,.$$

Let $g_1(t) \geqq T_3$ for $t \geqq T_4$. Multiplying (37) by $c^{\alpha}(y_1(g_1(t)))^{-\alpha}$ and then integrating from T_4 to u, we get

$$\frac{c^\alpha}{\alpha - 1} (y_1(g_1(T_4)))^{1-\alpha} \geqq \int_{T_4}^{u} (p_1(g_1(t))\, g_1'(t) \int_t^\tau p_{n-1}(x)\, f_{n-1}(x)\, f_{n-1}(M\, \phi_n(x))\,.$$

$$.\, I_{n-3}(x, g_1(t);\, p_{n-2}, \ldots, p_2)\, \mathrm{d}x)\, \mathrm{d}t\,,$$

which contradicts (27) as $u \to \infty$, $\tau \to \infty$.

II. Let $l = 1$ (n is odd). Then we proceed in the same way as in the proof of Theorem 1.

This completes the proof of the theorem.

References

[1] *Foltynska I., Werbowski J.:* On the oscillatory behaviour of solutions of system of differential equations with deviating arguments. Colloquia Math. Soc. J. B. 30, Qualitative theory of Diff. Eq. Szegéd, 1979, 243—256.

[2] *Kitamura Y., Kusano T.:* On the osillation of a class of nonlinear differential systems with deviating argument. J. Math. Annal. and Appl. *66,* 1978, 20—36.

[3] *Kitamura Y., Kusano T.:* Oscillation and a class of nonlinear differential systems with general deviating arguments. Nonlin. Analysis Theory, Math. & Apl. N 5, 1978, 537—551.

[4] *Marušiak P.:* On the oscillation of nonlinear differential systems with retarded arguments. Math. Slov. *34*, 1984, 73—88.

[5] *Varech N. V., Ševelo V. N.:* On the conditions of the oscillation of the solutions of differential systems with retarded arguments. Kačestvennye metody teorii differencial'nych uravnenij s otkloniajuščimsia argumentom, Kiev 1977, 26—44 (in Russian).

[6] *Ševelo V. N., Varech N. V.:* On some properties of solutions of a system of functional differential equations. Kačestvennoe isledovanie differencial'no-functional'nych uravnenij, Kiev, 1980, 153—171 (in Russian).

[7] *Ševelo V. N., Varech N. V.:* Oscillatory properties of solutions of some systems of functional-differential equations with general character of the deviation of the argument. Oscillyaciya i ustojčivosť rešenij diferencial'no-funkcional'nych uravnenij, Kiev, 1982, 128—137 (in Russian).

Author's address: 01 088 Žilina, Marxa-Engelsa 25, Czechoslovakia (Vysoká škola dopravy a spojov).