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INTEGRATION OF VECTOR-VALUED FUNCTIONS WITH RESPECT TO AN OPERATOR-VALUED MEASURE

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INTRODUCTION

The purpose of this paper is to develop an integration theory for the case of vector-valued functions with respect to an operator-valued measure. The idea of this type of integration has been introduced by several authors in [3], [5] and [11]. In all of these papers, either the integrands or the integrals or both have their values in normed spaces (particularly, Banach spaces in some cases). In this paper we consider the normed space valued integrands and locally convex space valued integrals, as in [3]. However, our idea of integrability is more general than that of [3] and generalizes [5] and [11] in a locally convex space setting.

In Section 1, we introduce the basic terminology. The concepts of total variation and semi-variation concerning an operator-valued measure are also presented here.

The theory of integration is developed in Section 2. Our integrability is defined by means of a linear functional approach in the sense of Pettis, as followed in [11] and [12]. The extension of the well-known Lebesgue dominated convergence theorem is also valid under an additional assumption.

In Section 3, the relationship between integrability with respect to a given measure and that with respect to its total variation is investigated.

The last section is concerned with the generalization of some results of [12],[11], [1], [6] and [2] on the representation of a weakly compact operator and the mapping properties of its representing measure in our setting.

1. NOTATIONS AND PRELIMINARIES

Throughout this paper, unless otherwise stated, τ is a δ-ring of subsets of a non-empty set T, that is, τ is a collection of subsets of T closed under relative complement, finite union and countable intersection. $C(\tau)$ is the σ-algebra of sets locally in τ. Let X be a normed linear space (n.l.s.) and Y a locally convex Hausdorff linear topological space (l.c.s.) generated by the family $\{q_\beta\}_{\beta\in J}$ of continuous semi-norms.

The scalar field of X and Y may be either the real or the complex numbers and is denoted by C. Let X' and Y' be the topological duals of X and Y, respectively, and $L(X, Y)$ the space of all continuous linear operators from X to Y, equipped with the topology of bounded convergence. The family of semi-norms

$$u \to \|u\|_\beta = \operatorname{Sup} \{q_\beta(u(x)) \colon \|x\| \leqq 1\}$$

generates the topology of bounded convergence on $L(X, Y)$ and under this topology $L(X, Y)$ becomes a l.c.s. [14].

Definition 1.1. An operator-valued measure $\mu \colon \tau \to L(X, Y)$ is an additive set function with

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

for all mutually disjoint sequences $\{E_n\} \subset \tau$ with $\bigcup_{n=1}^{\infty} E_n \in \tau$, the series being unconditionally convergent with respect to the topology of simple convergence.

Theorem 1.2. *If $\mu \colon \tau \to L(X, Y)$ is an operator-valued measure, then for each $x \in X$, the set function $\mu_x \colon \tau \to Y$, defined by $\mu_x(E) = \mu(E)\, x$ is a vector measure and conversely, if for each $x \in X$, $\mu(\cdot)\, x$ is a vector measure, then $\mu \colon \tau \to L(X, Y)$ is countably additive with respect to the topology of simple convergence in $L(X, Y)$.*

With the help of the above theorem, it can be easily proved that for each $y' \in Y'$, the set function $y'\mu \colon \tau \to X'$ defined by $(y'\mu)(E)\, x = y'(\mu(E)\, x)$ for each $E \in \tau$ is an X'-valued measure.

Definition 1.3. For each $\beta \in J$, we define the *β-variation of μ*, which is a non-negative, not necessarily finite, countably additive set function on $C(\tau)$, as

$$v_\beta(\mu, E) = \operatorname{Sup} \sum_{i=1}^{n} \|\mu(E \cap E_i)\|_\beta\,, \quad E \in C(\tau)$$

where the supremum is taken over all finite pairwise disjoint collections $\{E_i\} \subset \tau$.

For each $y' \in Y'$, we write $v(y'\mu, \cdot)$, the variation of $y'\mu$, as

$$v(y'\mu, E) = \operatorname*{Sup}_{i} \sum_{i=1}^{n} \|y'\mu(E \cap E_i)\|\,.$$

Definition 1.4. We define the *β-semi-variation of μ* as

$$\hat{\mu}_\beta(E) = \operatorname*{Sup}_{y' \leqq q_\beta} v(y'\mu, E)\,, \quad E \in C(\tau)\,,$$

which is non-negative and not necessarily finite.

Note that $\hat{\mu}_\beta(E) < \infty$ whenever $v(y'\mu, E) < \infty$ for each $y' \in Y'$ (cf. Lemma 1 of [6]).

2. INTEGRATION WITH RESPECT TO AN OPERATOR-VALUED MEASURE

In the remainder of this paper, μ is a fixed $L(X, Y)$-valued measure defined on τ with $v(y'\mu, E) < \infty$ for each $E \in \tau$ and $y' \in Y'$. Also the integrands are assumed to be measurable.

Definition 2.1. If $E \subset T$, then χ_E will always denote its characteristic function on T. By a τ-simple function f on T with values in X, we mean a function of the form

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

where $x_i \in X$, $E_i \in \tau$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, n$.

Definition 2.2. A function f: $T \to X$ is said to be *μ-integrable* if

(i) f is $y'\mu$-integrable (in the sense of [4]), and

(ii) for each $E \in C(\tau)$, there is an element $y_E \in Y$ such that

$$y'(y_E) = \int_E f(t)\, y'\mu(\mathrm{d}t) \quad \text{for each} \quad y' \in Y'.$$

If f is μ-integrable, we denote y_E by $\int_E f(t)\, \mu(\mathrm{d}t)$. It follows from Definition 2.2 that every simple function defined as in 2.1 is μ-integrable and the integral of such a function is given by

$$\int_E f(t)\, \mu(\mathrm{d}t) = \sum_{i=1}^{n} \mu(E \cap E_i)\, x_i.$$

Lemma 2.3. *If f: $T \to X$ is $y'\mu$-integrable, then $\|f\|$ is $v(y'\mu, \cdot)$-integrable.*

Proof. Since f is $y'\mu$-integrable in the sense of [4], there is a sequence $\{f_n\}$ of simple functions which converges to f $v(y'\mu, \cdot)$-a.e.. Then $\|f_n\|$ converges $v(y'\mu, \cdot)$-a.e. to $\|f\|$, which implies the result.

Lemma 2.4. *If f: $T \to X$ is $y'\mu$-integrable, then*

$$\left| \int_E f(t)\, y'\mu(\mathrm{d}t) \right| \leqq \int_E \|f(t)\|\, v(y'\mu, \mathrm{d}t)$$

for each $E \in C(\tau)$.

Proof. If f is $y'\mu$-integrable then by Lemma 2.3, $\|f\|$ is $v(y'\mu, \cdot)$-integrable. So, if $\{f_n\}$ is a defining sequence of simple functions for $y'\mu$-integrability of f, then $\|f_n\|$ is a defining sequence corresponding to the function $\|f\|$ and

$$\left| \int_E f_n(t)\, y'\mu(\mathrm{d}t) \right| \leqq \int_E \|f_n(t)\|\, v(y'\mu, \mathrm{d}t), \quad E \in C(\tau)$$

which yields the required inequality.

Theorem 2.5. *If f: $T \to X$ is a bounded μ-integrable function, then for each*

$\beta \in J$ *and* $E \in C(\tau)$,

$$q_\beta\left(\int_E f(t)\,\mu(\mathrm{d}t)\right) \leqq \|f\|_T\,\hat{\mu}_\beta(E)\,, \quad \text{where} \quad \|f\|_T = \operatorname*{Sup}_{t\in T}\|f(t)\|\,.$$

Proof. The theorem follows from the above lemma.

Theorem 2.6. *If* $f\colon T \to X$ *is* μ*-integrable, then the set function defined by*

$$\lambda(E) = \int_E f(t)\,\mu(\mathrm{d}t)$$

is a measure on $C(\tau)$.

Theorem 2.7. *Let* $g\colon T \to X$ *be a* μ*-integrable function with* $\lim_n \int_{E_n} \|g(t)\|\,.\,v(y'\mu, \mathrm{d}t) = 0$ *uniformly for* $y' \leqq q_\beta$ *for each* $\beta \in J$ *and* $E_n \searrow \emptyset$. *Let* $\{f_n\}$ *be a sequence of* μ*-integrable functions which converges pointwise to* f *on* T *and* $\|f_n(t)\| \leqq \|g(t)\|$ *for each* n. *Then* f *is* μ*-integrable whenever* Y *is sequentially complete.*

In this case

$$\int_E f(t)\,\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\,\mu(\mathrm{d}t)\,,$$

uniformly for each $E \in C(\tau)$.

Proof. By applying the dominated convergence theorem [4], we see that f is $y'\mu$-integrable and

$$\int_E f(t)\,y'\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\,y'\mu(\mathrm{d}t)$$

for each $E \in C(\tau)$.

For fixed $\varepsilon > 0$, let $F_n = \{t \in T\colon \|f(t) - f_n(t)\| > \varepsilon\|g(t)\|\}$ and $E_n = \bigcup_{k=n}^{\infty} F_k$. Then $\{E_n\}$ is a decreasing sequence of sets with $E_n \searrow \emptyset$.

Now for each $\beta \in J$,

$$q_\beta\left(\int_E f_n(t)\,(\mathrm{d}t) - \int_E f_m(t)\,\mu(\mathrm{d}t)\right) \leqq$$

$$\leqq \operatorname*{Sup}_{y'\leqq q_\beta}\left|\int_{E\sim E_n}(f - f_n)(t)\,y'\mu(\mathrm{d}t)\right| + \operatorname*{Sup}_{y'\leqq q_\beta}\left|\int_{E\cap E_n}(f - f_n)(t)\,y'\mu(\mathrm{d}t)\right| +$$

$$+ \operatorname*{Sup}_{y'\leqq q_\beta}\left|\int_{E\sim E_m}(f - f_m)(t)\,y'\mu(\mathrm{d}t)\right| + \operatorname*{Sup}_{y'\leqq q_\beta}\left|\int_{E\cap E_m}(f - f_m)(t)\,y'\mu(\mathrm{d}t)\right| \leqq$$

$$\leqq \varepsilon \operatorname*{Sup}_{y'\leqq q_\beta}\int_{E\sim E_n}\|g(t)\|\,v(y'\mu, \mathrm{d}t) + 2\operatorname*{Sup}_{y'\leqq q_\beta}\int_{E\cap E_n}\|g(t)\|\,v(y'\mu, \mathrm{d}t) +$$

$$+ \varepsilon \operatorname*{Sup}_{y'\leqq q_\beta}\int_{E\sim E_m}\|g(t)\|\,v(y'\mu, \mathrm{d}t) + 2\operatorname*{Sup}_{y'\leqq q_\beta}\int_{E\cap E_m}\|g(t)\|\,v(y'\mu, \mathrm{d}t)$$

for all n, m and $E \in C(\tau)$.

So $\{\int_E f_n(t)\,\mu(\mathrm{d}t)\}$ is Cauchy uniformly with respect to $E \in C(\tau)$. Since Y is sequentially complete, there is an element y_E in Y such that

$$y'(y_E) = y'\left(\lim_n \int_E f_n(t)\,\mu(\mathrm{d}t)\right) = \int_E f(t)\,y'\mu(\mathrm{d}t)\,.$$

Hence f is μ-integrable and $\int_E f(t)\,\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\,\mu(\mathrm{d}t)$.

Theorem 2.8. *If Y is sequentially complete and $\hat{\mu}_\beta(\cdot)$ is continuous at $\emptyset$ on $C(\tau)$ for each $\beta \in J$, then every bounded measurable function $f\colon T \to X$ is μ-integrable.*

Proof. Since f is a bounded measurable function, there is a sequence $\{f_n\}$ of simple functions such that $\{f_n\}$ converges pointwise to f on T and $\|f_n\|_T \leqq \|f\|_T$ for $n = 1, 2, \ldots$.

Let $\varepsilon > 0$ be fixed, $F_n = \{t \in T\colon \|f(t) - f_n(t)\| > \varepsilon\}$ and $E_n = \bigcup_{k=n}^{\infty} F_k$. So for each $y' \in Y'$ there exists a positive integer n_0 such that $v(y'\mu, E_n) < \varepsilon$ for all $n \geqq n_0$.

Then (we write $\|f\|_T = M$)

$$\int_E \|f(t) - f_n(t)\|\, v(y'\mu, \mathrm{d}t) \leqq$$

$$\leqq \int_{E\sim E_n} \|f(t) - f_n(t)\|\, v(y'\mu, \mathrm{d}t) + \int_{E\cap E_n} \|f(t) - f_n(t)\|\, (y'\mu, \mathrm{d}t) \leqq$$

$$\leqq \varepsilon v(y'\mu, E \sim E_n) + 2Mv(y'\mu, E \cap E_n) \leqq$$

$$\leqq \varepsilon(v(y'\mu, E \sim E_n) + 2M) \quad \text{for} \quad n \geqq n_0\,.$$

So f is $y'\mu$-integrable and $\int_E f(t)\,y'\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\,y'\mu(t)$ for each $y' \in Y'$.

Since $\hat{\mu}_\beta(\cdot)$ is continuous at $\emptyset$ on $C(\tau)$, there is a positive integer N such that $\hat{\mu}_\beta(E_n) < \varepsilon$ for $n \geqq N$ and therefore

$$q_\beta\left(\int_E f_n(t)\,\mu(\mathrm{d}t) - \int_E f_m(t)\,\mu(\mathrm{d}t)\right) \leqq$$

$$\leqq \varepsilon(\hat{\mu}_\beta(E \sim E_n) + 2M) + \varepsilon(\hat{\mu}_\beta(E \sim E_m) + 2M)$$

for all $n, m \geqq N$ and $E \in C(\tau)$.

This inequality establishes that f is μ-integrable.

Corollary 2.9. *Let $f\colon T \to X$ be a μ-integrable function such that* $\lim_n \int_E \|f(t)\|\,.$ $.\, v(y'\mu, \mathrm{d}t) = 0$ *uniformly for $y' \leqq q_\beta$, $E_n \searrow \emptyset$. If Y is sequentially complete, then $\phi\,.\,f$ is μ-integrable for every bounded scalar measurable function ϕ.*

Proof. Without loss of generality we may suppose that $|\phi(t)| \leqq 1$ for each $t \in T$ Since ϕ is scalar measurable, we can choose a sequence of scalar τ-simple functions $\{\phi_n\}$ which converges to ϕ on T, for which $|\phi_n(t)| \leqq 1$ for all $t \in T$.

For each n, let $E_n = \{t \in T\colon \|f(t)\| \leqq n\}$. If $f_n = f\chi_{E_n}$ then $\{f_n\}$ is a sequence of bounded integrable functions converging pointwise to f. So $\{\phi_n . f_n\}$ is a sequence of integrable functions which converges pointwise to $\phi . f$.

Moreover,

$$\|(\phi_n . f_n)(t)\| = |\phi_n(t)| \, \|f_n(t)\| \leqq \|f(t)\| .$$

Hence by applying Theorem 2.7 we see that $\phi . f$ is μ-integrable.

Theorem 2.10. *Let Y be sequentially complete and let $f: T \to X$ be $y'\mu$-integrable and such that $\lim_n \int_{E_n} \|f(t)\| \, v(y'\mu, \mathrm{d}t) = 0$ uniformly for each $y' \leqq q_\beta$ and $E_n \searrow \emptyset$. Then the following statements are equivalent:*

(i) *f is μ-integrable.*

(ii) *There is a sequence $\{f_n\}$ of bounded measurable functions which converges pointwise to f and for which $\{\int_E f_n(t) \, \mu(\mathrm{d}t)\}$ is Cauchy uniformly with respect to $E \in C(\tau)$.*

(iii) *There is a sequence $\{f_n\}$ of simple functions which converges pointwise to f and for which the integrals $\{\int_E f_n(t) \, \mu(\mathrm{d}t)\}$ form a Cauchy sequence uniformly with respect to $E \in C(\tau)$.*

(iv) *There is a sequence $\{f_n\}$ of simple functions which converges pointwise to f and for which the integrals $\int_E f_n(t) \, \mu(\mathrm{d}t)$, $n = 1, 2, \ldots$ are uniformly countably additive on $C(\tau)$.*

Proof. (i) $\Rightarrow$ (ii). If f is μ-inregrable then the sequence $\{f_n\}$ considered in Corollary 2.9 is a sequence of bounded measurable functions which satisfies the conditions given in Theorem 2.7 and (ii) follows.

It is easy to see that (ii) implies (iii).

(iii) $\Rightarrow$ (iv). Let $\{f_n\}$ be a sequence of simple functions which converges pointwise to f and for which $\{\int_E f_n(t) \, \mu(\mathrm{d}t)\}$ is Cauchy uniformly with respect to $E \in C(\tau)$. If $\lambda_n(E) = \int_E f_n(t) \, \mu(\mathrm{d}t)$ then $\{\lambda_n\}$ is a sequence of Y-valued measures for which $\lim_n \lambda_n(E)$ exists for each $E \in C(\tau)$ and which are μ-continuous. So by the Vitali-Hahn-Saks theorem [9], $\{\lambda_n\}$ is uniformly μ-continuous.

Let $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_j = \emptyset$ for $i \neq j$. If $F_j = \bigcup_{i=1}^{j} E_i$ then $\{F_j\}$ is an increasing sequence of sets with $\lim_j (E \sim F_j) = \emptyset$. So $q_\beta(\lambda_n(E \sim F_j)) \to 0$ uniformly (over n) as $j \to \infty$. This shows that $q_\beta(\lambda_n(E) - \sum_{i=1}^{j} \lambda_n(E_i)) \to 0$ as $j \to \infty$ uniformly on $C(\tau)$ for $n = 1, 2, \ldots$ and (iii) $\Rightarrow$ (iv) is proved.

(iv) $\Rightarrow$ (i). If λ_n are uniformly countably additive then by Theorem 3.9 of [9], λ_n are uniformly μ-continuous.

Let E_n be defined as in Theorem 2.8. For each $\varepsilon > 0$ and $\beta \in J$,

$$q_\beta\left(\int_E f_n(t) \, \mu(\mathrm{d}t) - \int_E f_m(t) \, \mu(\mathrm{d}t)\right) \leqq$$

$$\leq \operatorname*{Sup}_{y' \leq q_\beta} \int_{E \sim E_n} \|f(t) - f_n(t)\| \, v(y'\mu, \mathrm{d}t) + \operatorname*{Sup}_{y' \leq q_\beta} \int_{E \cap E_n} \|f(t)\| \, v(y'\mu, \mathrm{d}t) +$$

$$+ q_\beta\left(\int_{E \cap E_n} f_n(t)\, \mu(\mathrm{d}t)\right) + \operatorname*{Sup}_{y' \leq q_\beta} \int_{E \sim E_m} \|f(t) - f_m(t)\| \, v(y'\mu, \mathrm{d}t) +$$

$$+ \operatorname*{Sup}_{y' \leq q_\beta} \int_{E \cap E_m} \|f(t)\| \, v(y'\mu, \mathrm{d}t) + q_\beta\left(\int_{E \cap E_m} f_m(t)\, \mu(\mathrm{d}t)\right),$$

which shows that $\{\int_E f_n(t)\, \mu(\mathrm{d}t)\}$ is Cauchy for each $E \in C(\tau)$ and since Y is sequentially complete, f is μ-integrable.

Remark. For (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii), the sequential completeness is superfluous. The implication (iii) $\Rightarrow$ (i) shows that if f is integrable in the sense of Debieve [3] then it is also integrable in our sense. We give an example below to show that the converse is not true. Also we have established a relationship between our integrability and Dobrakov integrability [5]. This type of relationship has been studied by Swartz [15]. He has shown that the class of all integrable functions in the sense of [15] coincides with that of Dobrakov [5]. Our class of integrable functions forms a subclass of [15]. This is due to the fact that Swartz assumed the integrability of f with respect to $y'\mu$ in the sense of [5], whereas we assume the integrability of f with respect to $y'\mu$ in the sense of [4], which is a weaker condition. However, in [15], the domain of $y'\mu$ is a σ-algebra whereas in our case this is a δ-ring.

The following example enables us to get more integrable functions than Debieve.

Example. Let T be the set of all natural numbers, τ the σ-algebra of all subsets of T and $X = R$, the space of all real numbers, and $Y = c_0$. Let the set function μ be defined on τ with values in $L(X, Y)$ by $\mu(\{k\})\, x = xe_k$, where $x \in R$, $k \in T$ and $e_k = (0, 0, \ldots, 0, 1, 0, \ldots) \in c_0$, and let $\mu(E) = \sum_{k \in E} \mu(\{k\})$ for $E \in \tau$. Then μ is an operator-valued measure countably additive in the topology of simple convergence.

Let us define the function $f\colon T \to X$ by $f(k) = 1/k$ and the functions f_n by $f_n(k) = (1/k)\, \chi_{E_n}(k)$, where $E_n = \{1, 2, \ldots, n\}$, for all $k \in T$. Then $\{f_n\}$ is a sequence of $y'\mu$-integrable functions converging pointwise to f. So by Theorem 3 of [4] (p. 136), f is $y'\mu$-integrable and $\int_E f(t)\, y'\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\, y'\mu(\mathrm{d}t)$ for each $E \in \tau$. Since $\lim_n \int_T f_n(t)\, \mu(\mathrm{d}t)$ exists, it is clear that f is μ-integrable. But f is not μ-integrable in the sense of Debieve, since $\hat{\mu}(\cdot)$ is not continuous at $\emptyset$, as $\hat{\mu}(E) = 1$ for all $E \in \tau$ (cf. Proposition 6 [3]).

3. RELATION BETWEEN μ-INTEGRABILITY AND $v_\beta(\mu, \cdot)$-INTEGRABILITY

Definition 3.1. A function $f\colon T \to X$ is said to be *integrable with respect to a non-negative measure* v if there is a sequence of τ-simple functions $\{f_n\}$ converging to f ν-a.e. such that

$$\lim_{n,m} \int_E \|f_n(t) - f_m(t)\| \, \nu(\mathrm{d}t) = 0$$

for each $E \in C(\tau)$.

Theorem 3.2. *If f is $v_\beta(\mu, \cdot)$-integrable for each $\beta \in J$, then f is μ-integrable whenever Y is sequentially complete.*

Moreover,

$$v_\beta\left(\left(\int f(t)\, \mu(\mathrm{d}t), E\right)\right) \leqq \int_E \|f(t)\| \, v_\beta(\mu, \mathrm{d}t)$$

for each $E \in C(\tau)$ ($v_\beta(\cdot)$ denotes the total variation of the integral on the left hand side).

Proof. We shall first show that f is $y'\mu$-integrable for each $y' \in Y'$. For each $y' \in Y'$ there exists $M > 0$ such that $v(y'\mu, E) \leqq M v_\beta(\mu, E)$ for all $E \in C(\tau)$. So it is clear that f is $v(y'\mu, \cdot)$-integrable for each $y' \in Y'$ and therefore, there exists a Cauchy sequence $\{f_n\}$ of $v(y'\mu, \cdot)$-integrable simple functions converging to f $v(y'\mu, \cdot)$-a.e. for which

$$\lim_{n,m} \int_E \|f_n(t) - f_m(t)\| \, v(y'\mu, \mathrm{d}t) = 0 \, .$$

Thus f is $y'\mu$-integrable and $\int_E f(t)\, y'\mu(\mathrm{d}t) = \lim_n \int_E f_n(t)\, y'\mu(\mathrm{d}t)$.

Since for each $\beta \in J$ and $y' \leqq q_\beta$ we have $v(y'\mu, E) \leqq v_\beta(\mu, E)$, and since $\lim_{n,m} \int_E \|f_n(t) - f_m(t)\| \, v_\beta(\mu, \mathrm{d}t) = 0$, it is clear that $\{\int_E f_n(t)\, \mu(\mathrm{d}t)\}$ is Cauchy for each $E \in C(\tau)$. Hence f is μ-integrable.

If $\lambda(E) = \int_E f(t)\, \mu(\mathrm{d}t)$ then $y'\lambda(E) = \int_E f(t)\, y'\mu(\mathrm{d}t)$ and

$$q_\beta(\lambda(E)) \leqq \int_E \|f(t)\| \, v_\beta(\mu, \mathrm{d}t) \quad \text{for each} \quad \beta \in J \, .$$

So

$$v_\beta(\lambda, E) \leqq \operatorname*{Sup}_i \sum_{i=1}^{n} \int_{E \cap E_i} \|f(t)\| \, v_\beta(\mu, \mathrm{d}t) \leqq \int_E \|f(t)\| \, v_\beta(\mu, \mathrm{d}t) \, .$$

4. APPLICATION TO WEAKLY COMPACT OPERATORS

In this section we assume that T is a compact Hausdorff topological space and $C(\mathscr{B})$ is the smallest σ-algebra containing $\mathscr{B}$, where $\mathscr{B}$ denotes the δ-ring of all compact subsets of T. Let us recall that X is a normed linear space and Y is a locally convex Hausdorff linear topological space generated by the family of semi-norms $\{q_\beta\}_{\beta \in J}$. Let X'' and Y'' denote the biduals of X and Y, respectively.

Let $C(T, X)$ be the space of all continuous functions from T to X endowed with the topology $\mathscr{I}$ of the usual supremum norm. We shall write $C(T)$ in place of $C(T, X)$ when $X = C$.

Definition 4.1. A measure $\mu: C(\mathscr{B}) \to L(X, Y)$ is said to be *regular* if for each $\varepsilon > 0$ and $E \in C(\mathscr{B})$ there exists a compact set A and an open set B such that $A \subset E \subset B$ and $\hat{\mu}_\beta(B \sim A) < \varepsilon$ for all $\beta \in J$.

We recall that a linear operator is weakly compact if it maps bounded subsets into weakly relatively compact subsets.

Theorem 4.2. *Let a continuous linear operator U: $C(T, X) \to Y$ be weakly compact. Then there is a unique measure μ: $C(\mathscr{B}) \to L(X, Y)$ such that*

(i) *μ is X'-regular, that is, $y'\mu$ is regular for each $y' \in Y'$,*

(ii) *the set $Q = \{ \sum_{i \in I} \mu(E_i) x_i$, I finite, $E_i \in C(\mathscr{B})$ disjoint, $x_i \in X$, $\|x_i\| \leqq 1\}$ is weakly relatively compact,*

(iii) *every bounded Borel function defined on T is μ-integrable,*

(iv) *$Uf = \int_T f(t)\, \mu(\mathrm{d}t)$ for $f \in C(T, X)$,*

and

(v) *$U'y' = y'\mu$ for $y' \in Y'$.*

Conversely, if μ is an $L(X, Y)$-valued measure which satisfies (i), (ii) *and* (iii), *then* (iv) *defines a weakly compact operator which satisfies* (v).

Proof. Since the dual of $C(T, X)$ is isometrically isomorphic to rcab v$(C(\mathscr{B}), X')$, the space of all regular X'-valued measures of finite variations on $C(\mathscr{B})$, the equation

$$g''(m) = \int_T g(t)\, m(\mathrm{d}t)$$

defines an element of $C(T, X)''$ for each bounded Borel function g.

Now, if U: $C(T, X) \to Y$ is weakly compact then U'', the second adjoint of U, maps $C(T, X)''$ into Y. Let us define

$$\mu(E)\, x = U''(x\chi_E)'' \quad \text{for each} \quad E \in C(\mathscr{B}) .$$

It is clear that $\mu(E): X \to Y$ is linear for each $E \in C(\mathscr{B})$. Also for each $y' \in Y'$, $U'y' = \mu_{y'}$ is a measure in rcab v$(C(\mathscr{B}), X')$. If $y' \in Y'$ and $x \in X$ then

$$y'\, \mu(E)\, x = y'(U''(x\chi_E)'') = (U'y')\, (x\chi_E)'' = \mu_{y'}(E)\, x \quad \text{for each} \quad E \in C(\mathscr{B}) .$$

Thus $y'\mu \in$ rcab v$(C(\mathscr{B}), X')$ for each $y' \in Y'$.

For each $E \in C(\mathscr{B})$ and $\beta \in J$, $q_\beta(\mu(E)\, x) \leqq \hat{\mu}_\beta(E)\, \|x\|$ shows that $\mu(E): X \to Y$ is continuous. It is also clear that μ is countably additive and $U'y' = \mu_{y'} = y'\mu$, which implies that μ is X'-regular.

Let $V = \{f: f \in C(T, X)$ and $\|f\|_T \leqq 1\}$. Then V is a $\mathscr{I}$-bounded subset of $C(T, X)$ and V^0 is a neighbourhood of zero in $C(T, X)'$ with respect to the strong topology. Let $G = \{(\sum_{i \in I} x_i \chi_{E_i})''$: I finite, $E_i \cap E_j = \emptyset$, $i \neq j$, $E_i \in C(\mathscr{B})$, $\|x_i\| \leqq 1\}$. We shall show that $G \subset V^{00}$. Let $(\sum_{i\in I} x_i\chi_{E_i})'' \in G$. For each $f' \in V^0$, let $f' \leftrightarrow m$ where $m \in$ $\in \operatorname{rcabv}(C(\mathscr{B}), X')$. Then we have $|(\sum_{i\in I} x_i\chi_{E_i})''(f')| \leqq 1$. So $(\sum_{i\in I} x_i\chi_{E_i})'' \in V^{00}$ and consequently, $G \subset V^{00}$. Hence G is equicontinuous (Prop. 6, [10], p. 200 and therefore by (2a) of Theorem 9.3.1 of [8], Q is weakly relatively compact.

Let $g = \sum_{i=1}^{k} x_i\chi_{E_i}$ be any X-valued simple function defined on T. Then $U''g'' =$ $= \int_T g(t)\,\mu(\mathrm{d}t)$. Thus $y'(U''f'') = \int_T f(t)\,y'\mu(\mathrm{d}t)$ holds for every bounded Borel function f and for each $y' \in Y'$ and therefore it is μ-integrable.

Since U'' is the extension of U, $Uf = U''f = \int_T f(t)\,\mu(\mathrm{d}t)$ for all $f \in C(T, X)$.

As concerns the uniqueness of μ, suppose that μ_1 is another $L(X, Y)$-valued measure such that $\int_T f(t)\,\mu_1(\mathrm{d}t) = Uf$ for all $f \in C(T, X)$. Then for each $y' \in Y'$ we have

$$\int_T f(t)\,y'\mu(\mathrm{d}t) = \int_T f(t)\,y'\mu_1,(\mathrm{d}t)$$

for all $f \in C(T, X)$ and consequently, $y'\mu = y'\mu_1$. Hence $\mu = \mu_1$ since Y is a l.c.s..

Conversely, if μ satisfies (i), (ii) and (iii), then $U: C(T, X) \to Y$ defined by (iv) is a continuous linear operator, since for each $\beta \in J$,

$$q_\beta(Uf) = q_\beta\left(\int_T f(t)\,\mu(\mathrm{d}t)\right) \leqq \|f\|_T\,\hat{\mu}_\beta(T)\,.$$

Also $U'y' = y'\mu$ for each $y' \in Y'$, since μ is X'-regular.

To prove that U is weakly compact, let V be the closed convex balanced hull of Q. Then V^0 is a neighbourhood of zero in Y' with respect to the Mackey topology $\mathscr{T}(Y', Y)$. If $W = \{f \in C(T, X): \|f\|_T \leqq 1\}$, then W^0 is a neighbourhood of zero in $C(T, X')$ with respect to the $\mathscr{C}$-topology, where $\mathscr{C}$ is the collection of all bounded subsets of $C(T, X)$.

We shall show that U' is continuous with respect to the $\mathscr{T}(Y', Y)$-topology and the $\mathscr{C}$-topology. Let $y' \in V^0$. Then $|\langle y', \sum_{i\in I} \mu(E_i)\,x_i\rangle \leqq 1$ for all $\sum_{i\in I} \mu(E_i)\,x_i \in Q$.

Therefore, for each $f \in W$,

$$|\langle U'y', f\rangle| = |\langle y', Uf\rangle| = \left|\int_T f(t)\,y'\mu(\mathrm{d}t)\right| \leqq 1$$

which shows that $U'y' \in W^0$. So $U'(V^0) \subset W^0$ and consequently, U' is continuous with respect to the $\mathscr{T}(Y', Y)$-topology and the $\mathscr{C}$-topology. Hence by Theorem 9.3.1 of [8], U is weakly compact.

Theorem 4.3. *Let $U: C(T, X) \to Y$ be a continuous linear operator with μ its representing measure. If U is weakly compact then $\mu(E): X \to Y$ is weakly compact for each $E \in C(\mathscr{B})$.*

Conversely, if X' and X'' have the Radon-Nikodym property, $\hat{\mu}_\beta(\cdot)$ is continuous at $\emptyset$ and $\mu(E): X \to Y$ is a weakly compact operator for each $E \in C(\mathscr{B})$, then U is a weakly compact operator whenever Y is quasi-complete.

Proof. For the "necessary" part, we recall that $\mu(E)\, x = U''(x\chi_E)''$, where χ_E denotes the characteristic function of E. To complete the proof it is enough to show that for each $E \in C(\mathscr{B})$, the set $P_E = \{\mu(E)\, x: \|x\| \leqq 1\}$ is weakly relatively compact in Y.

If $V = \{m: m \in \operatorname{rcab} v(C(\mathscr{B}), X'),\ |m|\,(T) \leqq 1\}$ then V is a neighbourhood of zero in $\operatorname{rcab} v(C(\mathscr{B}), X')$, where $|m|\,(\cdot)$ denotes the total variation of m. Let $G_E = \{(x\chi_E)'': \|x\| \leqq 1,\ x \in X\}$, $E \in C(\mathscr{B})$. Then G_E is a set of continuous linear functionals defined on $\operatorname{rcab} v(C(\mathscr{B}), X')$.

Now, for $E \in C(\mathscr{B})$ and $x \in X$ with $\|x\| \leqq 1$ we have

$$|\langle (x\chi_E)'', m\rangle| = |m(E)\, x| \leqq 1 \quad \text{whenever} \quad m \in V.$$

This shows that $G_E \subset V^0$ and consequently, G_E is equicontinuous. Hence by Theorem 9.3.1 of [8], $\{U''(x\chi_E)'': \|x\| \leqq 1\}$ is weakly relatively compact, and therefore P_E is weakly relatively compact.

Conversely, suppose that X' and X'' have the Radon-Nikodym property and $\mu(E): X \to Y$ is a weakly compact operator for each $E \in C(\mathscr{B})$. Here $U: C(T, X) \to Y$ is defined by $Uf = \int_T f(t)\, \mu(dt)$ for all $f \in C(T, X)$. Since $C(T, X)' \cong \operatorname{rcab} v(C(\mathscr{B}), X')$ we have $U'y' \leftrightarrow \mu_{y'}$ for each $y' \in Y'$, where $\mu_{y'} \in \operatorname{rcab} v(C(\mathscr{B}), X')$. Let M be any equicontinuous subset of Y'. By Prop. 6 of [10], p. 200 there exists a neighbourhood V of zero in Y such that $M \subset V^0$. Let us consider the set $K = \{U'y': y' \in V^0\}$. Since $U'(M) \subset U'(V^0)$, it is enough to show that K is weakly relatively compact.

Now K is bounded since $\operatorname{Sup}_{y' \in V^0} |\mu_{y'}|\,(T) < \infty$. Since $\hat{\mu}_\beta(\cdot)$ is continuous at $\emptyset$ for each $\beta \in J$, Lemma 3.1 and Prop. 3.1 of [2] imply that $\{|\mu_{y'}|: y' \in V^0\}$ is uniformly countably additive. Also, since $\mu(E): X \to Y$ is weakly compact, $\{\mu(E)'\, y': y' \in V^0\} = \{\mu_{y'}(E): y' \in V^0\}$ is weakly relatively compact in X'. So by Prop. 3.1 of [2], K is weakly relatively compact in $\operatorname{ca}(C(\mathscr{B}), X')$. Thus, by Theorem 9.3.2 of [8], U is weakly compact.

Remark. This theorem generalizes Theorem 4.1 of [2] when Y is a l.c.s..

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