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TOURNAMENTS WITH THE SAME NEIGHBOURHOODS

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A *tournament* T is a directed graph in which every pair of vertices is joined by exactly one arc. If there is an arc from the vertex u to the vertex v in T, we write $(u, v) \in T$ and we say that u *dominates* v. The set of vertices dominated by u is denoted by $N_T(u)$ and the tournament induced on $N_T(u)$ is called the *neighbourhood of u*. The score of the vertex u is $|N_T(u)|$, the cardinality of $N_T(u)$. The score sequence of T is the sequence $(s_1, s_2, \ldots, s_{|T|})$ of scores of the vertices of T in non-decreasing order.

In 1963, A. A. Zykov [2] suggested a problem concerning the characterization of graphs with a constant neighbourhood. B. Zelinka [1] studied the tournament variant of this problem, namely:

Characterize the tournaments T with the property that there exists a tournament $\bar{T}$ such that $N_{\bar{T}}(u) \cong T$ for each vertex of $\bar{T}$, i.e. a tournament $\bar{T}$ with a constant neighbourhood T.

He obtained a partial solution of this problem. Denote by $T(n)$ the class of all tournaments with the following structure. For any tournament $T \in T(n)$ there exists an n-subset $S(T)$ of the set $\{1, 2, \ldots, 2n\}$ with the properties:

(i) $a + b \neq 2n + 1$ for any two elements a, b of $S(T)$,

(ii) the vertices of T can be labelled by the elements of $S(T)$ in such a way that for each arc $(u, v) \in T$ the labelling of v minus the labelling of u is congruent with an element of $S(T)$ modulo $2n + 1$.

B. Zelinka [1] proved that if $T \in T(n)$ then there exists a tournament $\bar{T}$ with the constant neighbourhood T.

Further, he expressed a conjecture that the converse assertion is true. This note disproves his conjecture.

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A tournament T is said to be *point-symmetric* if the automorphism group of T acts transitively on T. It is obvious that every point-symmetric tournament has a constant neighbourhood. We give an example of a point-symmetric tournament $\bar{T}$ on 21 vertices with a constant neighbourhood on 10 vertices.

The incidence matrix of $\overline{T}$ is given by the following matrix of order 21:

$$
\begin{matrix}
0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&1&1&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&1&1\\
0&0&0&1&1&1&0&0&0&0&0&1&0&0&1&0&1&1&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&1\\
1&0&0&0&0&1&1&0&1&0&0&0&0&0&1&1&0&1&1&1&1\\
1&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&1&1&1&1&0\\
1&1&1&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&1&1\\
0&1&1&1&1&1&1&0&1&1&1&0&0&0&1&0&0&0&0&0&0\\
1&1&1&1&0&1&1&0&0&1&1&1&0&0&0&0&1&0&0&0&0\\
1&0&1&1&1&1&1&0&0&0&1&1&1&0&0&0&0&0&1&0&0\\
1&1&1&1&1&0&1&0&0&0&0&1&1&1&0&0&0&0&0&0&1\\
1&1&0&1&1&1&1&1&0&0&0&0&1&1&0&1&0&0&0&0&0\\
1&1&1&1&1&1&0&1&1&0&0&0&0&1&0&0&0&1&0&0&0\\
1&1&1&0&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1&1&1&0&0&0\\
0&0&1&0&0&0&0&1&1&1&1&0&1&1&0&0&1&1&1&0&0\\
0&0&0&0&1&0&0&1&0&1&1&1&1&1&0&0&0&1&1&1&0\\
0&0&0&0&0&0&1&1&1&1&1&1&0&1&0&0&0&0&1&1&1\\
0&1&0&0&0&0&0&1&1&0&1&1&1&1&1&0&0&0&0&1&1\\
0&0&0&1&0&0&0&1&1&1&1&1&1&0&1&1&0&0&0&0&1\\
0&0&0&0&0&1&0&1&1&1&0&1&1&1&1&1&1&0&0&0&0
\end{matrix}
$$

The vertex-set of $\overline{T}$ is the set $\{1, 2, \ldots, 21\}$. It is easy to verify that two permutations

$$\alpha = (1\ 8\ 15)\ (2\ 9\ 16)\ (7\ 14\ 21)\,,$$

$$\beta = (1\ 7\ 6\ 5\ 4\ 3\ 2)\ (8\ 13\ 11\ 9\ 14\ 12\ 10)\ (15\ 18\ 21\ 17\ 20\ 16\ 19)$$

are automorphisms of $\overline{T}$ and generate a group which acts transitively on $\overline{T}$. Thus $\overline{T}$ has a constant neighbourhood T on 10 vertices with the score sequence (3, 3, 4, 4, 4, 4, 5, 5, 6, 7). The incidence matrix of T is given by the following matrix of order 10:

$$
\begin{matrix}
0&1&1&0&1&1&1&0&1&1\\
0&0&1&0&0&1&1&1&1&1\\
0&0&0&0&1&1&1&1&0&1\\
1&1&1&0&0&0&0&0&0&0\\
0&1&0&1&0&1&1&1&0&0\\
0&0&0&1&0&0&1&1&1&0\\
0&0&0&1&0&0&0&1&1&1\\
1&0&0&1&0&0&0&0&1&1\\
0&0&1&1&1&0&0&0&0&1\\
0&0&0&1&1&1&0&0&0&0
\end{matrix}
$$

By straightforward and tedious computation one can verify that T is not in $T(10)$, i.e. there is no labelling of T by numbers $1, 2, \ldots, 20$ with the properties (i) and (ii). By aid of computer even the following surprising result was obtained: there is no tournament in $T(10)$ with the score sequence $(3, 3, 4, 4, 4, 4, 5, 5, 6, 7)$, that is, the score sequence of T.

Since all tournaments with a constant neighbourhood known until now are point-symmetric, the following problem seems to be very interesting:

Problem. Does there exist a non-point-symmetric tournament with a constant neighborhood?

References

[1] *B. Zelinka:* Neighbourhood tournaments, (to appear).
[2] *A. A. Zykov:* Theory of graphs and its applications, Proc. Symp. Smolenice 1963 ed. M. Fiedler, Prague 1964, 164—165.

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