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HOMOMORPHISMS OF JORDAN ALGEBRAS AND HOMOMORPHISMS OF PROJECTIVE PLANES

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There is a well-known relation between Moufang planes and certain classes of Jordan algebras (as described e.g. by H. Freudenthal, T. A. Springer, N. Jacobson, F. Veldkamp). In this connection it is natural to ask what is the relation between homomorphisms of Moufang planes and homomorphisms of the corresponding Jordan algebras. This paper deals with this question. Our research uses the known result about projective plane homomorphisms and places of coordinate octonian algebras ([1], [3]). We prove in Theorem 1 and its Corollary that every Jordan homomorphism with $(1\,[ij])^\sigma = 1[ij]'$ implies a projective plane homomorphism. In Theorem 2 we start from a projective plane homomorphism θ, construct a mapping σ of subsets of Jordan algebras and derive the properties of σ. Theorem 3 is the converse of Theorem 2.

Let C be an octonion algebra (or Cayley division algebra) over a commutative field K with a characteristic $\neq 2, 3$. C is an alternative not necessarily associative algebra. We consider Jordan algebras $A = A(C, \gamma_i)$ of matrices

$$x = \begin{Vmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\bar{b} \\ \gamma_2^{-1}\gamma_1\bar{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\bar{a} & \alpha_3 \end{Vmatrix}, \tag{1}$$

where $\alpha_i \in K$, $a, b, c \in C$ (C is called the *coefficient algebra of* A), $\bar{a}$ denotes the conjugate element to a and γ_i are fixed elements $\neq 0$ in K. N. Jacobson proved ([4], p. 128) that there is no loss of generality in assuming that $\gamma_1 = 1$. The multiplication in A is the Jordan multiplication $x\,.\,y = \frac{1}{2}(xy + yx)$, where xy is the matrix product A is a commutative not necessarily associative algebra over K with a unit element e. Throughout the paper we shall assume that $i, j \in \{1, 2, 3\}$. As usual, let e_{ij} denote the matrix having 1 in the (i, j)-position and 0's elsewhere. Thus $e = \Sigma e_{ii}$. If we put

$$a[ij] = ae_{ij} + \gamma_j^{-1}\gamma_i\bar{a}e_{ji} \tag{2}$$

then (1) yields

$$x = \Sigma\alpha_i e_{ii} + a[23] + b[31] + c[12]\,. \tag{3}$$

If $t(x) = \Sigma\alpha_i$ is a trace in A, then $Q(x) = \frac{1}{2}\, t(x^2)$ is a quadratic form on A and $Q(x + y) - Q(x) - Q(y) = t(x \,.\, y) = t(x, y)$ is the corresponding bilinear form.

We note that this Jordan algebra A is an exceptional simple reduced Jordan algebra with the reducing set of primitive idempotents e_{11}, e_{22}, e_{33} ([4]).

Besides the ordinary product, a *crossproduct* in A is defined ([2], [7]):

$$x \times y = x \,.\, y - \tfrac{1}{2}\, t(y)\, x - \tfrac{1}{2}\, t(x)\, y - \tfrac{1}{2}\, t(x, y)\, e + \tfrac{1}{2}\, t(x)\, t(y)\, e\,. \tag{4}$$

This product is related to the symmetric trilinear form (x, y, z) on A defined by

$$(x, y, z) = t(x \times y, z)\,. \tag{5}$$

Inserting (4) we obtain

$$(x, y, z) = t(x \,.\, y \,.\, z) - \tfrac{1}{2}\, t(x)\, t(y \,.\, z) - \tfrac{1}{2}\, t(y)\, t(x \,.\, z) - \\ - \tfrac{1}{2}\, t(z)\, t(x \,.\, y) + \tfrac{1}{2}\, t(x)\, t(y)\, t(z)\,.$$

From the trilinear form (x, y, z) we determine a norm $n(x)$ on A by

$$(x, x, x) = 3\, n(x)\,. \tag{6}$$

An element $x \in A$ is said to be of ***rank one*** if $x \neq 0$ and $x \times x = 0$.

The following result is known ([9]):

Proposition 1. *If $x \in A$, then $x \times x = 0$ if and only if either x is a scalar multiple of a primitive idempotent or $x^2 = 0$.*

Recall that an *incidence structure* is an ordered triple (π, λ, I), where π and λ are non-empty sets of elements called *points* and *lines*, respectively, and I is a binary relation from π to λ called an *incidence relation.*

Denote

$$\Pi = \{x \in A;\ x \neq 0,\ x \times x = 0\}\,,$$

$$\langle \Pi \rangle = \{\langle x \rangle = K^* x;\ x \in \Pi,\ K^* = K \setminus \{0\}\}\,.$$

We now define an incidence structure $\mathfrak{T}_{A,\gamma_i} = (\pi, \lambda, I)$ by putting $\pi = \lambda = \langle \Pi \rangle$ and $x\, I\, y$ iff $t(x, y) = 0,\ x, y \in \Pi$.

The following proposition are known ([4], [7]):

Proposition 2. *If x and y are linearly independent elements of Π, then $x \times y \in \Pi$.*

Proposition 3. *If $x, y, z \in \Pi$, where x, y are linearly independent, then $t(x, z) = t(y, z) = 0$ if and only if z is a multiple of $x \times y$.*

A consequence of these propositions is that if x and y are linearly independent elements of Π, then $x \times y$ is either the line incident with the points x, y or the point of intersection of the lines x, y.

It can be easily proved that the incidence structure $\mathfrak{T}_{A,\gamma_i}$ is a projective plane ([4]). T. A. Springer showed that if Jordan algebras A and A' have isomorphic coefficient algebras C and C', then the incidence structures $\mathfrak{T}_{A,\gamma_i}$ and $\mathfrak{T}_{A',\gamma_i'}$ are isomorphic. Thus $\mathfrak{T}_{A,\gamma_i}$ depends only on the coefficient algebra C. This projective plane $\mathfrak{T}_{A,\gamma_i}$ is isomorphic to the plane introduced previously in a different manner by R. Moufang. The above construction of $\mathfrak{T}_{A,\gamma_i}$ has its origin in the starting ideas of P. Jordan ([5]) and H. Freudenthal ([2]) for K – the field of reals and was adapted by T. A. Springer ([6], [7], [8]) for an arbitrary field K of a characteristic $\neq 2, 3$. Recall that an octonion plane is a projective plane over an octonion algebra C with points (a, b), (m), (∞), lines $[m, c]$, $[a]$, $[\infty]$ and the incidence relation such that

(∞) lies on $[\infty]$ and $[a]$,

(m) lies on $[\infty]$ and $[m, c]$ and

(a, b) lies on $[a]$ and (a, b) lies on $[m, c]$ provided $b = ma + c$ with $a, b, c, m \in C$.

We shall denote this projective plane by P_C.

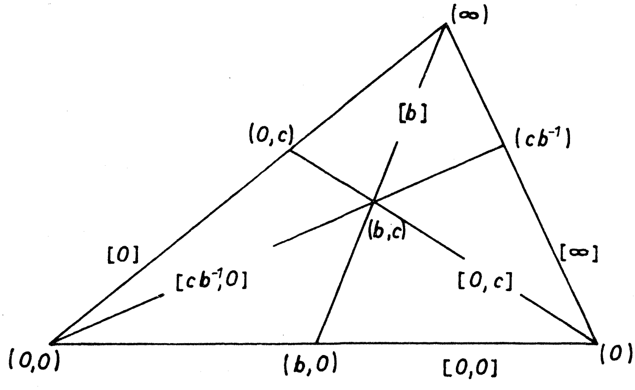
Now we shall find a convenient isomorphism between $\mathfrak{T}_{A,\gamma_i}$ and P_C. Choose in A a primitive idempotent $u = e_{11}$. Then in accordance with T. A. Springer ([7]) we associate with the point (b, c) the class $\langle x \rangle$ of all scalar multiples of the element $x \in A$ such that

$$(7) \qquad x = p(y) = u + \tfrac{1}{2}\, Q(y)\,(e - u) + y^2 - \tfrac{1}{2}\, Q(y)\,(e + u) + y = \\ = u - Q(y)\, u + y^2 + y\,,$$

where $y = b[31] + c[12]$. It can be verified that

$$(8) \qquad x = \begin{Vmatrix} 1 & c & \gamma_1^{-1}\gamma_3\bar{b} \\ \gamma_2^{-1}\gamma_1\bar{c} & \gamma_2^{-1}\gamma_1 c\bar{c} & \gamma_2^{-1}\gamma_3\bar{c}\bar{b} \\ b & bc & \gamma_1^{-1}\gamma_3 b\bar{b} \end{Vmatrix},$$

where $x \times x = 0$ or $x \in \Pi$. So we have $(b, c) \leftrightarrow \langle x \rangle$. By $\|(b, c)\|$ we shall denote a chosen matrix of the class $\langle x \rangle$. The line passing through the points (b_1, c_1) and (b_2, c_2) is associated with the matrix $\|(b_1, c_1)\| \times \|(b_2, c_2)\|$. In detail, we have



$$\|(b, c)\| = \left\| \begin{matrix} 1 & c & \gamma_1^{-1}\gamma_3\bar{b} \\ \gamma_2^{-1}\gamma_1\bar{c} & \gamma_2^{-1}\gamma_1 c\bar{c} & \gamma_2^{-1}\gamma_3\bar{c}\bar{b} \\ b & bc & \gamma_1^{-1}\gamma_3 b\bar{b} \end{matrix} \right\|$$

for all $b, c \in C$. This enables us to evaluate $\|(1, 1)\|$, $\|(0, 1)\|$, $\|(1, 0)\|$, $\|(0, 0)\| = e_{11}$. Using the incidence relation I we obtain

$$\|[0]\| = \|(0, 0)\| \times \|(0, 1)\| = e_{33}\,,$$

$$\|[0, 0]\| = \|(0, 0)\| \times \|(1, 0)\| = e_{22}\,.$$

For $b, c \in C$ ($b \neq 0$ and, if necessary, also $c \neq 0$) we have

$$\|[cb^{-1}, 0]\| = \|(0, 0)\| \times \|(b, c)\| = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & \gamma_1^{-1}\gamma_3 b\bar{b} & -\gamma_2^{-1}\gamma_3\bar{c}\bar{b} \\ 0 & -bc & \gamma_2^{-1}\gamma_1 c\bar{c} \end{matrix} \right\|,$$

$$\|[b]\| = \|(b, 1)\| \times \|(b, 0)\| = \left\| \begin{matrix} \gamma_1^{-1}\gamma_3 b\bar{b} & 0 & -\gamma_1^{-1}\gamma_3\bar{b} \\ 0 & 0 & 0 \\ -b & 0 & 1 \end{matrix} \right\|,$$

$$\|[0, c]\| = \|(0, c)\| \times \|(1, c)\| = \left\| \begin{matrix} \gamma_2^{-1}\gamma_1 c\bar{c} & -c & 0 \\ -\gamma_2^{-1}\gamma_1\bar{c} & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|,$$

$$\|(0)\| = \|[0, 0]\| \times \|[0, 1]\| = e_{33}\,,$$

$$\|(\infty)\| = \|[0]\| \times \|[1]\| = e_{22}\,,$$

$$\|[\infty]\| = \|(0)\| \times \|(\infty)\| = e_{11}\,,$$

$$\|(cb^{-1})\| = \|[\infty]\| \times \|[cb^{-1}, 0]\| = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & \gamma_2^{-1}\gamma_1 c\bar{c} & \gamma_2^{-1}\gamma_3\bar{c}\bar{b} \\ 0 & bc & \gamma_1^{-1}\gamma_3 b\bar{b} \end{matrix} \right\|,$$

$$\|[-cb^{-1}, c]\| = \|(0, c)\| \times \|(b, 0)\| =$$

$$= \left\| \begin{matrix} (b\bar{b})(c\bar{c}) & -\gamma_1^{-1}\gamma_2(b\bar{b})\,c & -(c\bar{c})\,\bar{b} \\ -(b\bar{b})\,\bar{c} & \gamma_1^{-1}\gamma_2 b\bar{b} & \bar{c}\bar{b} \\ -\gamma_3^{-1}\gamma_1(c\bar{c})\,b & \gamma_3^{-1}\gamma_2 bc & \gamma_3^{-1}\gamma_1 c\bar{c} \end{matrix} \right\|.$$

Putting $b = -m^{-1}c$ in the last expression we get $\|[m, c]\|$.

Various forms of $\mathfrak{T}_{A,\gamma_i}$ depend on the choice of the starting idempotent u, but all are isomorphic.

Let A and A' be Jordan algebras. The mapping $\sigma: A \to A'$ satisfying $(x + y)^\sigma = x^\sigma + y^\sigma$ is called

(i) *semilinear* if

(9) $$(\alpha x)^\sigma = \alpha^\varkappa x^\sigma \quad \text{for} \quad x \in A\,, \quad \alpha \in K\,, \quad \text{where} \quad \varkappa: K \to K' \quad \text{is an associated isomorphism}\,,$$

(ii) *a Jordan algebra homomorphism from A into A′ if*

$$(10)\qquad (x\,.\,y)^\sigma = x^\sigma\,.\,y^\sigma \quad \text{for all} \quad x, y \in A\,.$$

Now we shall prove the following

Lemma. *Let σ be a semilinear mapping of A into A' with an associated isomorphism $\varkappa\colon K \to K'$ satisfying $e^\sigma = e'$. Then σ is a Jordan algebra homomorphism if one of the two equivalent conditions is valid:*

$$(11)\qquad n(x^\sigma) = \varrho\, n(x)^\varkappa\,,$$

$$(12)\qquad x^\sigma \times y^\sigma = (x \times y)^\sigma\,, \quad \varrho \in K'^*\,.$$

Proof. If we put $x = e$ in (11) then we get $1 = n(e') = n(e^\sigma) = \varrho\, n(e)^\varkappa = \varrho$. Thus we have

$$(11')\qquad n(x^\sigma) = n(x)^\varkappa\,.$$

Using $(x, y, z) = t(x \times y, z)$ and $(x, x, x) = 3\, n(x)$ we obtain

$$(x, y, z) = \tfrac{1}{2}(n(x + y + z) - n(x + y) - n(x + z) - n(y + z) + {} \\ {} + n(x) + n(y) + n(z))\,.$$

Then (11′) implies

$$(13)\qquad (x^\sigma, y^\sigma, z^\sigma) = (x, y, z)^\varkappa\,.$$

Now $t(x) = t(e\,.\,x) = t(e \times e, x) = (e, e, x)$, $e^\sigma = e'$ and (13) yield

$$(14)\qquad t(x^\sigma) = t(x)^\varkappa\,.$$

Further $t(x \times y) = t(x \times y, e) = (x, y, e)$, (13) and (14) imply

$$t(x^\sigma \times y^\sigma) = (x^\sigma, y^\sigma, e^\sigma) = (x, y, e)^\varkappa = t(x \times y)^\varkappa = t((x \times y)^\sigma)$$

and since $t(x)$ is a nondegenerate form we have

$$(15)\qquad (x \times y)^\sigma = x^\sigma \times y^\sigma\,.$$

So we have proved that (11) implies (12).

It remains to show that σ is a Jordan algebra homomorphism. From the definition of the cross-product it follows that

$$(16)\qquad y \times e = \tfrac{1}{2}(t(y)\, e - y)\,.$$

Using the trilinear form we get $(y, y, e) = (y, e, y)$, $t(y \times y, e) = t(y \times e, y)$ or

$$(17)\qquad t(y \times y) = t((y \times e)\,.\,y)\,.$$

Then $t(y \times y)^\varkappa = t((y \times e)\,.\,y)^\varkappa$, (14) and (15) give

$$t(y^\sigma \times y^\sigma) = t((y \times e)\,.\,y)^\varkappa\,.$$

After arranging the left and the right hand sides of the last equation with help of (14)–(17) the left hand side assumes the form

$$t(y^\sigma \times y^\sigma) = t((y^\sigma \times e') \,.\, y^\sigma) = \tfrac{1}{2}t((t(y^\sigma)\, e' - y^\sigma) \,.\, y^\sigma) =$$
$$= \tfrac{1}{2}t(t(y^\sigma)\, y^\sigma - (y^\sigma)^2) = \tfrac{1}{2}(t(y^\sigma)\, t(y^\sigma) - t((y^\sigma)^2))$$

and the right hand side

$$t((y \times e) \,.\, y)^\varkappa = t(((y \times e) \,.\, y)^\sigma) = \tfrac{1}{2}t((t(y)\, y - y^2)^\sigma) =$$
$$= \tfrac{1}{2}t(t(y^\sigma)\, y^\sigma - (y^2)^\sigma) = \tfrac{1}{2}(t(y^\sigma)\, t(y^\sigma) - t((y^2)^\sigma))\,.$$

Thus we obtain

$$t((y^\sigma)^2) = t((y^2)^\sigma) = t(y^2)^\varkappa\,. \tag{18}$$

Hence we get that the quadratic form $Q(x) = \frac{1}{2}\, t(x^2)$ and the bilinear form $t(x, y) = Q(x + y) - Q(x) - Q(y)$ satisfy

$$\begin{cases} Q(x^\sigma) = Q(x)^\varkappa\,, \\ t(x, y)^\varkappa = t(x^\sigma, y^\sigma)\,. \end{cases} \tag{19}$$

Finally, the definition of the cross-product together with (15) and (19) gives the required relation $(x \,.\, y)^\sigma = x^\sigma \,.\, y^\sigma$.

Now we prove that (12) implies (11). The definition of the cross-product yields $e \times e = e$. Then (12) implies $e' = e^\sigma \times e^\sigma = \varrho e^\sigma = \varrho e'$ and $\varrho = 1$. Again using (16) we obtain $\frac{1}{2}(t(y)^\varkappa\, e^\sigma - y^\sigma) = (y \times e)^\sigma = y^\sigma \times e^\sigma = \frac{1}{2}(t(y^\sigma)\, e^\sigma - y^\sigma)$ and

$$t(y^\sigma) = t(y)^\varkappa\,. \tag{20}$$

Further, $(x, y, e) = t((x \times y) \,.\, e) = t(x \times y)$, (12) and (20) give $(x, y, e)^\varkappa = t(x \times y)^\varkappa = t((x \times y)^\sigma) = t(x^\sigma \times y^\sigma) = (x^\sigma, y^\sigma, e')$. We also know that $(x, y, e) = (x, e, y) = t((x \times e) \,.\, y) = \frac{1}{2}t((t(x)\, e - x) \,.\, y) = \frac{1}{2}t(t(x)\, y - x \,.\, y)$. Hence successively obtain

$$\tfrac{1}{2}t(t(x^\sigma)\, y^\sigma - x^\sigma \,.\, y^\sigma) = (x^\sigma, e', y^\sigma) = (x, e, y)^\varkappa =$$
$$= \tfrac{1}{2}t(t(x)\, y - x \,.\, y)^\varkappa = \tfrac{1}{2}t(t(x)^\varkappa\, y^\sigma - (x \,.\, y)^\sigma)\,,$$
$$t(x^\sigma)\, y^\sigma - x^\sigma \,.\, y^\sigma = t(x^\sigma)\, y^\sigma - (x \,.\, y)^\sigma\,,$$
$$(x \,.\, y)^\sigma = x^\sigma \,.\, y^\sigma\,,$$
$$(x^\sigma, y^\sigma, z^\sigma) = t((x^\sigma \times y^\sigma) \,.\, z^\sigma) = t((x \times y)^\sigma \,.\, z^\sigma) =$$
$$= t(((x \times y) \,.\, z)^\sigma) = t((x \times y) \,.\, z)^\varkappa = (x, y, z)^\varkappa\,.$$

Finally, $(x, x, x) = 3\, n(x)$ yields the required relation

$$n(x^\sigma) = n(x)^\varkappa\,.$$

N. Jacobson proved ([4], p. 130)

Theorem. *Let $A(C, \gamma_i)$ and $A'(C', \gamma_i')$ be Jordan algebras. Suppose η is an octonion* *algebra homomorphism of C into C' such that $\gamma_i^\eta = \gamma_i'$. Then the mapping $\sigma: A \to A'$ defined by $x^\sigma = \|x_{ij}^\eta\|$ ($x \in A$, $x_{ij} \in C$) is a Jordan algebra homomorphism satisfying $(1[ij])^\sigma = 1[ij]'$, where $a[ij] \in A$, $a'[ij]' \in A'$, $a \in C$, $a' \in C'$. Conversely, if $\sigma: A \to A'$ is a Jordan algebra homomorphism sytisfying $(1[ij])^\sigma = 1[ij]'$ then the mapping $\eta: C \to C'$ defined by $a^\eta[ij]' = (a[ij])^\sigma$ is an octonion algebra homomorphism satisfying $\gamma_i^\eta = \gamma_i'$.*

Let $\mathfrak{T} = (\pi, \lambda, I)$ and $\mathfrak{T}' = (\pi', \lambda', I')$ be two incidence structures. The mapping $\theta: \mathfrak{T} \to \mathfrak{T}'$ is called a *homomorphism* of $\mathfrak{T}$ into $\mathfrak{T}'$ if $\pi^\theta \subseteq \pi'$, $\lambda^\theta \subseteq \lambda'$ and $x\,I\,y$ implies $x^\theta\,I'\,y^\theta$.

Let $A(C, \gamma_i)$ and $A'(C', \gamma_i')$ be Jordan algebras with octonion coefficient algebras C, C' and let $\mathfrak{T}_{A,\gamma_i}$ and $\mathfrak{T}_{A',\gamma_i'}$ be the corresponding projective planes. Now we shall prove

Theorem 1. *Let $\sigma: A(C, \gamma_i) \to A'(C', \gamma_i')$ be a semilinear mapping such that*

$$(21) \qquad n(x^\sigma) = \varrho\, n(x)^\varkappa\,, \quad x \in A\,, \quad \varrho \in K'^*\,, \quad \varkappa: K \to K'$$

is an associated isomorphism ,

$$(22) \qquad (1[ij])^\sigma = 1[ij]'\,.$$

Then the mapping $\theta: \mathfrak{T}_{A,\gamma_i} \to \mathfrak{T}_{A',\gamma_i'}$ defined by $\|X^\theta\| = x^\sigma$, where X is a point or a line in $\mathfrak{T}_{A,\gamma_i}$ and $x \in A$ is its corresponding expression in A, is a projective plane homomorphism satisfying

$$(23) \qquad (0, 0)^\theta = (0, 0)'\,, \quad (1, 1)^\theta = (1, 1)'\,, \quad (0)^\theta = (0)'\,, \quad (\infty)^\theta = (\infty)'\,.$$

Proof. From (22) we get $e_{ii}^\sigma = e_{ii}'$, $i = 1, 2, 3$, and $e^\sigma = \Sigma e_{ii}^\sigma = \Sigma e_{ii}' = e'$. Now we can use Lemma, by which σ is a Jordan algebra homomorphism. Further, in accordance with Jacobson's theorem there exists an octonion algebra homomorphism $\eta: C \to C'$ with $a^\eta[ij] = (a[ij])^\sigma$ and $\gamma_i^\eta = \gamma_i'$. This all implies (23).

We recall that $(x, y, z) = t(x \times y, z)$ which implies that $x \in \Pi$ if and only if $x \neq 0$ and $(x, x, y) = 0$ for all $y \in A$. Now from $(x^\sigma, y^\sigma, z^\sigma) = (x, y, z)^\varkappa$ it is clear that $x \in \Pi$ if and only if $x^\sigma \in \Pi'$. Lemma also gives $(x \times y)^\sigma = x^\sigma \times y^\sigma$ which guarantees that θ reproduces the incidence or θ is a projective plane homomorphism. This completes the proof.

From Theorem 1 and Lemma we have the following

Corollary. *Let $\sigma: A(C, \gamma_i) \to A'(C', \gamma_i')$ be a Jordan algebra homomorphism such that $(1[ij])^\sigma = 1[ij]'$. Then the mapping $\theta: \mathfrak{T}_{A,\gamma_i} \to \mathfrak{T}_{A',\gamma_i'}$ defined by $\|X^\theta\| = x^\sigma$ (with X, x as in Theorem 1) is a projective plane homomorphism satisfying* (23).

Let $C_\infty = C \cup \{\infty\}$ and let us extend the ring structure on the octonion algebra C to C_∞ by setting $\infty^{-1} = 0$, $0^{-1} = \infty$, $a + \infty = \infty + a = \infty$ for $a \in C$ and $a\infty = \infty a = \infty$ for $0 \neq a \in C$. Similarly we extend another octonion algebra C'. Note that $\infty + \infty$, 0∞ and $\infty 0$ are not defined.

A *place* on C is a mapping φ of C_∞ into C'_∞ which satisfies

(i) $1^\varphi = 1$,

(ii) if $a^\varphi + b^\varphi$ is defined for $a, b \in C$, so is $a + b$ and $(a + b)^\varphi = a^\varphi + b^\varphi$,

(iii) if $a^\varphi b^\varphi$ is defined for $a, b \in C$, so is ab and $(ab)^\varphi = a^\varphi b^\varphi$.

Note that $(a^{-1})^\varphi = (a^\varphi)^{-1}$, $0^\varphi = 0$, $\infty^\varphi = \infty$. If $a^\varphi = \infty$, $b^\varphi \neq \infty$ then $(a \pm b)^\varphi = \infty$. Also $b^\varphi = 0$ is equivalent to $\bar{b}^\varphi = 0$, $c^\varphi = \infty$ is equivalent to $\bar{c}^\varphi = \infty$ and $(-c)^\varphi = \infty$.

V. Havel ([3]) and J. R. Faulkner - J. C. Ferrar ([1]) proved

Theorem. *Let θ: $P_C \to P_{C'}$ be a projective plane homomorphism satisfying*

(i) $(0, 0)^\theta = (0, 0)'$, $(0)^\theta = (0)'$, $(\infty)^\theta = (\infty)'$, $(1, 1)^\theta = (1, 1)'$.

Then there exists a place φ: $C_\infty \to C'_\infty$ (in [3], a place is called a pseudohomomorphism) with

(ii) $(m)^\theta = (m^\varphi)$,

(iii) $(a, b)^\theta = (a^\varphi, b^\varphi)$ *for* $a \neq \infty \neq b$, $a^\varphi \neq \infty \neq b^\varphi$,

(iv) $(a, b)^\theta = ((ba^{-1})^\varphi)$ *for* $a \neq \infty \neq b$, *but* $\infty \in \{a^\varphi, b^\varphi\}$,

(v) $[a]^\theta = [a^\varphi]$,

(vi) $[m, c]^\theta = [m^\varphi, c^\varphi]$ *for* $m \neq \infty \neq c$, $m^\varphi \neq \infty \neq c^\varphi$,

(vii) $[m, c]^\theta = [(-m^{-1}c)^\varphi]$ *for* $m \neq \infty \neq c$, *but* $\infty \in \{m^\varphi, c^\varphi\}$.

Conversely, every place φ: $C_\infty \to C'_\infty$ of octonion algebras with (ii)–(vii) *induces a homomorphism θ: $P_C \to P_{C'}$ of projective planes with* (i).

Now we shall prove

Theorem 2. *Let θ: $P_C \to P_{C'}$ be a projective plane homomorphism satisfying*

(24) $$(0, 0)^\theta = (0, 0)', \quad (1, 1)^\theta = (1, 1)', \quad (0)^\theta = (0)', \quad (\infty)^\theta = (\infty)'$$

and let φ: $C_\infty \to C'_\infty$ be the corresponding place in accordance with Havel-Faulkner-Ferrar's theorem. Then the mapping σ: $\Pi \to \Pi'$ constructed by

(25) $\|(c)\|^\sigma = \|(c)^\theta\| = \|(c^\varphi)\|$,

(26) $\|[b]\|^\sigma = \|[b]^\theta\| = \|[b^\varphi]\|$,

(27) $\|(b, c)\|^\sigma = \|(b, c)^\theta\| = \|(b^\varphi, c^\varphi)\|$ *for* $b, c, b^\varphi, c^\varphi \neq \infty$,

(28) $\|[m, c]\|^\sigma = \|[m, c]^\theta\| = \|[m^\varphi, c^\varphi]\|$ *for* $m, c, m^\varphi, c^\varphi \neq \infty$,

(29) $\|(b, c)\|^\sigma = \|(b, c)^\theta\| = \|((cb^{-1})^\varphi)\|$ *for* $b, c \neq \infty$, *but* $\infty \in \{b^\varphi, c^\varphi\}$,

(30) $\|[m, c]\|^\sigma = \|[m, c]^\theta\| = \|[(-m^{-1}c)^\varphi]\|$ *for* $m, c \neq \infty$, *but* $\infty \in \{m^\varphi, c^\varphi\}$

satisfies

(31) $$(\delta x)^\sigma = \delta^\varphi x^\sigma, \quad \textit{where} \quad \delta \in K^*, \quad \delta^\varphi \in K'^*, \quad x \in \Pi,$$

(32) $$x^\sigma \times y^\sigma = \varrho(x \times y)^\sigma \quad \textit{for} \quad x, y \in \Pi, \quad \varrho \in K'^*.$$

For every $X \in P_C$ we can choose such a matrix $x \in \Pi$ that

$$\|X^\theta\| = x^\sigma = \|x_{ij}\|^\sigma = \|x_{ij}^\varphi\|\,, \quad \textit{where all} \quad x_{ij}^\varphi \neq \infty\,. \tag{33}$$

If we extend the mapping σ on a subset B of A, where $B = \{y \in A;\ y_{ij}^\varphi \neq \infty\}$, by $y^\sigma = \|y_{ij}^\varphi\|$, then

$$x^\sigma + y^\sigma = (x + y)^\sigma \quad \textit{for} \quad x, y \in B\,, \tag{34}$$

$$(1[ij])^\sigma = 1[ij]'\,. \tag{35}$$

Proof. In accordance with T. A. Springer we associate with a point or a line $X \in P_C$ the class $\langle x \rangle$ of all scalar multiples of the element $x \in \Pi$, where x is a matrix satisfying $x = \|X\|$. Thus δx, $\delta \in K^*$, is also a matrix expression of X and for every $y \in \langle x \rangle$ we have $y^\sigma \in \langle x^\sigma \rangle$. Therefore $(\delta x)^\sigma = \delta' x^\sigma$, $\delta' \in K'^*$. Here it is sufficient to choose $\delta' = \delta^\varphi$ and we get (31).

From the assumption that θ is a projective plane homomorphism and from (24) we get $[0,0]^\theta = [0,0]'$, $[\infty]^\theta = [\infty]'$, $[0]^\theta = [0]'$, $[1,0]^\theta = [1,0]'$, $[0,1]^\theta = [0,1]'$, $[1]^\theta = [1]'$, $(1,0)^\theta = (1,0)'$, $(0,1)^\theta = (0,1)'$, $(1)^\theta = (1)'$. Then $(0,0)^\theta = (0,0)'$, $(\infty)^\theta = (\infty)'$ and $(0)^\theta = (0)'$ imply $e_{ii}^\sigma = e_{ii}'$.

Now $1^\varphi = 1$, $0^\varphi = 0$ and (31) imply (33) in the cases (27), (28) and in (25), (26) with $c^\varphi \neq \infty \neq b^\varphi$. So in these cases we get

$$x^\sigma + y^\sigma = \|x_{ij}^\varphi\| + \|y_{ij}^\varphi\| = \|x_{ij}^\varphi + y_{ij}^\varphi\| = \|(x_{ij} + y_{ij})^\varphi\| = (x + y)^\sigma\,.$$

Then

$$\begin{aligned} 1[12]' &= \|(0,1)'\| - \|(0,0)'\| - \gamma_2'^{-1}\gamma_1'\|(\infty)'\| = \\ &= \|(0,1)\|^\sigma - \|(0,0)\|^\sigma - (\gamma_2^{-1}\gamma_1\|(\infty)\|)^\sigma = (1[12])^\sigma\,. \end{aligned}$$

Similarly

$$1[31]' = \|(1,0)'\| - \|(0,0)'\| - \gamma_1'^{-1}\gamma_3'\|(0)'\| = (1[31])^\sigma\,,$$

$$1[32]' = \|(1)'\| - \gamma_2'^{-1}\gamma_1'\|(\infty)'\| - \gamma_1'^{-1}\gamma_3'\|(0)'\| = (1[32])^\sigma\,.$$

Using $1[ji] = \gamma_i^{-1}\gamma[ij]$ we get $(1[ij])^\sigma = 1[ij]'$ for all $i, j \in \{1, 2, 3\}$. Thus (35) holds.

Now we shall prove (33) in the remaining cases.

If in (25) $c^\varphi = \infty$ we choose

$$\begin{aligned} \|(c)\| &= (c\bar{c})^{-1}(\gamma_2^{-1}\gamma_1 c\bar{c} e_{22} + \gamma_1^{-1}\gamma_3 e_{33} + b[32]) = \\ &= \gamma_2^{-1}\gamma_1 e_{22} + \gamma_1^{-1}\gamma_3 (c\bar{c})^{-1} e_{33} + (c\bar{c})^{-1} b[32] \end{aligned}$$

and then $x^\sigma = \|(c)\|^\sigma = \|(c^\varphi)\| = \|x_{ij}^\varphi\| = \|(\infty)'\|$.

Similarly, if in (26) $b^\varphi = \infty$ we choose

$$\|[b]\| = (b\bar{b})^{-1}(\gamma_1^{-1}\gamma_3 b\bar{b} e_{11} + e_{33} - b[31])$$

and $x^\sigma = \|[b]\|^\sigma = \|[b^\varphi]\| = \|x_{ij}^\varphi\| = \|[\infty]'\|$.

In the case (29) for $b^\varphi = \infty$ we choose

$$\|(b, c)\| = (b\bar{b})^{-1}(e_{11} + \gamma_2^{-1}\gamma_1 c\bar{c}e_{22} + \gamma_1^{-1}\gamma_3 b\bar{b}e_{33} + c[12] + b[31] + bc[32])$$

and $x^\sigma = \|(b, c)\|^\sigma = \|((cb^{-1})^\varphi)\| = \|x_{ij}^\varphi\|$, $x_{ij}^\varphi \neq \infty$, for $c^\varphi = \infty$ we select

$$\|(b, c)\| = (c\bar{c})^{-1}(e_{11} + \gamma_2^{-1}\gamma_1 c\bar{c}e_{22} + \gamma_1^{-1}\gamma_3 b\bar{b}e_{33} + c[12] + b[31] + bc[32])$$

and we have again $x^\sigma = \|(b, c)\|^\sigma = \|x_{ij}^\varphi\|$, $x_{ij}^\varphi \neq \infty$.

In the case (30), substituting $b = -m^{-1}c$ we get $m = -cb^{-1}$ and

$$\|[m, c]\| = \|[-cb^{-1}, c]\| = \delta((b\bar{b})(c\bar{c})\, e_{11} + \gamma_1^{-1}\gamma_2 b\bar{b}e_{22} + \\ + \gamma_3^{-1}\gamma_1 c\bar{c}e_{33} + \bar{c}\bar{b}[23] - (c\bar{c})\,\bar{b}[13] - (b\bar{b})\,\bar{c}[21])\,.$$

The relations $m^\varphi = \infty$, $b^\varphi \neq 0$ imply $c^\varphi = \infty$, so that we choose $\delta = (c\bar{c})^{-1}$. Then

$$x^\sigma = \|[m, c]\|^\sigma = \|[(-m^{-1}c)^\varphi]\| = \|x_{ij}^\varphi\|\,, \quad x_{ij}^\varphi \neq \infty\,.$$

The relations $c^\varphi = \infty$, $b^\varphi = \infty$ imply $m = -cb^{-1} \neq \infty$, $m^\varphi \neq \infty$. Choosing $\delta = (c\bar{c})^{-1}(b\bar{b})^{-1}$ we have again

$$x^\sigma = \|x_{ij}^\varphi\|\,, \quad x_{ij}^\varphi \neq \infty\,.$$

Thus we have proved (33).

Now using (33) we can extend the mapping σ on a subset $B \subset A$ by $y^\sigma = \|y_{ij}^\varphi\|$, where $y \in B$, $B = \{y \in A;\ y_{ij}^\varphi \neq \infty\}$. Then it is clear that (34) is fulfilled.

The assumption that θ preserves the incidence and the construction of σ lead to $\|Z\| = \|X\| \times \|Y\|$, $\varrho\|Z^\theta\| = \|X^\theta\| \times \|Y^\theta\|$, $\varrho(\|X\| \times \|Y\|)^\sigma = \|X\|^\sigma \times \|Y\|^\sigma$, where X, Y are points (lines) and Z is their joining line (intersection point), $\varrho \in K'^*$. Therefore (32) is fulfilled. This completes the proof.

The last theorem shows that the suppositions of Theorem 1 are too strong. Therefore we now present a theorem with weaker conditions as a converse of Theorem 2.

Theorem 3. *Let $A(C, \gamma_i)$ and $A'(C', \gamma_i')$ be exceptional simple reduced Jordan algebras, Π and Π' sets of all their elements of rank one and $\sigma: \Pi \cup \{1[ij]\} \to \Pi' \cup \{1[ij]'\}$ a mapping satisfying*

$$(36)\qquad (\delta x)^\sigma = \delta' x\,, \quad \text{where} \quad x \in \Pi\,, \quad \delta \in K^*\,, \quad \delta' \in K'^*\,,$$

$$(37)\qquad x^\sigma \times y^\sigma = \varrho(x \times y)^\sigma\,, \quad x, y \in \Pi\,, \quad \varrho \in K'^*\,,$$

$$(38)\qquad (1[ij])^\sigma = 1[ij]'\,,$$

$$(39)\qquad x_1^\sigma + x_2^\sigma + \ldots + x_n^\sigma = (x_1 + x_2 + \ldots + x_n)^\sigma\,, \quad \text{where} \\ x_k \in \Pi \cup \{1[ij]\}\,, \quad k = 1, \ldots, n\,.$$

Then the mapping $\theta: \mathfrak{T}_{A,\gamma_i} \to \mathfrak{T}_{A',\gamma_i'}$ defined by $\|X^\theta\| = x^\sigma$, where X is a point or a line in $\mathfrak{T}_{A,\gamma_i}$ and $x \in \Pi$ is the corresponding matrix, is a projective plane homomorphism satisfying $(0, 0)^\theta = (0, 0)'$, $(0)^\theta = (0)'$, $(\infty)^\theta = (\infty)'$, $(1, 1)^\theta = (1, 1)'$.

Proof. Using our fundamental correspondence between X and $\langle x \rangle$ we verify by (36) that $y^\sigma \in \langle x^\sigma \rangle$ for all $y \in \langle x \rangle$.

From (38) we obtain $e_{ii}^\sigma = e'_{ii}$, which gives $(0, 0)^\theta = (0, 0)'$, $(\infty)^\theta = (\infty)'$, $(0)^\theta = (0)'$. Using

$$\|(1, 1)\| = 1[12] + 1[31] + 1[32] + e_{11} + \gamma_2^{-1}\gamma_1 e_{22} + \gamma_1^{-1}\gamma_3 e_{33},$$

with help of (36), (38) and (39) we deduce $(1, 1)^\theta = (1, 1)'$.

Finally, (37) guarantees that θ preserves the incidence and this completes the proof.

Theorems 2 and 3 are certain analogues of the fundamental theorem of projective geometry ([4], p. 406). Here we have investigated a projective plane homomorphism while in the fundamental theorem we have a projective isomorphism.

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