Antonín Vrba

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THE PERMANENT OF THE LAPLACIAN MATRIX OF A BIPARTITE GRAPH

ANTONÍN VRBA, Praha

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INTRODUCTION

Many interesting relations between combinatorial properties of a nondirected graph G and algebraic properties of subdeterminants, the characteristic polynomial, eigenvalues or eigenvectors of its Laplacian matrix $L(G)$ have been studied by W. N. Anderson, M. Fiedler, T. D. Morley, J. Sedláček, H. M. Trent, the author and others [1–7]. The permanent and the permanental characteristic polynomial of the Laplacian matrix have been investigated recently. At the Czechoslovak conference on combinatorics in 1979, R. Merris conjectured that per $L(G)$ attains its minimum over connected graphs G if and only if G is the star. This conjecture was later published in the survey [8] and in a short time affirmed by its author himself [9] in a rather complicated way. In the present paper three simple proofs of this conjecture are given as corollaries of Theorems 1, 2, 3. The first two theorems give lower bounds for per $L(G)$ in terms of the number of vertices and the number of edges of G. Theorem 3 expresses per $L(G)$ by means of a certain collection of subgraphs of a bipartite graph G and shows the combinatorial nature of per $L(G)$ in another way than it has been done for a tree in [9].

PRELIMINARIES

By a graph we understand a finite nondirected graph without loops or multiple edges. To avoid useless complications with singular cases let us consider only graphs without isolated vertices. We shall see that this restriction does not cause any loss of generality. Particularly, a single vertex is not considered to be a tree in this paper. The number of edges incident with a vertex is called the valence of the vertex.

A graph is called bipartite if its vertex set can be decomposed into two disjoint nonempty parts such that its edges do not connect vertices in the same part. It is well known that a graph is bipartite if and only if any of its circuits has even length. Apparently, trees are a special kind of bipartite graphs. Further, we shall call a graph monocyclic if it is connected and contains exactly one circuit. It is easy to see that a connected graph is monocyclic if and only if the number of vertices equals that of edges.

Let $G = (V, E)$ be a graph, $V = (1, 2, \ldots, v)$. We can assign to G the $v \times v$ matrix $L(G)$ whose off-diagonal entries are $a_{ik} = -1$ if $(i, k) \in E$ or 0 if $(i, k) \notin E$, and the diagonal entries a_{ii} are equal to the valence of the vertex i. This matrix, usually called the Laplacian matrix of G, is symmetric and singular (row sums are zero), $\det L(G) = 0$.

Let $A = (a_{ik})$ be an $n \times n$ matrix. The permanent of A is the sum

$$\operatorname{per} A = \sum_p \prod_{i=1}^{n} a_{ip(i)}$$

taken over all permutations of the indices $1, 2, \ldots, n$. By $|A|$ we mean the $n \times n$ matrix $(|a_{ik}|)$.

RESULTS

Proposition 1. *Let G be a graph and $d(G)$ the product of the valences of its vertices. Then*

$$\det |L(G)| + \operatorname{per} |L(G)| \geqq 2\, d(G)\,.$$

Equality is attained if and only if G is the star.

Proof. For any nonnegative $n \times n$ matrix M,

$$\det M + \operatorname{per} M = 2 \sum \prod_{i=1}^{n} m_{ip(i)}$$

where the summation is taken over all even permutations of the indices $1, 2, \ldots, n$. All the summands on the right hand side are nonnegative and one of them is the product of the diagonal entries $\prod_{i=1}^{n} m_{ii}$. Consequently,

$$\det M + \operatorname{per} M \geqq 2 \prod_{i=1}^{n} m_{ii}\,.$$

For $M = |L(G)|$ we obtain the desired inequality.

If G is the star on v vertices then $\det |L(G)| = 0$, $\operatorname{per} |L(G)| = 2(v-1)$, $d(G) = v - 1$ and equality is attained. If G is the triangle then $\det |L(G)| = 4$, $\operatorname{per} |L(G)| = 16$, $d(G) = 8$ and the inequality is strict. In all the other cases there exist two disjoint edges (p, q), (r, s) in G. The product of two transpositions $p \leftrightarrow q$, $r \leftrightarrow s$ is an even permutation, the corresponding term in $\det |L(G)|$ is positive and the inequality is strict.

Proposition 2. *Let G be a bipartite graph and M a principal submatrix of $L(G)$. Then*

$$\det |M| = \det M\,, \quad \operatorname{per} |M| = \operatorname{per} M\,.$$

Proof. Let $\emptyset \neq K \subset \{1, 2, \ldots, v\}$. Take a nonzero product $\prod_{i \in K} m_{ip(i)}$ where p is a permutation on K and decompose p into cycles. Cycles of length 1 correspond to the diagonal entries which are positive for both M and $|M|$. Every cycle of length greater than 1 has an even length because G is bipartite and so the corresponding product of an even number of nondiagonal entries is positive in both cases M, $|M|$.

Proposition 3. *Let $a_1, a_2, \ldots, a_k$ be positive integers, $\sum_{i=1}^{k} a_i = n$. Then*

$$\prod_{i=1}^{k} a_i \geqq n - k + 1$$

with equality if and only if $a_i = n - k + 1$ for one index i while the other a_i's are 1.

Proof. The assertion is a consequence of the inequality

$$(a + 1)(b - 1) < ab$$

which holds for any $a \geqq b \geqq 0$.

Proposition 4. *Let G be a graph and $K(G)$ the collection of all its subgraphs the connected components of which are circuits or single edges. (The empty subgraph also belongs to $K(G)$.) For each $H \in K(G)$ denote by $c(H)$ the number of circuits in H and by $d(H)$ the product of the valences of the vertices of G which are not contained in H. Then*

$$\operatorname{per} |L(G)| = \sum_{H \in K(G)} 2^{c(H)} d(H) .$$

Proof. The assertion follows immediately from the definitions of the permanent and of $L(G)$. Actually, every permutation can be decomposed into cycles and so each term of the permanent is a product of cyclic products. The nonzero cyclic products of per $|L(G)|$ mutually correspond:

(1) to circuits in G (each circuit to two oppositely directed cycles with product 1),

(2) to edges of G (each edge to one cycle of length 2 with product 2),

(3) to vertices of G (each vertex to one cycle of length 1 with product equal to the diagonal entry, i.e. to the valence of the vertex).

Theorem 1. *Let G be a bipartite graph with v vertices and e edges. Then*

$$\operatorname{per} L(G) \geqq 3e - v + 1 .$$

Equality is attained if and only if G is the star.

Proof. Let $G = (V, E)$ and let d_i denote the valence of the vertex i. The empty subgraph as well as the subgraphs consisting of one edge belong to the collection $K(G)$ introduced in Proposition 4. According to Propositions 2 and 4,

$$\operatorname{per} L(G) = \operatorname{per} |L(G)| \geqq \prod_{i=1}^{v} d_i \Big(1 + \sum_{(j,k) \in E} d_j^{-1} d_k^{-1}\Big) \geqq \prod_{i=1}^{v} d_i + e .$$

Obviously, $\sum_{i=1}^{v} d_i = 2e$ and with respect to Proposition 3,

$$\prod_{i=1}^{v} d_i \geqq 2e - v + 1$$

and so

$$\text{per } L(G) \geqq 3e - v + 1 .$$

Equality is attained if and only if it is attained in all the preceding inequalities, i.e. for the star.

Theorem 2. *Let G be a bipartite graph with v vertices and e edges. Then*

$$\text{per } L(G) \geqq 2(2e - v + 1) .$$

Equality is attained if and only if G is the star.

Proof. We have already mentioned that $\det L(G) = 0$. According to Propositions 1 and 2,

$$\text{per } L(G) = \text{per } L(G) + \det L(G) = \text{per } |L(G)| + \det |L(G)| \geqq 2\, d(G)$$

where $d(G)$ is the product of valences of all vertices. With respect to Proposition 3,

$$d(G) \geqq 2e - v + 1$$

which gives the desired inequality with equality for the star.

Proposition 5. *Let $G = (V, E)$ be a graph, $V = (1, 2, \ldots, v)$, $E = (w_1, w_2, \ldots, w_e)$. Let U be a $v \times e$ matrix whose (i, k)-entry is 1 if $i \in w_k$ and 0 otherwise. Then*

$$L(G) = UU^T .$$

Proof. The scalar product of the i-th and k-th rows of U, $i \neq k$, is 0 for $(i, k) \notin E$ and 1 for $(i, k) \in E$. If $i = k$ then this scalar product is equal to the number of 1's in the i-th row of U, i.e. to the valence of the vertex i.

Proposition 6 (Binet-Cauchy formula for permanents). *Let us denote by $G_{k,n}$ the set of all nondecreasing ordered k-tuples of integers $1, 2, \ldots, n$, $G_{k,n} = \{(t_1, t_2, \ldots, t_k) | 1 \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_k \leqq n\}$. For each k-tuple $T \in G_{k,n}$ let $m_j(T)$ denote the number of occurrences of the number j in T and $m(T) = \prod_{j=1}^{n} m_j(T)!$.*

Let $A = (a_{ik})$ be a $p \times q$ matrix, $U = (u_1, \ldots, u_p) \in G_{r,p}$, $V = (v_1, \ldots, v_s) \in G_{s,q}$. Then $A(U, V)$ denotes the $r \times s$ matrix whose (i, k)-entry is $a_{u_i v_k}$.

If B and C are $k \times n$ and $n \times k$ matrices and $K = (1, 2, \ldots, k)$ then

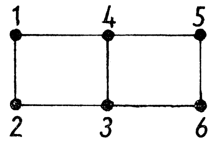
$$\text{per } BC = \sum_{T \in G_{k,n}} \frac{1}{m(T)} \text{per } B(K, T) \text{ per } C(T, K) .$$

Proof. See [10] or [11].

Theorem 3. *Let G be a bipartite graph and let $S(G)$ denote the collection of its* *subgraphs H such that H covers all vertices of G and each connected component of H is either a tree or a monocyclic graph. To a tree with h edges let us assign the weight 2h and to a monocyclic graph the weight 4. Further, to each $H \in S(G)$ assign the weight $w(H)$ equal to the product of the weights of its connected components. Then*

$$\operatorname{per} L(G) = \sum_{H \in S(G)} w(H) .$$

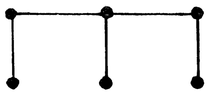
Example 1. Let G be the graph



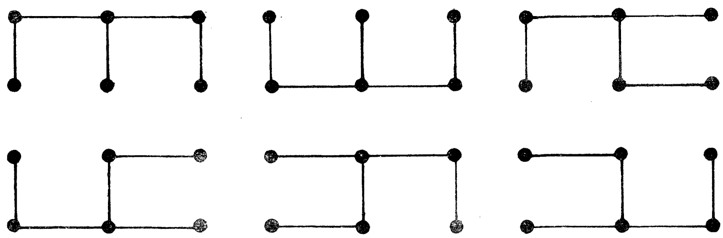
We have

$$L(G) = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

and $\operatorname{per} L(G) = 410$. The expansion of $\operatorname{per} L(G)$ according to Theorem 3 is demonstrated in Table 1. Each family of isomorphic subgraphs is represented by one specimen only to make the table less extensive. For instance, the figure



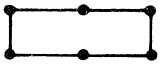
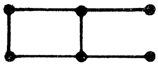
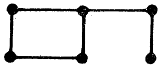
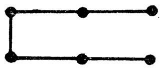
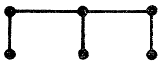
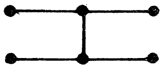
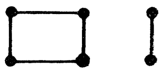
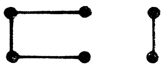
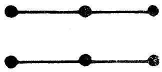
represents six subgraphs



Proof of Theorem 3. Let G have v vertices and e edges. According to Propositions 2, 5 and 6,

$$\operatorname{per} L(G) = \sum_{T \in G_{v,e}} \frac{1}{m(T)} \operatorname{per}^2 U(V, T) .$$

Tab. 1

$H \in S(G)$	$w(H)$	number of isomorphic graphs	contri-bution
	4	1	4
	4	2	8
	4	4	16
	10	8	80
	10	6	60
	10	1	10
	4 . 2	2	16
	6 . 2	12	144
	4 . 4	3	48
	2 . 2 . 2	3	24

per $L(G)$ = 410

Each v-tuple of edges $T \in G_{v,e}$ can be regarded as a multigraph T with v vertices and v edges. If some connected component of T contains more vertices than edges or more edges than vertices then $U(V, T)$ contains a zero $p \times q$ submatrix where $p + q > v$

and per $U(V, T) = 0$. Consequently, if per $U(V, T) \neq 0$ then each connected component of T consists of an equal number of vertices and edges and per $U(V, T)$ is equal to the product of the permanents of the square submatrices of $U(V, T)$ corresponding to the connected components of T. Obviously, $m(T)$ is the product of the values of m for the connected components of T as well. As to the connected components of T with per $U(V, T) \neq 0$ there are only two possibilities:

(1) A monocyclic graph. The subpermanent is 2 and m has the value 1.

(2) A tree with one double edge. The subpermanent is 2 and m has the value 2. A simple computation completes the proof.

Corollary. *If G is a tree on v vertices then*

$$\operatorname{per} L(G) \geqq 2(v - 1)$$

with equality if and only if G is the star.

Proof 1. Put $e = v - 1$ in Theorem 1.

Proof 2. Put $e = v - 1$ in Theorem 2.

Proof 3. The star is the only tree T such that the collection $S(T)$ from Theorem 3 contains T only.

Theorem 4. *Let $G = (V, E)$ be a bipartite graph and let $L_K(G)$ denote the principal submatrix of $L(G)$ corresponding to $\emptyset \neq K \subset V$. Let $S_K(G)$ denote the collection of the subgraphs H of G such that H covers at least the vertices of K and each connected component of H is either*

(1) *a tree with vertices in K, or*

(2) *a monocyclic graph with vertices in K, or*

(3) *a tree with exactly one vertex out of K.*

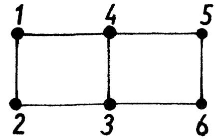
To a tree of type (1) with h edges let us assign the weight $2h$, to a monocyclic graph the weight 4 and to a tree of type (3) the weight 1. Further, to each $H \in S_K(G)$ assign the weight $w(H)$ equal to the product of the weights of its connected components. Then

$$\operatorname{per} L_K(G) = \sum_{H \in S_K(G)} w(H) .$$

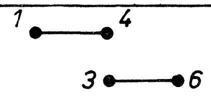
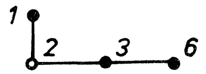
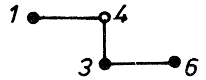
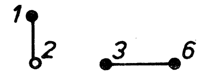
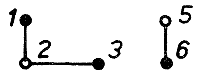
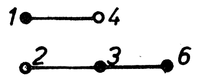
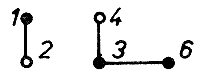
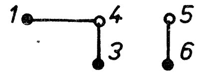
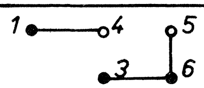
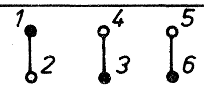
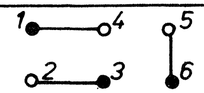
Proof. The argument is the same as in the proof of Theorem 3.

Remark. It is well known that det $L_K(G)$ is equal to the number of the subgraphs from $S_K(G)$ whose all connected components are of type (3). In other words, det $L_K(G)$ equals the number of the subgraphs $H \in S_K(G)$ such that $w(H) = 1$.

Example 2. Let G be once more the graph

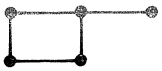
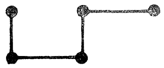
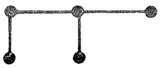
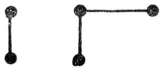
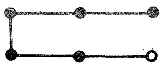
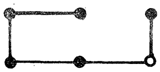
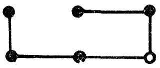
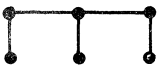
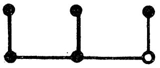
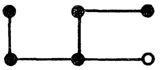
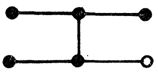


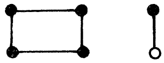
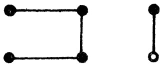
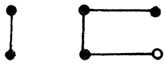
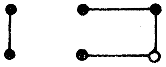
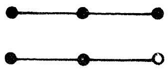
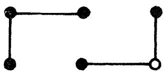
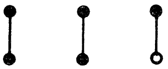
Tab. 2

$H \in S_{(1,3,6)}(G)$	$w(H)$
	1 . 2
	1
	1
	1 . 2
	1 . 1
	1 . 1
	1 . 1
	1 . 1
	1 . 1
	1 . 1
	1 . 1 . 1
	1 . 1 . 1

per $L_{(1,3,6)}(G) = 14$

Tab. 3

$H \in S_{(1,2,3,4,5)}(G)$	$w(H)$	number of isomorphic graphs	contribution
	4	1	4
	8	2	16
	8	2	16
	2.4	3	24
	1	3	3
	1	3	3
	1	2	2
	1	3	3
	1	2	2
	1	1	1
	1	1	1

	4 . 1	1	4
	6 . 1	5	30
	2 . 1	4	8
	2 . 1	3	6
	4 . 1	2	8
	4 . 1	1	4
	2 . 2 . 1	3	12

$\text{per } L_{(1,2,3,4,5)}(G) = 147$

Table 2 demonstrates the expansion of

$$\text{per } L_{(1,3,6)}(G) = \text{per} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 14$$

and Table 3 the expansion of

$$\text{per } L_{(1,2,3,4,5)}(G) = \text{per} \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} = 147$$

according to Theorem 4. The vertices of $H \in S_K(G)$ are drawn full if they belong to K and empty if not. Notice that there are 10 subgraphs H with $w(H) = 1$ in Table 2 and 15 of them (spanning trees) in Table 3. Indeed, $\det L_{(1,3,6)}(G) = 10$ and $\det L_{(1,2,3,4,5)}(G) = 15$.

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Author's address: 115 67 Praha 1, Žitná 25, Czechoslovakia (Matematický ústav ČSAV).