Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 4, 565–616

Persistent URL: http://dml.cz/dmlcz/102051

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

EXTENDED SHANNON ENTROPIES II

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(Received July 6, 1984)

This is a continuation of Part I (Czechosl. Math. J. *33* (*108*) (1983), 564–601). The numbering of sections is consecutive in Part I and II. Terminology and notation of Part I is used, as a rule, without reference.

We continue the examination of extended Shannon entropies (semientropies) defined on the class of sets endowed with a finite measure and a measurable semimetric, or on a suitable subclass. In particular, we investigate, for normal gauge functionals τ, the C_τ-entropy and C_τ^*-semientropy, introduced in Part I.

The main results are as follows. Both $C_\tau(P)$ and $C_\tau^*(P)$ are positive unless P is trivial (or τ fails to satisfy a fairly natural condition) and they are finite whenever P satisfies a certain boundedness condition. If a mild persistence condition is imposed, then, under a set-theoretic assumption, there are not too many distinct extended Shannon entropies on the class of all Borel metrized probability spaces. Under certain assumptions, C_τ and C_τ^* satisfy a condition of the Lipschitz type on the collection of all subspaces of a given semimetrized measure space. As a consequence, for any given finite non-void set Q, both C_τ and C_τ^* are continuous on the space of all metrized measure spaces on Q.

The results just mentioned provide answers to the questions posed in 2.30. Namely, Theorem X answers the questions Ib and IVb, Theorems V, VI and VII answer (partially) the questions III and VI, and Proposition 11.4 answers the questions II and V.

The paper is organized as follows. There are five sections (7 through 11). Each of the sections 7, 8, 9 and 10 contains a short introduction describing the topic. In Section 11, the main results of the present Part II are summarized.

The following facts, which will not be formulated as propositions, are worth mentioning. Many "qualitative" properties, such as, e.g., $C_\tau^*\langle Q, \varrho, \mu\rangle < \infty$ or $\operatorname{Ded}\langle Q, \varrho, \mu\rangle < \infty$ (see 8.13) are not preserved under transition to a uniformly equivalent metric (see, e.g., 8.20). On the other hand, some properties, e.g., those just mentioned, are invariant with respect to the Lipschitz equivalence of metrics (semimetrics). Thus, besides investigating extended Shannon entropies etc., we examine, in fact, the following kinds of spaces: in most cases, metric or semimetric spaces (in contrast to merely metrizable or semimetrizable ones) endowed with a finite measure; in a few cases, Lipschitz spaces equipped with a finite measure; and sometimes, see, e.g., 7.23 and 7.25, also metrizable topological spaces endowed with a finite measure.

Acknowledgment. The author is indebted to J. Hejcman for numerous valuable comments which enhanced the presentation and helped to avoid some mistakes; also, for lemma 8.11 and the examples 10.8.1 – 4.

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In this section, we consider some fairly general properties of metric W-spaces. In particular, we show (7.25) that any metric W-space is the sum of a subspace such that almost no point has a neighborhood of measure zero and a subspace almost all points of which possess a neighborhood of measure zero. We also give an estimate (see 7.39) of the number of distinct, with respect to a certain equivalence, weakly Borel (see 7.19) metric W-spaces of a "not too large" topological weight, and an estimate (see 7.57) of the number of persistent (in a broad sense, see 7.49) extended Shannon semientropies on a fairly wide class of metric W-spaces.

7.1. We now recall some simple facts concerning what we call $\bar{\boldsymbol{R}}$-measures, i.e., measures in the usual sense (the value ∞ possible), and introduce W-spaces of the form $P \upharpoonright T$ (see 7.8).

7.2. Definition. Let Q be a non-void set. If $\mathscr{B}$ is a σ-algebra on Q, μ is a σ-additive non-negative function on $\mathscr{B}$ (possibly assuming the value ∞) and $\mu(\emptyset) = 0$, then μ will be called an *$\bar{\boldsymbol{R}}$-measure* on Q. An $\bar{\boldsymbol{R}}$-measure on Q will be called *σ-finite* if there exist sets $A_k \in \operatorname{dom} \mu$, $k \in \boldsymbol{N}$, such that $\bigcup(A_k\colon k \in \boldsymbol{N}) = Q$ and $\mu(A_k) < \infty$ for all $k \in \boldsymbol{N}$.

7.3. The completion of an $\bar{\boldsymbol{R}}$-measure and the product $\mu_1 \times \mu_2$ of two $\bar{\boldsymbol{R}}$-measures are defined in the same way as the corresponding concepts for measures. Therefore we can omit explicit definitions.

7.4. Notation. Let Q be a set and let μ be a function such that $\operatorname{dom} \mu \subset \exp Q$. Let $T \subset Q$. For any Y such that $Y = X \cap T$ for some $X \in \operatorname{dom} \mu$, pur $\nu(Y) = \inf \{\mu(X)\colon X \in \operatorname{dom} \mu, X \cap T = Y\}$. Then the function ν will be denoted by $\mu \upharpoonright T$, provided there is no danger of confusion.

7.5. Fact and notation. Let μ be a measure (an $\bar{\boldsymbol{R}}$-measure) on Q and let $T \subset Q$. If $T \neq \emptyset$, then $\mu \upharpoonright T$ is a measure (an $\bar{\boldsymbol{R}}$-measure) on T and $[\mu \upharpoonright T] = \mu \upharpoonright T$. The outer measure of T, i.e., $(\mu \upharpoonright T)(T)$, will be sometimes denoted by $\mu_e(T)$.

7.6. Fact. For $i = 1, 2$, let μ_i be a measure (an $\bar{\boldsymbol{R}}$-measure) on Q_i and let $\emptyset \neq T_i \subset Q_i$. Put $T = T_1 \times T_2$, $\mu = \mu_1 \times \mu_2$, $\nu_i = \mu_i \upharpoonright T_i$, $\nu = \mu \upharpoonright T$. Then $\nu = \nu_1 \times \nu_2$, $\bar{\mu} \upharpoonright T = [\nu_1 \times \nu_2]$.

Since 7.5 and 7.6 are well-known facts (though in a slightly different setting) of measure theory and can be proved in a straightforward way, we omit the proofs.

7.7. Fact and notation. If $S = \langle Q, \varrho \rangle$ is a semimetric space and $T \subset Q$, then $\langle T, \varrho \restriction T \rangle$ is a semimetric space, which will be denoted by $S \restriction T$. – For the notation $\varrho \restriction T$ see 1.1 D.

7.8. Fact and notation. If $P = \langle Q, \varrho, \mu \rangle$ is a W-space and $\emptyset \neq T \subset Q$, then $\langle T, \varrho \restriction T, \mu \restriction T \rangle$ is a W-space, which will be denoted by $P \restriction T$.

Proof. We have to show only that the semimetric $\varrho \restriction T$ is $[(\mu \restriction T) \times (\mu \restriction T)]$ – measurable. This follows immedaitely from 7.5 and 7.6.

7.9. Now we give a simple characterization of measures μ such that some $\langle Q, \varrho, \mu \rangle$ is a metric W-space (see 7.19 below), in other words, of probability spaces admitting a metric. Although the result is possibly known, we will prove it in full, omitting only the proof of a well-known proposition (see 7.12) of measure theory.

7.10. Definition. Let μ be a measure on Q. If, for any $X \in \operatorname{dom} \mu$, $\mu X = 0$ or $\mu X = \mu Q$, then μ is called a *two-valued measure*. If, for any $X \in \operatorname{dom} \mu$, the set $\{\mu Y: Y \in \operatorname{dom} \mu, Y \subset X\}$ is equal to the interval $[0, \mu X]$, then μ is said to have *the Darboux property*.

7.11. Lemma. *Let* $\langle Q, \varrho, \mu \rangle$ *be a W-space. If* μ *is a two-valued measure, then there exists a non-void set* $A \in \operatorname{dom} \mu$ *such that* (i) $\mu A = \mu Q$, (ii) *if* $x, y \in A$, *then* $\varrho(x, y) = 0$.

Proof. Clearly we can assume that $\mu Q = 1$. If $A_k, B_k \in \operatorname{dom} \mu$, $\bigcup(A_k \times B_k: k \in N) \supset \{(x, y): \varrho(x, y) = 0\}$, then, with $E_k = A_k \cap B_k$, we have $\bigcup(E_k \times E_k: k \in N) \supset \{(x, x): x \in Q\}$, hence $\bigcup E_k = Q$ and therefore $\mu E_n = 1$ for some $n \in N$. This implies $\Sigma(\mu A_k . \mu B_k: k \in N) \geqq 1$, hence $[\mu \times \mu] \{(x, y) \in Q \times Q: \varrho(x, y) = 0\} = 1$, $[\mu \times \mu](G) = 0$, where $G = \{(x, y): \varrho(x, y) > 0\}$. Consequently, there exist sets $X_k, Y_k \in \operatorname{dom} \mu$, $k \in N$, such that $\bigcup(X_k \times Y_k: k \in N) \supset G$, $\Sigma(\mu X_k . \mu Y_k: k \in N) = 0$. Put $K = \{k: \mu X_k = 1\}$; put $A = (Z \setminus \bigcup(X_k: k \text{ non} \in K) \setminus \bigcup(Y_j: j \in K)$, where $Z = \bigcup(X_k: k \in K)$ if $K \neq \emptyset$, $Z = Q$ if $K = \emptyset$. Due to $\Sigma \mu X_i . \mu Y_i = 0$, we have $\mu Y_j = 0$ whenever $j \in K$. This implies $\mu A = 1$.

Suppose A contains points x, y such that $\varrho(x, y) > 0$. Then, for some $n, (x, y) \in X_n \times Y_n$. If $n \in K$, then $y \in \bigcap(Y_i: i \in K)$, hence $y \text{ non} \in A$. If $n \text{ non} \in K$, then $x \in \bigcap(X_i: i \text{ non} \in K)$, hence $x \text{ non} \in A$. In both cases we get a contradiction.

7.12. Proposition. *Let a measure* μ *satisfy the following condition: if* $X \in \operatorname{dom} \mu$ *and* $\mu X > 0$, *then there exists a set* $Y \subset X$ *such that* $Y \in \operatorname{dom} \mu$ *and* $0 < \mu Y < \mu X$. *Then* μ *has the Darboux property.*

This is well known (see, e.g., [2], § 2, Proposition 7).

7.13. Proposition. *Let* μ *be a measure on a set* Q. *Then the following conditions are equivalent:* (1) $\langle Q, 1, \mu \rangle \in \mathfrak{W}$, (2) *there exists a metric* ϱ *on* Q *such that*

$\langle Q, \varrho, \mu\rangle \in \mathfrak{W}$, (3) *there exists a countable set* $A \subset Q$ *such that* $\mu \restriction (Q \setminus A)$ *has the Darboux property and* $\{x\} \in \operatorname{dom} \mu$ *for all* $x \in A$.

Proof. Trivially, (1) implies (2). – Assume (2). Let A be the set of all $x \in Q$ such that $\{x\} \in \operatorname{dom} \mu$, $\mu\{x\} > 0$. Then A is countable. Put $\nu = \mu \restriction (Q \setminus A)$. If $X \in \operatorname{dom} \nu$, $\nu X > 0$, then $\nu \restriction X$ is not two-valued, for otherwise, by 7.11, there would exist an $x \in X$ such that $\{x\} \in \operatorname{dom} \nu$, $\nu\{x\} = \nu X > 0$, which is a contradiction since $X \cap A = \emptyset$. Hence, for any $X \in \operatorname{dom} \nu$ such that $\nu X > 0$, there exists a $Y \in \operatorname{dom} \nu$ such that $Y \subset X$ and $0 < \nu Y < \nu X$. By 7.12, this proves that $\mu \restriction (Q \setminus A)$ has the Darboux property. – Assume (3). We are going to prove that $\langle Q, 1, \mu\rangle \in \mathfrak{W}$. Since clearly $\{(x, x): x \in A\}$ is $(\mu \times \mu)$-measurable, it is enough to show that, for any $\varepsilon > 0$, there is a set $G_\varepsilon \in \operatorname{dom}(\mu \times \mu)$ such that $G_\varepsilon \supset \{(x, x): x \in Q \setminus A\}$, $(\mu \times \mu)(G_\varepsilon) < \varepsilon$. If $\mu(Q \setminus A) = 0$, we can put $G_\varepsilon = (Q \setminus A) \times (Q \setminus A)$. If $\mu(Q \setminus A) > 0$, then, since $\mu \restriction (Q \setminus A)$ has the Darboux property, it is easy to see that there exists a partition $(X_k: k \in K)$ of $Q \setminus A$ such that, for all $k \in K$, $X_k \in \operatorname{dom} \mu$ and $\mu X_k < \varepsilon/\mu Q$. Put $G_\varepsilon = \bigcup(X_k \times X_k: k \in K)$. Then $(\mu \times \mu)(G_\varepsilon) = \Sigma((\mu X_k)^2: k \in K) <$ $< (\varepsilon/\mu Q)\, \Sigma(\mu X_k: k \in K) = \varepsilon$.

7.14.1. Example. Let Q be an uncountable set. Let $\mathscr{A}$ be the smallest σ-algebra on Q containing all finite sets. For $X \in \mathscr{A}$, put $\mu X = 0$ if X is countable, $\mu X = 1$ if not. Then μ is a measure which does not satisfy the condition (3) from 7.13.

7.14.2. Fact. *If* $P = \langle Q, \varrho, \mu\rangle \in \mathfrak{W}$ *and* μ *is a two-valued measure, then* $d(P) = 0$.

Proof. Let $A \subset Q$ be a set with properties described in 7.11. Then $[\mu \times \mu]$ . . $\{(x, y): \varrho(x, y) = 0\} \geqq [\mu \times \mu]\,(A \times A) = (\mu A)^2 = (\mu Q)^2 = (\mu \times \mu)\,(Q \times Q)$, hence $d(P) = 0$.

7.14.3. Fact. *If a measure* μ *on* Q *is not two-valued, then there exists a semimetric* ϱ *such that* $P = \langle Q, \varrho, \mu\rangle \in \mathfrak{W}$, $d(P) > 0$.

Proof. Choose $X \in \operatorname{dom} \mu$ such that $0 < \mu X < \mu Q$ and, for $x, y \in Q$, put $\varrho(x, y) = 1$ if (x, y) or (y, x) is in $X \times (Q \setminus X)$, $\varrho(x, y) = 0$ if not. Clearly, $(\mu \times \mu)\,\{(x, y): \varrho(x, y) = 1\} = 2\mu X\, \mu(G \setminus X) > 0$.

7.15. Now we are going to prove a proposition (or rather two versions of a proposition, see 7.23 and 7.25) which plays an important role in proving various results (see e.g. 7.28, 7.35 and 7.39) on metric W-spaces and extended Shannon semientropies.

7.16. Definition. If P is a semimetric space $\langle Q, \varrho\rangle$ or a W-space $\langle Q, \varrho, \mu\rangle$, then for any $x \in Q$ and any positive number t we put $B(x, t) = \{y \in Q: \varrho(x, y) < t\}$, $\bar{B}(x, t) = \{y \in Q: \varrho(x, y) \leqq t\}$. A set of the form $B(x, t)$ or $\bar{B}(x, t)$ will be called a *ball* in P (with a center at x). – Note that a ball can have more than one center.

7.17. Fact. *Let* μ *be a measure on* Q. *Let* $Z \subset Q \times Q$ *be* $[\mu \times \mu]$*-measurable. Then there exists a* μ*-measurable set* $A \subset Q$ *such that* $\mu(Q \setminus A) = 0$ *and, for any* $a \in A$, $\{x: (a, x) \in Z\} \in \operatorname{dom} \bar{\mu}$. *If, in addition,* $[\mu \times \mu]\,(Z) = 0$, *then* A *can be chosen in such a way that* $\bar{\mu}\{x: (a, x) \in Z\} = 0$ *for each* $a \in A$.

This is well known.

7.18. Lemma. *Let* $P = \langle Q, \varrho, \mu\rangle$ *be a W-space. Then there exists a set* $A \in \operatorname{dom} \mu$ *such that* (1) $\mu(Q \setminus A) = 0$, (2) *any ball in P with a center in A is* $\bar{\mu}$*-measurable,* (3) *in the W-space* $P \restriction A$, *all balls are* $[\mu \restriction A]$*-measurable.*

Proof. For any positive number t, the set $\{(x, y) \in Q \times Q: \varrho(x, y) < t\}$ is $[\mu \times \mu]$-measurable. Hence, by 7.17, for any rational $t > 0$ there exists a set $A_t \in \operatorname{dom} \mu$ such that $\mu(Q \setminus A_t) = 0$ and, for any $a \in A_t$, $B(a, t)$ is $\bar{\mu}$-measurable. Put $A = \bigcap(A_t: t > 0 \text{ rational})$. Then $A \in \operatorname{dom} \mu$, $\mu(Q \setminus A) = 0$, and every $B(a, t)$ where $a \in A$ and t is rational positive, is $\bar{\mu}$-measurable. This implies assertion (2). Assertion (3) follows by 7.5.

7.19. Definition. A W-space $\langle Q, \varrho, \mu\rangle$ will be called *metric* if ϱ is a metric. A metric W-space $\langle Q, \varrho, \mu\rangle$ will be called *weakly Borel* if every Borel set $X \subset Q$ is $\bar{\mu}$-measurable. The class of all metric W-spaces will be denoted by $\mathfrak{W}_M$, that of all weakly Borel metric W-spaces by $\mathfrak{W}_{MB}$.

7.20. Definition. Let $P = \langle Q, \varrho, \mu\rangle$ be a metric W-space. Then (1) the (topological) weight of $\langle Q, \varrho\rangle$ (i.e., the least cardinality of an open base) will be called the *topological weight* (abbreviated t. weight) of P and will be denoted by $\operatorname{tw}(P)$; (2) the minimum of the (topological) weights of $\langle Q, \varrho\rangle \restriction T$, where $T \in \operatorname{dom} \mu$ and $\mu(Q \setminus T) = 0$, will be called the *reduced topological weight* (abbreviated r.t. weight) of P and will be denoted by $\operatorname{rtw}(P)$; (3) P will be called *second-countable* if $\langle Q, \varrho\rangle$ is second-countable, i.e., if it has a countable open base.

7.21. Lemma. *Let* $P = \langle Q, \varrho\rangle$ *be a metric space. Let* μ *be a measure on Q. Assume that every* $x \in Q$ *has arbitrarily small* $\bar{\mu}$*-measurable neighborhoods (open or not). Let V consist of all* $x \in Q$ *such that* $\bar{\mu}X > 0$ *for any* $\bar{\mu}$*-measurable neighborhood X of x. Then the set V is* $\bar{\mu}$*-measurable closed and the space* $P \restriction V$ *is second-countable.*

Proof. Clearly, V is closed. – Suppose that $P \restriction V$ is not second-countable. Then there exists a number $\varepsilon > 0$ and an uncountable set $Y \subset V$ such that $\varrho(y_1, y_2) > 2\varepsilon$ whenever $y_1 \in Y$, $y_2 \in Y$, $y_1 \neq y_2$. For each $y \in Y$, choose a $\bar{\mu}$-measurable neighborhood $Z_y \subset B(y, \varepsilon)$. Then the sets Z_y are disjoint and $\bar{\mu}Z_y > 0$ for every $y \in Y$. Since Y is uncountable, this is a contradiction, which proves that $P \restriction V$ is second-countable. – For any $n = 1, 2, \ldots$ and any $x \in V$, let $M(n, x)$ be a $\bar{\mu}$-measurable neighbourhood of x contained in $B(x, 1/n)$. Since $P \restriction V$ is second-countable, there exists for every $n = 1, 2, \ldots$ a countable set $A_n \subset V$ such that $\bigcup(M(n, x): x \in A_n) \supset V$. Put $B_n = \bigcup(M(n, x): x \in A_n)$. Clearly, $V = \bigcap(B_n: n = 1, 2, \ldots)$, hence V is $\bar{\mu}$-measurable.

7.22. Lemma. *Let* $\langle Q, \varrho, \mu\rangle$ *be a metric W-space. Then, for any* $x \in Q$, *either* (i) $\bar{\mu}G = 0$ *for some open neighborhood G of x, or* (ii) *any ball with a center at x is* $\bar{\mu}$*-measurable.*

Proof. Let $x \in Q$. Let A be a set with the properties described in 7.18 (thus, in particular, all balls with a center in A are $\bar{\mu}$-mesaruable). If $A \cap B(x, \varepsilon) = \emptyset$ for some $\varepsilon > 0$, then $B(x, \varepsilon) \subset Q \setminus A$, hence $\bar{\mu}(B(x, \varepsilon)) = 0$. If $A \cap B(x, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$, then there exist $a_n \in A$, $n = 1, 2, \ldots$, such that $\varrho(a_n, x) < 1/n$. It is easy to see that if $0 < t < \infty$, then $B(x, t) = \bigcap(B(a_n, t - 1/n)\colon n = 1, 2, \ldots, nt > 1)$ and therefore $B(x, t)$ is $\bar{\mu}$-measurable.

7.23. Proposition. *Let $P = \langle Q, \varrho, \mu \rangle$ be a metric W-space. Let U be the union of all open (in $\langle Q, \varrho \rangle$) sets G such that $G \in \operatorname{dom} \bar{\mu}$, $\bar{\mu}G = 0$. Let $V = Q \setminus U$. Then (1) U and V are $\bar{\mu}$-measurable, U is open, V is closed, (2) the W-space $P \restriction V$ is second-countable weakly Borel, (3) every ball in P with a center in V is $\bar{\mu}$-measurable, (4) if the reduced topological weight of P is countable, then $\bar{\mu}U = 0$.*

Proof. The assertions (1), (2) and (3) follow at once from 7.21 and 7.22. – Let $\operatorname{rtw}(P)$ be countable. Then there exists a set $T \in \operatorname{dom} \mu$ such that $\mu(Q \setminus T) = 0$ and $\langle Q, \varrho \rangle \restriction T$ is second-countable. Every $x \in U \cap T$ has an open (in $\langle Q, \varrho \rangle$) neighbourhood G_x such that $\bar{\mu}G_x = 0$. Since $\langle Q, \varrho \rangle \restriction T$ is second-countable, the open cover $(G_x\colon x \in U \cap T)$ of $U \cap T$ contains a countable subcover and therefore $\bar{\mu}(U \cap T) = 0$. Since $U \subset (U \cap T) \cup (Q \setminus T)$ and $\mu(Q \setminus T) = 0$, we get $\bar{\mu}U = 0$.

7.24. Proposition. *Let $P = \langle Q, \varrho, \mu \rangle$ be a metric W-space. If the reduced topological weight of P is countable (in particular, if P is second-countable), then P is weakly Borel.*

This is an immediate consequence of 7.23.

7.25. Proposition. *For any metric W-space $P = \langle Q, \varrho, \mu \rangle$, let $Z(P)$ denote the set of all $x \in Q$ such that $\mu(G) = 0$ for some neighborhood $G \in \operatorname{dom} \mu$ of x. Let $\mathscr{X}_+$ and $\mathscr{X}_0$ denote the classes of all metric W-spaces $S = \langle T, \sigma, \nu \rangle$ such that $\bar{\nu}(Z(S)) = 0$ and $\bar{\nu}(T \setminus Z(S)) = 0$, respectively. Then (1) every metric W-space $P = \langle Q, \varrho, \mu \rangle$ has exactly one partition (P_+, P_0) such that $P_+ \in \mathscr{X}_+$, $P_0 \in \mathscr{X}_0$, (2) $P_+ = V . P$, $P_0 = U . P$, where $U = \bigcup\{G\colon G \text{ open}, \bar{\mu}G = 0\}$, $V = G \setminus U$.*

Proof. I. Put $\mu_+ = V . \mu$, $\mu_0 = U . \mu$, $P_+ = V . P$, $P_0 = U . P$. Put $K = Z(P_+) \cap \cap V$. By 7.21, the space $\langle K, \varrho \restriction K \rangle$ is second-countable. Hence it is easy to see that $\bar{\mu}_+(K) = 0$. Since $\bar{\mu}_+(U) = 0$, we have $\bar{\mu}_+(U \cup K) = 0$. Clearly, $Z(P_+) \subset U \cup K$, hence $\bar{\mu}_+(Z(P_+)) = 0$, $P_+ \in \mathscr{X}_+$. Evidently, $U \subset Z(P_0)$, hence $Q \setminus Z(P_0) \subset V$, $\bar{\mu}_0(Q \setminus Z(P_0)) \leqq \bar{\mu}_0(V) = 0$, and therefore $P_0 \in \mathscr{X}_0$. – II. Let (S_+, S_0) be a partition of P and let $S_+ = \langle Q, \varrho, \nu_+ \rangle \in \mathscr{X}_+$, $S_0 = \langle Q, \varrho, \nu_0 \rangle \in \mathscr{X}_0$. Clearly, $Z(S_+) \supset \supset Z(P) = U$ and therefore $\bar{\nu}_+(U) = 0$, hence $\nu_+ \leqq V . \mu$. Put $M = V \cap Z(S_0)$. Since $\langle M, \varrho \restriction M \rangle$ is second-countable, we have $\bar{\nu}_0(M) = 0$. Since $\bar{\nu}_0(Q \setminus Z(S_0)) = 0$, this proves that $\nu_0(V) = 0$, hence $\nu_+(V) = \mu(V)$. Together with $\nu_+ \leqq V . \mu$, this yields $\nu_+ = V . \mu$ and therefore $\nu_0 = U . \mu$.

7.26. Let us recall some concepts and theorems of set theory. The terminology and (partly) notation is that of [1].

7.26.1. A cardinal α is called *real-measurable* (*Ulam-measurable*) if there exists a measure (a two-valued measure) μ on a set Q of cardinality α such that dom $\mu =$ $= \exp Q$, $\mu\{x\} = 0$ for any $x \in Q$, and $\mu Q > 0$.

7.26.2. *Every Ulam-measurable cardinal is real-measurable. If a cardinal α is real-measurable (Ulam-measurable), then so is every cardinal $\beta > \alpha$. The cardinal ω is not real-measurable. If a cardinal α is not real-measurable (Ulam-measurable), then neither is α^+, the smallest cardinal greater than α. If* card $A = \alpha$, *α is not real-measurable (Ulam-measurable), ξ_a, where $a \in A$, are cardinals and no ξ_a is real-measurable (Ulam-measurable), then neither is $\Sigma(\xi_a\colon a \in A)$.*

7.26.3. *If a cardinal α is not Ulam-measurable, then neither is 2^α.*

7.27. Lemma. *Let $\langle Q, \varrho\rangle$ be a metric space. Then the following conditions are equivalent: (1) if μ is a measure (a two-valued measure) on Q such that every Borel (in $\langle Q, \varrho\rangle$) set is $\bar{\mu}$-measurable, then there exists a closed set $X \subset Q$ such that for any open $G \subset Q$, $\bar{\mu}G > 0$ iff G intersects X; (2) if μ is a measure (a two-valued measure) on Q such that every Borel set is $\bar{\mu}$-measurable, then for any collection $\mathscr{G}$ of open sets G such that $\bar{\mu}G = 0$ we have $\bar{\mu}(\bigcup\mathscr{G}) = 0$; (3) the weight of $\langle Q, \varrho\rangle$ is not real-measurable (is not Ulam-measurable).*

This follows at once from the results contained in [6] (in particular, Theorems II and III).

7.28. Proposition. *Let $P = \langle Q, \varrho, \mu\rangle$ be a metric W-space. If P is weakly Borel and the topological weight of P is not real-measurable, then the reduced topological weight of P is countable. In more detail: there exists a μ-measurable set Z such that (1) $\mu(Q \setminus Z) = 0$; (2) the space $P \upharpoonright Z$ is second-countable; (3) if $G \subset Q$ is open and $\bar{\mu}G = 0$, then $G \cap Z = \emptyset$.*

Proof. Let U and V be as in 7.23. Then, by 7.27, $\bar{\mu}U = 0$. Choose a set $Z \in$ dom μ such that $Z \subset V$, $\mu(Q \setminus Z) = 0$.

7.29.1. If, in 7.28, the assumption of P being weakly Borel is omitted and "real-measurable" is replaced by "Ulam-measurable", then the proposition does not hold. An example: Q is the interval $[0, 1]$, λ is the Lebesgue measure on Q, $P = \langle Q, 1, \lambda\rangle$. Clearly, P is not weakly Borel. The t. weight of P is 2^ω, hence it is not Ulam-measurable. The r.t. weight of P is 2^ω as well.

7.29.2. Fact. *Let Q be a set. If* card Q *is real-measurable but not Ulam-measurable, then there exists a measure μ on Q such that $P = \langle Q, 1, \mu\rangle$ is a weakly Borel metric W-space and the reduced topological weight of P is real-measurable.*

Proof. Since card Q is real-measurable, there exists a measure μ on Q such that dom $\mu = \exp Q$, $\mu Q > 0$, $\mu\{x\} = 0$ for all $x \in Q$. Since card Q is not Ulam-measurable, no $\mu \upharpoonright T$, where $T \subset Q$, $\mu T > 0$, is two-valued. Hence, by 7.12, μ has the Darboux property and therefore, by 7.13, $P = \langle Q, 1, \mu\rangle$ is a W-space. Clearly, P is weakly Borel. If $T \subset Q$, $\mu T = 0$, then $\mu(Q \setminus T) = \mu Q > 0$ and therefore card $(Q \setminus T)$ is real-measurable. Consequently, the r.t. weight of P is real-measurable.

Remark. I do not know whether 7.28 remains true if we omit the assumption that P is weakly Borel while retaining the assumption on the topological weight of P.

7.30.1. It is an almost immediate consequence of 7.28 that if $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_{MB}$ and the topological weight of P is not real-measurable, then P satisfies the following condition: (∗) if $d(P) > 0$, then there are μ-measurable sets X, Y such that $\mu X > 0$, $\mu Y > 0$, $\inf\{\varrho(x, y): x \in X, \ y \in Y\} > 0$. There are examples of non-metric W-spaces not satisfying (∗), see 10.19 and 10.20. On the other hand, I do not know whether every $P \in \mathfrak{W}_{MB}$ (or even every $P \in \mathfrak{W}_M$) satisfies (∗). However, the following statement is easy to prove.

7.30.2. Fact. *Let $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_{MB}$ and let $d(P) > 0$. Assume that the topological weight of P is not Ulam-measurable. Then there are μ-measurable sets X and Y such that $\mu X > 0$, $\mu Y > 0$, $\inf\{\varrho(x, y): x \in X, \ y \in Y\} > 0$.*

Proof. Let $\mathscr{B}$ denote the σ-algebra of all Borel subsets of Q. Suppose that for any $Z \in \mathscr{B}$, $\bar{\mu}Z = 0$ or $\bar{\mu}Z = \mu Q$. Let ν denote the measure $\bar{\mu}$ restricted to $\mathscr{B}$. If there exists an $x \in Q$ such that $\nu G > 0$ for any open set G containing x, then we easily get $\nu\{x\} = \nu Q$, hence $\nu(Q \setminus \{x\}) = 0$ and therefore $d(P) = 0$, which is a contradiction. If every $x \in Q$ has an open neighborhood G such that $\nu G = 0$, then, by 7.27, we get $\nu Q = 0$, which contradicts $d(P) > 0$. Thus we have shown that there exists a Borel set Z such that $0 < \bar{\mu}Z < \mu Q$. Hence there exists a closed set $A \subset Z$ such that $0 < \bar{\mu}A$. For $n = 1, 2, \ldots$, put $G_n = \{y \in Q: \varrho(y, x) < 1/n$ for some $x \in A\}$. Clearly, $\bar{\mu}G_n \to \bar{\mu}A$ and therefore, for some n, we have $\bar{\mu}(Q \setminus G_n) > 0$. Obviously, $\varrho(x, y) \geqq 1/n$ if $x \in A$, $y \in Q \setminus G_n$. Now choose μ-measurable sets $X \subset A$, $Y \subset Q \setminus G_n$ such that $\mu X > 0$, $\mu Y > 0$.

7.31. We will introduce two closely related concepts, namely PC-injections and SC-injections. The former are closely connected (see 7.34) with identity mappings of the form $i: P \restriction T \to P$ whereas SC-injections are (cf. 7.33) straightforward generalizations of injective conervative mappings (see 2.7) of FW-spaces.

7.32. Definition. Let $P_i = \langle Q_i, \varrho_i, \mu_i \rangle$, $i = 1, 2$, be W-spaces. Let $f: P_1 \to P_2$ be an injective mapping. We will say that f satisfies SCI or that f is an SC-injection if the following conditions are satisfied:

(1) if $Y \in \operatorname{dom} \mu_2$, then $f^{-1}Y \in \operatorname{dom} \mu_1$, $\mu_1(f^{-1}Y) = \mu_2 Y$;

(2) if $X \in \operatorname{dom} \mu_1$, then $fX \in \operatorname{dom} \mu_2$;

(3) $[\mu_1 \times \mu_1]\{(x, y) \in Q_1 \times Q_1: \varrho_2(fx, fy) \neq \varrho_1(x, y)\} = 0$.

We will say that f satisfies PCI or that f is a PC- injection if (1), (2) hold, and (4) $\varrho_2(fx, fy) = \varrho_1(x, y)$ for all $x, y \in Q$.

If f is bijective and satisfies PCI, we will say that f is a PC-bijection.

If P and S are W-spaces and there exist W-spaces $T_0, \ldots, T_n$ such that $T_0 = P$, $T_n = S$ and, for each $k = 1, \ldots, n$, we have either a PC-injection (SC-injection) $f: T_{k-1} \to T_k$ or a PC-injection (SC-injection) $f: T_k \to T_{k-1}$, then we will say that P is PCI-equivalent (SCI-equivalent) to S.

Remarks. 1) This terminology is provisional. In Part III, various kinds of "conservative" mappings (not necessarily injective) and correspondences will be considered and a more systematic terminology will be introduced. — 2) It will be shown later (7.56) that if P and S are SCI-equivalent, then $C_\tau^*(P) = C_\tau^*(S)$, $C_\tau(P) = C_\tau(S)$ whenever τ is a gauge functional of the form $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$.

7.33. Fact. *Let P_1 and P_2 be FW-spaces. An injective mapping $f: P_1 \to P_2$ satisfies* SCI *iff it is conservative in the sense of* 2.7.

7.34. Fact. *Let $P_j = \langle Q_j, \varrho_j, \mu_j \rangle$, $j = 1, 2, 3$, be W-spaces. If $\emptyset \neq T \subset Q_1$, then the identity mapping $i: P_1 \restriction T \to P_1$ is a* PC*-injection iff $T \in \operatorname{dom} \mu_1$. If $f: P_1 \to P_2$ is a* PC*-injection, then $f: P_1 \to P_2 \restriction f(Q_1)$ is a* PC*-bijection. If $f: P_1 \to P_2$ and $g: P_2 \to P_3$ satisfy* SCI (PCI, *respectively), then so does $g \circ f: P_1 \to P_3$.*

7.35. Proposition. *If P is a weakly Borel metric W-space and the topological weight of P is not real-measurable, then there exists a second-countable metric W-space S and a* PC*-injection $f: S \to P$.*

Proof. See 7.28 and 7.34.

7.36. Although $\mathfrak{W}$, $\mathfrak{W}_M$, etc., are proper classes, the following question is meaningful (and sometimes important): what is the cardinality of a given class $\mathscr{X} \subset \mathfrak{W}$, up to a certain given equivalence. To put it precisely: we ask whether there exists a set $\mathscr{S} \subset \mathscr{X}$ such that every $P \in \mathscr{X}$ is equivalent in a specified sense to some $S \in \mathscr{S}$, and (if the answer is affirmative) what is the least cardinality of an $\mathscr{S}$ of this kind.

We will prove a proposition concerning the cardinality, up to a certain fairly natural equivalence, of the class of those $P \in \mathfrak{W}_{MB}$ the topological weight of which is not real-measurable.

7.37. Notation. For any cardinal α, we put $\exp \alpha = 2^\alpha$, $\exp^{(2)} \alpha = \exp \exp \alpha$, $\exp^{(3)} \alpha = \exp(\exp^{(2)} \alpha)$, etc.

7.38. Fact. *Let Q be an infinite set. Then* $\operatorname{card}\{P \in \mathfrak{W}: |P| = Q\} \leqq \exp^{(2)}(\operatorname{card} Q)$.

Proof. Put $\alpha = \operatorname{card} Q$. Clearly, there are at most $\exp^{(2)} \alpha$ σ-algebras on Q. If $\mathscr{A}$ is a σ-algebra on Q, then there are at most $\exp(\operatorname{card} \mathscr{A}) \leqq \exp^{(2)} \alpha$ measures μ such that $\operatorname{dom} \mu = \mathscr{A}$. Clearly, there are $\exp \alpha$ semimetrics on Q.

7.39. Proposition. *There exists a set $\mathscr{S}$ of cardinality $\leqq \exp^{(3)} \omega$ consisting of second-countable metric W-spaces and satisfying the following condition: if P is a weakly Borel metric W-space and the topological weight of P is not real-measurable, then there exists an $S \in \mathscr{S}$ and a* PC*-injection $f: S \to P$. Informally: up to* PC*-injections, there are at most* $\exp^{(3)} \omega$ *weakly Borel metric W-spaces with topological weights not real-measurable.*

Proof. By 7.38, it is easy to see that there exists a set $\mathscr{S}$ of cardinality $\leqq \exp^{(3)} \omega$ consisting of second-countable (hence, by 7.24, weakly Borel) metric W-spaces and such that for any second-countable $P \in \mathfrak{W}_M$ there is a PC-bijection $f: S \to P$ where $S \in \mathscr{S}$. By 7.35, $\mathscr{S}$ has the properties stated in the proposition.

Remark. It can be shown that, up to PCI-equivalence, there are exactly $\exp^{(3)} \omega$ weakly Borel metric W-spaces P such that $\operatorname{tw}(P)$ is not real-measurable (to be precise: there is a set $\mathscr{S}$ of W-spaces of this kind such that card $\mathscr{S} = \exp^{(3)} \omega$, no distinct $S_1, S_2 \in \mathscr{S}$ are PCI-equivalent and every W-space of the kind mentioned is PCI-equivalent to some $S \in \mathscr{S}$). However, at this stage we are interested mainly in the fact that there are "not too many" $P \in \mathfrak{W}_{MB}$ with $\operatorname{tw}(P)$ not real-measurable.

7.40. We intend to show that "there are at most 2^{ω} extended Shannon semientropies on $\mathfrak{W}_F$" and also to prove a result (7.57) of this sort concerning PCI-persistent (see 7.49) extended (b.s.) Shannon semientropies on a certain subclass of $\mathfrak{W}$.

However, since $\mathfrak{W}_F$ is a proper class, the functionals φ on $\mathfrak{W}_F$ do not form a class. Hence we have to give an exact meaning to the expression in quotation marks. This can be done in various ways. For instance, we can consider, for any infinite set Ω, the set $W(\Omega) = \{P \in \mathfrak{W}_F : |P| \subset \Omega\}$ and the cardinality $\varkappa(\Omega)$ of the set of all $\varphi \restriction W(\Omega)$, where φ is an extended Shannon entropy on $\mathfrak{M}_F$. It turns out that $\varkappa(\Omega)$ does not depend on Ω.

7.41. In the present paper we prefer a different approach, described below, which seems to be more general. This approach is well known (though in a sligthly different setting) and therefore we describe it only briefly and in a rather informal manner. A full formalization is easy though somewhat lengthy.

7.41.1. The axiomatic system we use is a version of GB, the Gödel-Bernays system, such that all objects (in particular, all sets) are classes.

7.41.2. For $n = 1, 2, \ldots$, a formula in GB containing exactly n free variables will be called an *n-ary condition*. We assume that for any $n = 1, 2, \ldots$, if $X_1, \ldots, X_n$ are objects and F is an n-ary condition, then the meaning of "X_1 satisfies F" (for $n = 1$) and of "$\langle X_1, \ldots, X_n \rangle$ satisfies F" (for $n > 1$) is clear. We will use the abbreviations $F\langle X \rangle$ for "X satisfies F", $F\langle X_1, \ldots, X_n \rangle$ for "$\langle X_1, \ldots, X_n \rangle$ satisfies F". – We note that (A) "X_1 satisfies F", etc., are not statements in GB, but in an appropriate metalanguage of GB, (B) if some X_i is a proper class, then $\langle X_1, \ldots, X_n \rangle$ does not denote any object and is not meaningful unless it stands in a context like "$\langle X_1, \ldots, X_n \rangle$ satisfies ...".

7.41.3. Let P and S be 1-ary conditions. A 2-ary condition F will be called *bijective* with respect to P and S if (1) for any X satisfying P, there is exactly one Y such that $S\langle Y \rangle$ and $F\langle X, Y \rangle$, (2) for any Y satisfying S, there is exactly one X such that $P\langle X \rangle$ and $F\langle X, Y \rangle$.

7.41.4. Let P and S be 1-ary conditions. If there exists a 2-ary condition F bijective with respect to P and S, we say that P and S are *equipollent* or that there are as many objects satisfying P as there are objects satisfying S (abbreviation: Card P = Card S). If there exists a condition S' implying S and equipollent to P, we will say that there are most as many objects satisfying P as there are those satisfying S or that there are at least as many objects satisfying S as there are those satisfying P (abbreviations:

Card $P \leqq$ Card S or Card $S \geqq$ Card P). – We note that the expressions "Card P", etc., are not meaningful unless being in a context like "Card $P =$ Card S", etc. (cf. 7.41.6).

7.41.5. We introduce the following convention: if "A" is a name for objects satisfying a certain condition P, then, e.g., "there are as many A's as ..." stands for "there are as many objects satisfying P as ...", etc.

7.41.6. If P is a 1-ary condition, M is a set, card $M = \varkappa$ and Card $P =$ $=$ Card $(x \in M)$ (or Card $P \leqq$ Card $(x \in M)$ or Card $P \geqq$ Card $(x \in M)$), then we will say that there are *exactly* $\varkappa$ (respectively, *at most* $\varkappa$ or *at least* $\varkappa$) objects satisfying P (abbreviations: Card $P = \varkappa$, Card $P \leqq \varkappa$, and Card $P \geqq \varkappa$). – We recall that "Card P" is not meaningful unless being in a context like "Card $P = \ldots$", "Card $P \geqq$ $\geqq \ldots$", etc.

7.42. Fact. *If P is a 1-ary condition, α is a cardinal,* Card $P \leqq \alpha$ *and* Card $P \geqq$ $\geqq \alpha$, *then* Card $P = \alpha$.

7.43.1. Before proceeding to the proof of Card (φ is an extended Shannon semi-entropy on $\mathfrak{W}_F$) $\leqq 2^\omega$, we state explicitly what a mapping is (till now, this has not been necessary).

7.43.2. A mapping is either I) a class F such that every member of F is an ordered pair, and if $(x, y_1) \in F$ and $(x, y_2) \in F$, then $y_1 = y_2$, or II) a triple $F = \langle f, \mathscr{X}, \mathscr{Y} \rangle$ such that f is a mapping (in the sense I), $\mathscr{X}$ and $\mathscr{Y}$ are sets or sets endowed with a structure (e.g., W-spaces, semimetric spaces, etc.), and dom f is equal to the underlying set of $\mathscr{X}$. – Note that we examine only very few kinds of sets endowed with a structure, and therefore it is superfluous to define "sets endowed with a structure" ("structured sets") in a general manner.

7.43.3. As a rule, it is clear from the context whether "mapping" is used in the sense I or II, and if $F = \langle f, \mathscr{X}, \mathscr{Y} \rangle$ is a mapping, whether dom F means $\mathscr{X}$ or dom f.

7.43.4. Clearly, since $\mathfrak{W}_F$ is a proper class, an extended (b.s.) Shannon semi-entropy on $\mathfrak{W}_F$ (or on $\mathfrak{W}$) is a mapping in the sense I, hence a class. Therefore, expressions like "there are exactly $\varkappa$ extended Shannon semientropies ..." have an exact meaning, described in 7.41.

7.44. Fact. *Let K_m, $m = 1, 2, \ldots$, be sets, card $K_m = m$. If φ and ψ are extended Shannon semientropies on $\mathfrak{W}_F$ and for any $m = 1, 2, \ldots$, $\varphi \upharpoonright \mathfrak{W}_F(K_m) = \psi \upharpoonright$ $\upharpoonright \mathfrak{W}_F(K_m)$, then $\varphi = \psi$.*

Proof. Let $P = \langle Q, \varrho, \mu \rangle$ be an FW-space. Put $m =$ card Q and let $g: K_m \to Q$ be a bijection. If $x, y \in K_m$, put $\varrho'(x, y) = \varrho(gx, gy)$. If $X \subset K_m$, put $\mu'(X) =$ $= \mu(g(X))$. Then $S = \langle K_m, \varrho', \mu' \rangle \in \mathfrak{W}_F(K_m)$ and hence $\varphi S = \psi S$. Since φ and ψ are regular (see 2.7 and 2.8), we have $\varphi P = \varphi S$, $\psi P = \psi S$.

7.45. Fact. *Let Q be a finite non-void set. Then the set of all feebly continuous* (see 2.12) *functions $f: \mathfrak{W}_F(Q) \to R_+$ is of cardinality 2^ω.*

The proof is standard and can be omitted.

7.46. Proposition. *There are at most 2^ω extended Shannon semientropies on $\mathfrak{W}_F$.*

Proof. Choose sets K_n, $n = 1, 2, \ldots$, such that card $K_n = n$. Let A be the set of all sequences of the form $(\varphi \upharpoonright \mathfrak{W}_F(K_n): n = 1, 2, \ldots)$, where φ is an e.S. semientropy on $\mathfrak{W}_F$. Since e.S. semientropies are feebly continuous (see 2.19, 2.11, 2.12) on every $\mathfrak{W}_F(Q)$ where Q is finite non-void, we have card $A \leqq 2^\omega$ by 7.45. By 7.44, for any $\alpha \in A$ there is exactly one e.S. semientropy φ on $\mathfrak{W}_F$ such that $\alpha = (\varphi \upharpoonright \mathfrak{W}_F(K_n): n = 1, 2, \ldots)$. We denote this semientropy by φ_α. Clearly, the 2-ary condition $\psi = \varphi_\alpha$ (where ψ and α are variables) is bijective for the 1-ary conditions (ψ is equal to some φ_α) and $\alpha \in A$. Thus, Card (ψ is an e.S. seminetropy on $\mathfrak{W}_F$) $\leqq$ card A.

Remark. In 10.10 it will be proved that there are exactly 2^ω extended Shannon entropies (hence also semientropies) on $\mathfrak{W}_F$.

7.47. It will be proved later (10.12) that there exist enormously many extended (b.s.) Shannon entropies on $\mathfrak{W}$, and examples will be exhibited (see 10.11) of e.(b.s.) Shannon entropies φ and W-spaces P, S such that there exists a PC-injection $f: P \to S$ whereas $\varphi P \neq \varphi S$. This shows that the concept of an extended (in the broad sense) Shannon entropy (semientropy) is too broad indeed, and a suitable restriction is desirable. In fact, if we impose certain fairly mild persistence (invariance) conditions, see 7.49 below, and consider extended Shannon entropies defined on a suitable class $\mathscr{M} \subset \mathfrak{W}$ (possibly coinciding with $\mathfrak{W}_{MB}$, cf. 7.60), then the number of distinct *e.* Shannon entropies on $\mathscr{M}$ is not too large (an estimate will be given in 7.57).

7.48. Definition. Let $\mathfrak{W}_F \subset \mathscr{X} \subset \mathfrak{W}$. A functional $\varphi: \mathscr{X} \to \bar{\boldsymbol{R}}_+$ will be called an *extended (in the broad sense) Shannon entropy or semientropy on $\mathscr{X}$* (abbreviation: e.(b.s.) S *entropy or semientropy on $\mathscr{X}$*) if φ is a hypoentropy (see 2.6) and $\varphi \upharpoonright \mathfrak{W}_F$ is respectively an e. Shannon entropy or semientropy on $\mathfrak{W}_F$. – We note that e.(b.s.) S. entropies (semientropies) on $\mathfrak{W}$ in the sense of the present definition are exactly the e.(b.s.) S. entropies (semientropies) on $\mathfrak{W}$ in the sense of 2.26.

7.49. Definition. Let $\mathscr{X} \subset \mathfrak{W}$. A functional $\varphi: \mathscr{X} \to \bar{\boldsymbol{R}}$ will be called *persistent with respect* to *injective mappings satisfying* PCI (*satisfying* SCI), abbreviated PCI-*persistent* (SCI-*persistent*), if $\varphi P_1 = \varphi P_2$ whenever $P_1 \in \mathscr{X}$, $P_2 \in \mathscr{X}$ and there exists an injective mapping $f: P_1 \to P_2$ satisfying PCI (satisfying SCI, respectively). A PCI-persistent functional will be also called *persistent in the broad sense.*

7.50. Fact. *Every extended Shannon semientropy φ on $\mathfrak{W}_F$ is* SCI-*persistent, hence* PCI-*persistent.*

Proof. If P_1 and P_2 are FW-spaces and an injective mapping $f: P_1 \to P_2$ satisfies SCI, then f is conservative in the sense of 2.7 and therefore, φ being regular, $\varphi P_1 = \varphi P_2$.

7.51. It is not immediately clear that there exist PCI-persistent e.S. entropies (semientropies) an $\mathfrak{W}$. However, it will be shown (see 7.56 below) that at least some of the functionals C_τ, C_τ^* (defined on $\mathfrak{W}$) are PCI-persistent, and even SCI-persistent.

7.52. Notation. If P is a W-space, then $\exp P$ ($\exp^* P$) denotes the set of all subspaces (pure subspaces, respectively) of P.

7.53. Fact and convention. *Let $P = \langle Q, \varrho, \mu \rangle$ and $S = \langle T, \sigma, \nu \rangle$ be W-spaces. Let $\psi: P \to S$ satisfy* SCI. *Then for any $U \leqq P$ there is exactly one subspace $V \leqq S$ such that if $V = g \,.\, S$, then $U = (g \circ \psi) \,.\, P$. The space V will be denoted by $\psi(U)$, and the mapping $U \mapsto \psi(U)$ will be also denoted by ψ, provided there is no danger of confusion.*

Proof. If $U \leqq P$, choose a $\bar{\mu}$-measurable function f such that $U = f \,.\, P$. Define g as follows: if $y \in \psi Q$, then $g(y) = f(\psi^{-1} y)$; if $y \in T \setminus \psi Q$, then $g(y) = 0$. Then g is $\bar{\nu}$-measurable; put $V = g \,.\, S$. Clearly, $g \circ \psi = f$, and if $V' \leqq S$, $V' = h \,.\, S \mathbin{\dot{+}} V$, then $U \mathbin{\dot{+}} (h \circ \psi) \,.\, P$.

7.54. Fact. *Let P and S be W-spaces and let $\psi: P \to S$ satisfy* SCI. *Then* (1) *the mapping $\psi: \exp P \to \exp S$ is bijective,* (2) *$U \leqq P$ is pure iff $\psi(U) \leqq S$ is pure,* (3) *if $U \leqq P$, $V \leqq P$, then $U \leqq V$ iff $\psi(U) \leqq \psi(V)$,* (4) *if $a, b \in \boldsymbol{R}_+$, $U \leqq P$, $V \leqq P$, $aU \mathbin{\dot{+}} bV \leqq P$, then $\psi(aU \mathbin{\dot{+}} bV) = a\,\psi(U) \mathbin{\dot{+}} b\,\psi(V)$,* (5) *if $U_k \leqq P$, $k \in K$, then $(U_k: k \in K)$ is a partition of P iff $(\psi(U_k): k \in K)$ is a partition of S,* (6) *if $T_x \leqq P$, $x \in D$, then $(T_x: x \in D)$ is a (pure) dyadic expansion of P iff $(\psi(T_x): x \in D)$ is a (pure) dyadic expansion of S.*

7.55. Proposition. *Let $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$. Let P and S be W-spaces and let $\psi: P \to S$ satisfy* SCI. *Then* (1) *if $U \leqq P$, $V \leqq P$, then $\tau(U, V) = \tau(\psi(U), \psi(V))$,* (2) *$C_r^*(P) = C_\tau^*(S)$, $C_\tau(P) = C_\tau(S)$.*

Proof. We prove (1) and (2) for $\tau = r_1 = r$. The remaining cases are similar.

I. Let $P = \langle Q, \varrho, \mu \rangle$, $S = \langle T, \sigma, \nu \rangle$; put $M = \psi(Q)$. Let $U \leqq P$, $V \leqq P$. Choose $\bar{\mu}$-measurable functions f, g such that $U = f \,.\, P$, $V = g \,.\, P$. For $y \in T$ put $F(y) = f(\psi^{-1} y)$, $G(y) = g(\psi^{-1} y)$ if $y \in M$, $F(y) = G(y) = 0$ if $y \in T \setminus M$. Then $\psi(U) = F \,.\, S$, $\psi(V) = G \,.\, S$. Define functions k and K as follows: $k(x, x') = f(x)\, g(x')$ for $x, x' \in Q$, $K(y, y') = F(y)\, G(y')$ for $y, y' \in T$. Clearly, we have

$$\int_{T \times T} \sigma \,\mathrm{d}(F \,.\, \nu \times G \,.\, \nu) = \int_{M \times M} \sigma K \,\mathrm{d}(\nu \times \nu) = \int_{Q \times Q} \varrho k d(\mu \times \mu)\,,$$

hence $\tau(\psi(U), \psi(V)) = \tau(U, V)$.

II. Since by 3.8, τ is an NGF, we have by Theorem III (in Section 6), $C_\tau(P) = \mathscr{F}_{\mathrm{De}}(P)\text{-}\lim \Gamma_\tau(\mathscr{P})$, $C_\tau(S) = \mathscr{F}_{\mathrm{De}}(S)\text{-}\lim \Gamma_\tau(\mathscr{S})$. Suppose $C_\tau(P) < C_\tau(S)$. Choose b such that $C_\tau(P) < b < C_\tau(S)$. Let $\mathscr{V} = (V_j: j \in J)$ be an arbitrary partition of S. Then by 7.54, $(\psi^{-1} V_j: j \in J)$ is a partition of P. Hence, due to $\mathscr{F}_{\mathrm{D}}(P)\text{-}\lim \Gamma_\tau(\mathscr{P}) = C_\tau(P) < b$, there exists a dyadic expansion $\mathscr{P} = (P_\alpha: x \in D)$ of P such that $\mathscr{P}''$ refines $(\psi^{-1} V_j: j \in J)$ and $\Gamma_\tau(\mathscr{P}) < b$.

By 7.54, $\mathscr{S} = (\psi(P_x): x \in D)$ is a dyadic expansion of S. By the assertion (1), already proved, we have $\Gamma_\tau(\mathscr{S}) = \Gamma_\tau(\mathscr{P}) < b$. Clearly, $\mathscr{S}''$ refines $\mathscr{V}$. Since $\mathscr{V}$ was arbitrary, we get $C_\tau(S) \leqq b$, which is a contradiction. Hence $C_\tau(P) \geqq C_\tau(S)$. In an analogous way, $C_\tau(S) \geqq C_\tau(P)$ is proved. Hence $C_\tau(P) = C_\tau(S)$. For C_τ^*, the proof is similar.

7.56. Proposition. *If $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$, then C_τ^* and C_τ are* SCI-*persistent and hence* PCI-*persistent.*

Proof. See 7.55.

7.57. Proposition. *There are at most* $\exp^{(4)} \omega$ *persistent (in the broad sense) extended Shannon semientropies on the class of all weakly Borel metric W-spaces P such that the topological weight of P is not real-measurable.*

Proof. Let $\mathscr{M}$ denote the class in question. Let $\mathscr{S}$ be a set with the properties described in 7.39. If φ_1 and φ_2 are PCI-persistent extended (b.s.) Shannon semientropies on $\mathscr{M}$ and $\varphi_1 \upharpoonright \mathscr{S} = \varphi_2 \upharpoonright \mathscr{S}$, then for any $P \in \mathscr{M}$ there exists a CP-injection $f: S \to P$, where $S \in \mathscr{S}$, and therefore we have $\varphi_1 P = \varphi_1 S = \varphi_2 S = \varphi_2 P$, hence $\varphi_1 = \varphi_2$. This shows that there are as many PCI-persistent extended (b.s.) Shannon semientropies on $\mathscr{M}$ as there are e.(b.s.) Shannon semientropies on $\mathscr{S}$. Since card $\mathscr{S} \leqq \exp^{(3)} \omega$, the proposition is proved.

7.58. In conclusion, we consider some questions of consistency. We note that some of the statements below can be expressed in a metalanguage of GB (cf. 7.41, in particular 7.41.1), but not in GB itself, and should be properly called metapropositions; however, this distinction can be disregarded here.

7.59. Convention. Let S be a statement expressible in GB, (a version of) the Gödel-Bernays axiomatic system. If either S can be proved in GB or its negation cannot be proved (in GB), we will say that S is consistent relative to GB or simply that it is admissible to assume S. – We note that the formulation we use (either S can be proved or non S cannot be proved) is, of course, equivalent to the usual one.

7.60. Proposition. *It is admissible to assume that there are no real-measurable cardinals.*

This is a well-known fact.

7.61. Fact. *If there are no real-measurable cardinals, then there are at most* $\exp^{(4)} \omega$ *persistent (in the broad sense) Shanonn semientropies on the class of all weakly Borel metric W-spaces.*

Proof. See 7.57.

7.62. Proposition. *It is admissible to assume (the conjunction of) the following statements: (1) the reduced topological weight of every weakly Borel metric W-space is countable; (2) there is a set $\mathscr{S}$ of cardinality $\leqq \exp^{(3)} \omega$ consisting of second-countable metric W-spaces and such that for any weakly Borel metric*

W-space P there is a PC*-injection $f: S \to P$ where $S \in \mathscr{S}$*; (3) *there are at most* $\exp^{(4)} \omega$ *persistent (in the broad sense) extended Shannon semientropies on the class of all weakly Borel metric spaces.*

Proof. For (1), see 7.60 and 7.28; for (2), see 7.60 and 7.39; for (3), see 7.61.

8

In this section, two questions (III and VI) posed in 2.30 are answered and it is proved that $C_\tau(P) \leqq C^*_\tau(P)$ holds for any normal gauge functional τ in a fairly wide class of metric W-spaces.

First we prove (see 8.7) that for any gauge functional $\tau \geqq r$ and any W-space P, $C^*_\tau(P)$ and $C_\tau(P)$ are positive unless $d(P) = 0$. Then we show (see 8.28 and 8.38) that for any normal gauge functional τ, $C^*_\tau(P)$ and $C_\tau(P)$ are finite whenever P is expansion-bounded (see 8.13) and $C_\tau(P) \leqq C^*_\tau(P)$ whenever P satisfies a certain condition (see 8.35) of the total boundedness type.

8.1. Definition and notation. Let $S = \langle Q, \varrho \rangle$ be a semimetric space. The infimum of all $a \in \bar{\boldsymbol{R}}_+$ such that $\varrho(x, y) \leqq a$ for each $x, y \in Q$ will be denoted by diam S. If $T \subset Q$, then diam $(S \restriction T)$ will be denoted by $\operatorname{diam}_S T$ or simply diam T. If diam $S < \infty$, we will say that S is *bounded*, and if $T \subset Q$, $\operatorname{diam}_S T < \infty$, we will say that the set T is *bounded* (in S).

8.2. Lemma. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let* diam $\langle Q, \varrho \rangle < \infty$. *Let $\varepsilon > 0$. Then there exists a pure partition $(U_k: k \in K)$ of P such that $\Sigma(\hat{r}(U_k): k \in K) < \varepsilon$.*

Proof. Choose $t \in \boldsymbol{R}$, $t >$ diam $\langle Q, \varrho \rangle$. Put $G = \{(x, y): \varrho(x, y) = 0\}$. Let ν denote the completion $[\mu \times \mu]$ of $\mu \times \mu$. Since G is ν-measurable, there exist $A_i \in \operatorname{dom} \mu$, $B_i \in \operatorname{dom} \mu$, $i \in N$, such that for $V = \bigcup(A_i \times B_i: i \in N)$ we have $V \supset G$, $\nu(V \setminus G) < \varepsilon/t$. Clearly, $\int_V \varrho \, \mathrm{d}(\mu \times \mu) < \varepsilon$. Put $X_i = A_i \cap B_i$, $Y_0 = X_0$, $Y_i = X_i \setminus \bigcup(X_j: j < i)$ for $i = 1, 2, \ldots$, $T = \bigcup(Y_i \times Y_i: i \in N)$. Then $T \subset \bigcup(X_i \times X_i: i \in N) \subset V$, hence $\int_T \varrho \, \mathrm{d}(\mu \times \mu) < \varepsilon$. Since Y_i are disjoint, we have $\int_T \mathrm{d}(\mu \times \mu) = \Sigma(\int_{Y_i \times Y_i} \varrho \, \mathrm{d}(\mu \times \mu): i \in N)$ and therefore $\Sigma(\hat{r}(Y_i): i \in N) < \varepsilon$.

Put $Z_n = \bigcup(Y_i: i > n)$. Then $Z_n \supset Z_{n+1}$, $\bigcup(Z_n: n \in N) = \emptyset$ and therefore $\hat{r}(Z_n) \to 0$. Hence there exists an $m \in N$ such that $\Sigma(\hat{r}(Y_i): i \leqq m) + \hat{r}(Z_m) < \varepsilon$. Clearly, $(Y_0, \ldots, Y_m, Z_m)$ is a pure partition of P.

8.3. Fact. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let $\mathscr{U} = (U_k: k \in K)$ and $\mathscr{V} = (V_m: m \in M)$ be partitions of P. If $\mathscr{U}$ is finer than $\mathscr{V}$, then $\Sigma(\hat{r}(U_k): k \in K) \leqq \Sigma(\hat{r}(V_m): m \in M)$.*

Proof. Let $(K_m: m \in M)$ be a partition of K such that $\Sigma(U_k: k \in K_m) = V_m$ for each $m \in M$. Let f_k, $k \in K$, be functions such that $U_k = f_k \, . \, P$ for all $k \in K$. For $m \in M$ put $g_m = \Sigma(f_k: k \in K_m)$. Define $u: Q \times Q \to \boldsymbol{R}_+$, $v: Q \times Q \to \boldsymbol{R}_+$ as follows: $u(x, y) = \Sigma(f_k(x) f_k(y): k \in K)$, $v(x, y) = \Sigma(g_m(x) \, g_m(y): m \in M)$. It is easy to see that $u(x, y) \leqq v(x, y)$ for all $x, y \in Q$. Clearly, $\int \varrho u \, \mathrm{d}(\mu \times \mu) = \Sigma(\hat{r}(U_k): k \in K)$, $\int \varrho v \, \mathrm{d}(\mu \times \mu) = \Sigma(\hat{r}(V_{\mathscr{M}}): m \in M)$. This proves the assertion.

8.4. Fact. *Let* $\mathscr{P} = (P_x: x \in D)$ *be a dyadic expansion of a W-space P. Then* $\Gamma_r(\mathscr{P}) \geqq \Sigma(4\hat{r}(P_{x0}, P_{x1})/wP_x: x \in D')$.

Proof. By 2.16.1, $H(wP_{x0}, wP_{x1}) \geqq 4wP_{x0} \,.\, wP_{x1}/wP_x$ for each $x \in D'$. Hence $\Gamma_r(\mathscr{P}) = \Sigma(H(wP_{x0}, wP_{x1})\, r(P_{x0}, P_{x1}): x \in D') \geqq \Sigma(4wP_{x0} \,.\, wP_{x1} \,.\, r(P_{x0}, P_{x1})/wP_x: x \in D') = \Sigma(4\hat{r}(P_{x0}, P_{x1})/wP_x: x \in D')$.

8.5. Fact. *Let* $\mathscr{P} = (P_x: x \in D)$ *be a dyadic expansion of a W-space P. Then* $\Gamma_r(\mathscr{P}) + 2\,\Sigma(\hat{r}(P_z): z \in D'')/wP \geqq 2\hat{r}(P)/wP$.

Proof. By 8.4, we get $\Gamma_r(\mathscr{P}) \geqq \Sigma(4\hat{r}(P_{x0}, P_{x1}): x \in D')/wP$. It is easy to see that $\Sigma(2\hat{r}(P_{x0}, P_{x1}): x \in D') + \Sigma(\hat{r}(P_z): z \in D'') = \hat{r}(P)$. This proves the assertion.

8.6. Proposition. *If τ is a gauge functional and $\tau \geqq r$, then for any W-space P,* $C_\tau^*(P) \geqq 2\hat{r}(P)/wP$, $C_\tau(P) \geqq 2\hat{r}(P)/wP$.

Proof. If $wP = 0$, then $\hat{r}(P) = 0$. Hence we may assume that $wP > 0$. Let $P = \langle Q, \varrho, \mu \rangle$. We first prove the proposition for the case $\tau = r$ and diam $\langle Q, \varrho \rangle < \infty$. Let $\varepsilon > 0$ be given. By 8.2 there exists a pure partition $\mathscr{U} = (U_k: k \in K)$ of P such that $\Sigma(\hat{r}(U_k): k \in K) < \varepsilon$. Let $\mathscr{P} = (P_x: x \in D)$ be a dyadic expansion of P such that $\mathscr{P}''$ refines $\mathscr{U}$. Then by 8.3 and 8.5, $\Gamma_r(\mathscr{P}) + 2\varepsilon/wP \geqq 2\hat{r}(P)/wP$. By Theorem III (in Section 6) we get $C_r(P) + 2\varepsilon/wP \geqq 2\hat{r}(P)/wP$, $C_r^*(P) + 2\varepsilon/wP \geqq 2\hat{r}(P)/wP$. Since $\varepsilon > 0$ was arbitrary, this proves the proposition for $\tau = r$ and diam $\langle Q, \varrho \rangle < \infty$.

Now, retaining $\tau = r$, we omit the assumption diam $\langle Q. \varrho \rangle < \infty$. We define the semimetrics ϱ_t, $t \in \boldsymbol{R}_+$, as follows: $\varrho_t(x, y) = \min(\varrho(x, y), t)$ for all $x, y \in Q$. We put $P_t = \langle Q, \varrho_t, \mu \rangle$. Clearly, for any $x, y \in Q$, $\varrho_t(x, y) \leqq \varrho_s(x, y)$ if $t \leqq s$, and $\varrho_t(x, y) \to \varrho(x, y)$ if $t \to \infty$. This implies $\hat{r}(P_t) \to \hat{r}(P)$ for $t \to \infty$. As we have shown, $C_r(P_t) \geqq 2\hat{r}(P_t)/wP$, $C_r^*(P_t) \geqq 2\hat{r}(P_t)/wP$ for every $t \in \boldsymbol{R}_+$. Since $\varrho_t \leqq \varrho$, this implies $C_r(P) \geqq 2\hat{r}(P)/wP$, $C_r^*(P) \geqq 2\hat{r}(P)/wP$.

Finally, if $\tau \geqq r$ is a gauge functional, then we apply Proposition 3.20.

Remark. There are examples (see 10.21) showing that the proposition is not valid if the assumption $\tau \geqq r$ is omitted. I do not know whether it remains true if instead of $\tau \geqq r$ we assume that τ is normal (in fact, I do not know whether there are normal gauge functionals τ not satisfying $\tau \geqq r$).

8.7. Proposition. *If P is a W-space, τ is a gauge functional and $\tau \geqq r$, then $C_\tau^*(P)$ and $C_\tau(P)$ are positive whenever $d(P) > 0$. If P is a W-space and $d(P) = 0$, then $C_\tau^*(P) = C_\tau(P) = 0$ for each gauge functional τ.*

Proof. The first assertion follows from 8.6 since $d(P) > 0$ implies $\hat{r}(P) > 0$. The second assertion follows from 3.14 and the definition (see 3.17 of C_τ^* and C_τ.)

Remark. An outline of proof of the first assertion has been given in [4], 3.9 and 3.10 (in fact, only for $\tau = r$; however, it is obvious that if $\tau \geqq r$, then $C_r(P) > 0$ and $C_r^*(P) > 0$ imply, respectively, $C_\tau(P) > 0$ and $C_\tau^*(P) > 0$).

8.8. Definition and notation. Let $S = \langle Q, \varrho \rangle$ be a semimetric space. An indexed set $(S_k: k \in K)$ will be called a *partition* of S if K is a finite non-void set and $S_k = S \upharpoonright T_k$ where $(T_k: k \in K)$ is a partition of the set Q. An indexed set $\mathcal{S} = (S_x: x \in D)$ will be called a *dyadic expansion* of S if there is a dyadic expansion (see 4.16) $(T_x: x \in D)$ of the set Q such that $S_x = S \upharpoonright T_x$ for all $x \in D$. We put $\mathcal{S}'' = (S_x: x \in D'')$.

8.9. Definition. A W-space P will be called *bounded* if $d(P) < \infty$. If P is a W-space (a semimetric space), then (I) we will say that P is *partition-fine* (in more detail, *d-partition-fine* or, respectively, diam-*partition-fine*) if for each $\varepsilon > 0$ there exists a partition $(U_k: k \in K)$ of P such that for any $k \in K$, $d(U_k) < \varepsilon$ (respectively, $\operatorname{diam} U_k < \varepsilon$), (II) we will say that P is *totally bounded* (in more detail, *d-totally bounded* or, respectively, diam-*totally bounded*) if P is partition-fine and bounded.

Remark. There are partition-fine W-spaces P and semimetric spaces S such that $d(P) = \infty$, $\operatorname{diam} S = \infty$. An example: $S = \langle N, \varrho \rangle$, $P = \langle N, \varrho, \mu \rangle$ where $\varrho(x, y) = 0$ if $x - y$ is even, $\varrho(x, y) = x + y$ if $x - y$ is odd, μ is any measure on N such that $\operatorname{dom} \mu = \exp N$, $\mu\{x\} > 0$ for all $x \in N$.

8.10. Fact. *A partition-fine metric space is bounded, hence totally bounded.*

8.11. We are now going to prove the following proposition, due to J. Hejcman: *for any W-space* $P = \langle Q, \varrho, \mu \rangle$, *if* $\langle Q, \varrho \rangle$ *is totally bounded, then so is* P (in fact, we prove a sligthly stronger assertion, see 8.11.3 below).

8.11.1. Fact. *Let* $P = \langle Q, \varrho, \mu \rangle$ *be a W-space. Let* $A \subset B \subset Q$, $B \in \operatorname{dom} \bar{\mu}$. *If* $\bar{\mu} B$ *is equal to* $\mu_e(A)$, *the outer measure of* A (see 7.5), *then* $d(B \,.\, P) = d(P \upharpoonright A) \leqq \operatorname{diam} A$.

Proof. Clearly, $d(P \upharpoonright A) \leqq d(B \,.\, P)$. We are going to prove $d(B \,.\, P) \leqq d(P \upharpoonright A)$. Put $v = \mu \times \mu$. By the definition of d, there exists a set $Z \in \operatorname{dom} v$ such that $vZ = 0$ and $\varrho(x, y) \leqq d(P \upharpoonright A)$ whenever $x, y \in A$, $(x, y) \operatorname{non} \in Z$. Let G consist of all $(x, y) \in B \times B$ such that $\varrho(x, y) \leqq d(P \upharpoonright A)$. Put $Z_1 = Z \cap (B \times B)$, $G_1 = G \cup Z_1$. Then $A \times A \subset G_1$, hence $\bar{v} G_1 \geqq v_e(A \times A)$ and therefore, by 7.6, $\bar{v} G_1 \geqq (\mu_e(A))^2$. This shows that $\bar{v} G_1 \geqq (\bar{\mu} B)^2$ and therefore, due to $G_1 \subset B \times B$, $\bar{v}(B \times B \setminus G_1) = 0$ and $\bar{v}(B \times B \setminus G) = 0$. Since $\varrho(x, y) \leqq d(P \upharpoonright A)$ whenever $(x, y) \in G$, we have shown that $d(B \,.\, P) \leqq d(P \upharpoonright A)$.

8.11.2. Lemma. *Let* $P = \langle Q. \varrho, \mu \rangle$ *be a W-space. Let* $(A_k: k \in K)$ *be a partition of* Q. *Then there exists a pure partition* $(U_k: k \in K)$ *of* P *such that* $d(U_k) \leqq d(P \upharpoonright A_k) \leqq \operatorname{diam} A_k$ *for each* $k \in K$.

Proof. Clearly, there are sets $B_k \in \operatorname{dom} \mu$ such that $A_k \subset B_k$, $\mu B_k = \mu_e(A_k)$ for each $k \in K$. By 8.11.1, $d(B_k \,.\, P) = d(P \upharpoonright A_k) \leqq \operatorname{diam} A_k$. Let $f: K \to \{0, \ldots, n\}$ be a bijection and put $T_k = B_k \setminus \bigcup(B_j: f(j) < f(k))$, $U_k = T_k \,.\, P$. Clearly, $(U_k: k \in K)$ is a pure partition and $d(U_k) \leqq d(B_k \,.\, P)$.

8.11.3. Lemma. *Let $P = \langle Q, \varrho, \mu \rangle$ qe a W-space. If for any $\varepsilon > 0$ there exists a partition $(A_k: k \in K)$ of Q such that $d(P \upharpoonright A_k) < \varepsilon$ for each $k \in K$, then P is partition-fine.*

This is an immediate consequence of 8.11.2.

8.11.4. Proposition. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. If $\langle Q, \varrho \rangle$ is totally bounded (respectively, partition-fine), then so is P.*

This follows at once from 8.11.3.

Remark. There exist W-spaces $\langle Q, \varrho, \mu \rangle$ such that (1) $\langle Q, \varrho \rangle$ is totally bounded, (2) if $(T_k: k \in K)$ is a partition of Q and all T_k are in dom $\bar{\mu}$, then $\max \{\text{diam } T_k: k \in K\} \geqq 1$. See 10.23.

8.12. There are examples (see 10.28) of totally bounded metric W-spaces P such that $C(P) = C^*(P) = \infty$. Therefore we introduce (see 8.13 and 9.27) some properties stronger than the d-total boundedness defined in 8.9.

8.13. Notation and definition. Let P be a W-space or a semimetric space. Then Ded (P) (De-diam P) will denote the infimum of all $a \in \bar{\boldsymbol{R}}_+$ such that for any $\varepsilon > 0$ there exists a dyadic exansion $(P_x: x \in D)$ satisfying the following conditions:

(1) $d(P_x) < \varepsilon$ (respectively, diam $P_x < \varepsilon$) for all $x \in D''$,

(2) $\Sigma(\max \{d(P_x): x \in \{0, 1\}^m \cap D\}: \{0, 1\}^m \cap D \neq \emptyset) \leqq a$ or,

respectively, $\Sigma(\max \{\text{diam } P_x: x \in \{0, 1\}^m \cap D\}: \{0, 1\}^m \cap D \neq \emptyset) \leqq a$.

If Ded $(P) < \infty$ (or De-diam $P < \infty$), we will say that P is *De-bounded.* – The following terminology will also be used provided there is no danger of confusion: Ded (P) (or De-diam P) will be called the *expansion-diameter* of P and a De-bounded space will be called *expansion-bounded.*

Remark. For examples of De-bounded spaces and of totally bounded spaces which are not De-bounded see 8.20.

8.14. Fact. *Every* De-*bounded W-space or semimetric space is totally bounded.*

8.15. Fact. *If S is a subspace of a semimetric space P, then* De-diam $S \leqq$ De-diam P. *If $P_i = \langle Q_i, \varrho_i \rangle$, $i = 1, 2$, are semimetric spaces and $f: Q_1 \to Q_2$ is a surjective mapping such that $\varrho_2(fx, fy) \leqq \varrho_1(x, y)$ for all $x, y \in Q_1$, then* De-diam $P_2 \leqq$ De-diam P_1. *If $P = \langle Q, \varrho \rangle$ is a metric space and S is a dense subspace, then* De-diam $S =$ De-diam P.

8.16. Proposition. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Then $d(P) \leqq$* diam $\langle Q, \varrho \rangle$ *and* Ded $(P) \leqq$ De-diam $\langle Q, \varrho \rangle$, *hence, in particular if $\langle Q, \varrho \rangle$ is expansion-bounded, then so is P.*

Proof. The first assertion is evident. To prove the second, let $b =$ De-diam $\langle Q, \varrho \rangle$. Choose an $a > b$. Let $\varepsilon > 0$ be given. Then there exists a dyadic expansion $(Q_x: x \in D)$ of Q such that (i) diam $Q_x < \varepsilon$ for each $x \in D''$, (ii) $\Sigma(\max \{\text{diam } Q_x: x \in \{0, 1\}^m \cap D\}: m \in K) \leqq a$, where $K = \{m \in N: \{0, 1\}^m \cap D \neq \emptyset\}$. – It is well known and easy to prove that the following assertion holds: $(*)$ if for $i = 1, 2$, $A_i \subset B_i \subset Q$, $B_i \in \operatorname{dom} \bar{\mu}$, $\bar{\mu}B_i = \mu_e(A_i)$, then $\bar{\mu}(B_1 \cup B_2) = \mu_e(A_1 \cup A_2)$. – Choose B_z, $z \in D''$, such that $B_z \in \operatorname{dom} \mu$, $B_z \supset Q_z$, $\mu B_z = \mu_e(Q_z)$ for each $z \in D''$. Put $M_x = \bigcap(B_z\colon z \in D(x) \cap D'')$ for each $x \in D$. Then, for any $x \in D$, we have $M_x \supset Q_x$ and, by $(*)$, $\mu M_x = \mu_e(Q_x)$. Consequently, by 8.11.1, $d(M_x . P) \leqq \operatorname{diam} Q_x$ for each $x \in D$. Now let $f\colon D'' \to \{0, \ldots, n\}$ be a bijection, put $U_z = B_z \setminus \bigcup(B_y\colon f(y) < f(z))$ for each $z \in D''$, and put $V_x = \bigcup(U_z\colon z \in D(x) \cap D'')$ for each $x \in D$. Clearly, $(V_x . P\colon x \in D)$ is a dyadic expansion of P and we have $V_x \subset M_x$, hence $d(V_x . P) \leqq \operatorname{diam} Q_x$ for each $x \in D$. This proves that $d(V_x . P) < \varepsilon$ for each $x \in D''$ and $\Sigma(\max\{d(V_x . P)\colon x \in \{0, 1\}^m \cap D\}\colon m \in K) \leqq a$. Since $a > b$ and $\varepsilon > 0$ were arbitrary, we have shown that $\operatorname{Ded}(P) \leqq b$.

8.17. Notation and conventions. Let $1 \leqq p \leqq \infty$. Let $n = 1, 2, \ldots$. If $x = (x_i\colon i < n) \in \boldsymbol{R}^n$ or $x = (x_i\colon i \in \boldsymbol{N}) \in \boldsymbol{R}^{\boldsymbol{N}}$, then we put $|x|_p = (\Sigma|x_i|^p)^{1/p}$ if $p < \infty$, $|x|_\infty = \sup |x_i|$. The Banach space consisting of all $x = (x_i\colon i < n) \in \boldsymbol{R}^n$ (or of all $(x_i\colon i \in N) \in \boldsymbol{R}^{\boldsymbol{N}}$ satisfying $|x|_p < \infty$) and endowed with the norm $|x| = |x|_p$ will be denoted by $\mathscr{L}_p(n)$ (respectively, $\mathscr{L}_p(\boldsymbol{N})$) and will be equipped, as usual, with the metric $(x, y) \mapsto |x - y|_p$. If $n = 1, 2, \ldots$, then the Banach space $\mathscr{L}_\infty(n)$ will also be denoted by $\boldsymbol{R}^n$. – If B is a normed linear space, then B will also denote the set of points of B, and if $Q \subset B$, then Q will also denote the metric space $\langle Q, \varrho\rangle$ where $\varrho(x, y)$ is equal to the norm of $x - y$ in B. – We recall that, by 1.6, if either $x = (x_i) \in \boldsymbol{R}^n$, $y = (y_i) \in \boldsymbol{R}^n$ or $x = (x_i) \in \boldsymbol{R}^{\boldsymbol{N}}$, $y = (y_i) \in \boldsymbol{R}^{\boldsymbol{N}}$, then $x \leqq y$ means that $x_i \leqq y_i$ for all i.

8.18. Lemma. *Let $m = 1, 2, \ldots$. Let $a = (a_i\colon i < m) \in \boldsymbol{R}^m$, $a \geqq 0$. Let P be the subspace of $\boldsymbol{R}^m$ consisting of all $x = (x_i\colon i < m)$ such that for any $i < m$, $x_i = 0$ or $x_i = a_i$. Then* De-diam $P = |a|_1$.

Proof. Clearly, we can assume that $a_0 \geqq \ldots \geqq a_{m-1} > 0$. Let Q be the set of all points of P. – I. Put $E_m = \bigcup(\{0, 1\}^j\colon j \leqq m)$. For any $z = (z_i\colon i < j) \in E_m$ let B_z consist of all $x = (x_i) \in Q$ such that $x_i = z_i a_i$ for all $i < j$. Then $(B_z\colon z \in E_m)$ is a dyadic expansion of Q, $\operatorname{diam} B_z = 0$ for any $z \in E_m''$, and for any $z \in \{0, 1\}^j$, $j < m$, $\operatorname{diam} B_z = a_j$. Hence De-diam $Q \leqq \Sigma(a_j\colon j < m) = |a|_1$. – II. Clearly, there are sets $Q_j \subset Q$, $j = 0, \ldots, m - 1$, such that card $Q_j = 2^{j+1}$ and $\varrho(x, y) \geqq a_j$ for all $x, y \in Q_j$, $x \neq y$. Choose a positive $\varepsilon < a_{m-1}$. Let $(B_z\colon z \in D)$ be a dyadic expansion of Q such that $\operatorname{diam} B_z < \varepsilon$ for all $z \in D''$. Then $D \cap \{0, 1\}^m \neq \emptyset$, for otherwise we would have card $D'' \leqq 2^{m-1}$, hence (due to card $Q_{m-1} = 2^m$) card $B_z \geqq 2$ for some $z \in D''$ and therefore $\operatorname{diam} B_z \geqq a_{m-1} > \varepsilon$. For $j = 0, \ldots, m - 1$ let M_j be the union of $D \cap \{0, 1\}^j$ and $D'' \cap (\bigcup(\{0, 1\}^i\colon i < j))$. It is easy to see that $0 <$ card $M_j \leqq 2^j$ and that $(B_z\colon z \in M_j)$ is a partition of Q. Hence there is a $z \in M_j$ such that card $(B_z \cap Q_j) \geqq 2$ and therefore $\operatorname{diam} B_z \geqq a_j$. Since $\operatorname{diam} B_z < \varepsilon < a_{m-1}$ whenever $z \in D''$, we have $z \in D \cap \{0, 1\}^j$ and therefore $\max\{\operatorname{diam} B_z\colon z \in D \cap \{0, 1\}^j\} \geqq a_j$ for $j < m$, hence $\Sigma(\max\{\operatorname{diam} B_z\colon z \in D \cap \{0, 1\}^j\}\colon j < m) \geqq \Sigma(a_j\colon j < m) = |a|_1$.

8.19. Proposition. *Let* $a = (a_i) \in \mathscr{L}_\infty(N)$, $a \geqq 0$, $a_i \to 0$ *for* $i \to \infty$. *Let* P *be the subspace of* $\mathscr{L}_\infty(N)$ *consisting of all* $x \in \mathscr{L}_\infty(N)$ *such that* $0 \leqq x \leqq a$. *Then* $|a|_1 = \leqq$ De-diam $P \leqq 2|a|_1$.

Proof. I. For $n = 1, 2, \ldots$ let S_n be the subspace of P consisting of $x = (x_i) \in P$ such that, for $i < n$, x_i is equal either to 0 or to a_i, and $x_i = 0$ for $i \geqq n$. Now, 8.18 and 8.15 imply that De-diam $S_n \geqq \Sigma(a_j : j < n)$ and therefore, n being arbitrary, De-diam $P \geqq |a|_1$.

II. We are going to prove De-diam $P \leqq 2|a|_1$. It is easy to show that it is enough to prove this inequality under the assumption that $(*)$ $a_i > 0$ for all $i \in N$, $\Sigma(a_i : i \in N) < \infty$, and $2^k a_m \neq a_n$ for each $m, n, k \in N$, $m \neq n$.

If $b = (b_i)$, $c = (c_i)$ are in $\mathscr{L}_\infty(N)$, $b_i < c_i$ for all $i \in N$, and $c_i - b_i \to 0$ for $i \to \infty$, put $T(b, c) = \{x = (x_i : i \in N) \in \mathscr{L}_\infty(N) : b_i \leqq x_i < c_i$ for all $i \in N\}$. To prove De-diam $P \leqq 2|a|_1$ it is sufficient, by 8.15, to show that De-diam $T(0, a) \leqq \leqq 2|a|_1$.

For any $x = (x_n) \in \mathscr{L}_\infty(N)$ such that $x \geqq 0$, $x_n \to 0$ for $n \to \infty$, put $K(x) = = \min\{h : x_h = \max\{x_n : n \in N\}\}$. If $X = T(b, c)$ for some b, c, we define $X^{(0)}, X^{(1)}$ in the following way. Let $s = (s_i)$, $s_i = (b_i + c_i)/2$ if $i = K(c - b)$, $s_i = c_i$ if $i \neq K(c - b)$. Put $X^{(0)} = T(b, s)$, $X^{(1)} = X \setminus X^{(0)} = T(b + c - s, c)$.

For $m \in N$ put $E_m = \bigcup(\{0, 1\}^j : j \leqq m)$. Put $E = \bigcup(\{0, 1\}^j : j \in N)$. By induction we define B_z for all $z \in E$. Put $B_\emptyset = T(0, a)$. If B_z are already defined for all $z \in E_m$, then for any $zi \in E_{m+1} \setminus E_m$ put $B_{zi} = B_z^{(i)}$. It is easy to see that for any $m \in N$, $(B_z : z \in E_m)$ is a dyadic expansion of $T(0, a)$.

Now we define $a_{m,n}$, $m \in N$, $n \in N$, in the following way. Put $a_{0n} = a_n$ for all $n \in N$. If $m \in N$ and $a_{m,n}$, $n \in N$, are already defined, we put $h = K(a_{m,n} : n \in N)$, $a_{m+1,h} = = a_{mh}/2$, $a_{m+1,n} = a_{mn}$ for $n \neq h$. Clearly, for each $m \in N$, $a_{mn} \to 0$ for $n \to \infty$. We put $\delta_m = \max(a_{mn} : n \in N)$ for each $m \in N$.

It is easy to show that for any $m = 1, 2, \ldots$ and any $z \in E_m \setminus E_{m-1}$ we have diam $B_z = \delta_m$. We are going to prove that $\Sigma(\delta_m : m \in N) \leqq 2\,\Sigma(a_n : n \in N)$, $\delta_m \to 0$ for $m \to \infty$.

Clearly, every a_{mn} is of the form $a_i 2^{-k}$, $k \in N$. Hence, by the assumption $(*)$, for any m, all a_{mn} are distinct. By the definition of $K(x_n : n \in N)$, every δ_m is equal to a_{mh} where $h = K(a_{mn} : n \in N)$. Since a_{mn} are distinct, we have $\delta_m > a_{mn}$ for $n \neq h$. Since $a_{m+1,h} = a_{mh}/2$, we get $\delta_{m+1} < \delta_m$ for all $m \in N$. Hence all δ_m are distinct numbers of the form $a_i 2^{-k}$, $k \in N$. This easily implies that $\Sigma(\delta_m : m \in N) \leqq \leqq \Sigma(a_i 2^{-k} : i \in N, k \in N) \leqq 2\,\Sigma(a_n : n \in N)$, and therefore $\delta_m \to 0$ for $m \to \infty$.

Now let $\varepsilon > 0$ be given. Choose m such that $\delta_m < \varepsilon$. Consider the dyadic expansion $(B_x : x \in E_m)$. Then diam $B_x < \varepsilon$ for $x \in E''_m = E_m \setminus E_{m-1}$ and $\max\{\text{diam } B_x : x \in E_m \cap \cap \{0, 1\}^j\} = \delta_j$ for $j = 0, \ldots, m$, hence $\Sigma(\max\{\text{diam } B_x : x \in E_m \cap \{0, 1\}^j\} : j \leqq m) \leqq \leqq \Sigma(\delta_j : j \leqq m) \leqq 2|a|_1$. Since $\varepsilon > 0$ was arbitrary, we obtain De-diam $T(0, a) \leqq \leqq 2|a|_1$. This proves the proposition.

8.20. Proposition. *Let $a = (a_i) \in \mathscr{L}_\infty(N)$, $a \geqq 0$, $a_i \to 0$ for $i \to \infty$. Let $P(a)$ be the (metric) subspace of $\mathscr{L}_\infty(N)$ consisting of all $x \in \mathscr{L}_\infty(N)$ such that $0 \leqq x \leqq a$. Then (1) $P(a)$ is totally bounded, (2) $P(a)$ is expansion-bounded if and only if $\Sigma a_i < \infty$.*

Proof. See 8.19.

8.21. Notation. Let τ be a gauge functional. If P is a W-space, then the supremum of all $\Gamma_\tau(S, T)$, where $S \leqq P$, $T \leqq P$, $S + T \leqq P$, will be denoted by Γ_τ-diam P. If $\mathscr{U} = (U_k: k \in K)$ is a partition of a W-space, then $\Sigma(\Gamma_\tau\text{-diam}\, U_k: k \in K)$ will be denoted by $\Sigma\Gamma_\tau$-diam $\mathscr{U}$.

8.22. Fact. *Let τ be a gauge functional. If P is a W-space, then Γ_τ-diam $P \leqq wP \,.\, d(P)$. If $\mathscr{U} = (U_k: k \in K)$ is a partition of a W-space, then $\Sigma\Gamma_\tau$-diam $\mathscr{U} \leqq wP \max\{d(U_k): k \in K\}$.*

Proof. Let $S + T \leqq P$. Then $\Gamma_\tau(S, T) = H(wS, wT)\, \tau(S, T)$, $H(wS, wT) \leqq wS + wT \leqq wP$ by 2.4, and $\tau(S, T) \leqq d(P)$ by (GF2). The second assertion is an evident consequence.

8.23. Lemma. *Let τ be a normal gauge functional. Let P be a W-space. For each $n \in N$ let $\mathscr{P}_n = (P_{n,x}: x \in D_n)$ be a dyadic expansion of P. If $\Sigma\Gamma_\tau$-diam $\mathscr{P}'' \to 0$ for $n \to \infty$, then $C_\tau(P) \leqq \lim \Gamma_\tau(\mathscr{P}_n)$. If, in addition, all $\mathscr{P}_n$ are pure, then $C^*_\tau(P) \leqq \lim \Gamma_\tau(\mathscr{P}_n)$.*

Proof. We can assume $\lim \Gamma_\tau(\mathscr{P}_n) < \infty$. Let $a \in \boldsymbol{R}_+$, $a > \lim \Gamma_\tau(\mathscr{P}_n)$. Let $\mathscr{U}$ be a partition of P. We can assume that $\mathscr{U} = (U_0, \ldots, U_{m-1})$, $m > 1$. Choose an $\varepsilon > 0$ such that $a - m\varepsilon > \lim \Gamma_\tau(\mathscr{P}_n)$. Choose $n \in N$ such that $\Sigma\Gamma_\tau$-diam $\mathscr{P}''_n < \varepsilon$, $\Gamma_\tau(\mathscr{P}_n) < a - m\varepsilon$; write D instead of D_n, P_z instead of $P_{n,z}$, $\mathscr{P}$ instead of $\mathscr{P}_n$. Choose functions g_z, $z \in D$, such that $P_z = g_z \,.\, P$.

Let E be the set of all sequences $(a_i: i < j) \in \bigcap(\{0, 1\}^j: j < m)$ such that $a_i = 1$ if $0 \leqq i < j - 1$. Put $T_\emptyset = P$. If $y \in E$, $y = (a_i: i < j)$, $j > 0$, put $T_y = U_{j-1}$ if $a_{j-1} = 0$ and $T_y = \Sigma(U_i: j \leqq i < m)$ if $a_{i-1} = 1$. Then, by 4.6.1, $\mathscr{T} = (T_y: y \in E)$ is a dyadic expansion of P and $\mathscr{T}''$ is equal to $\mathscr{U}$ re-indexed. It is easy to see that card $E' = m - 1$.

Let F consist of all $z \in D'$ and all $z \,.\, y$, where $z \in D''$, $y \in E$. Clearly, $F \in \Delta$. If $z \in D$, put $S_z = P_z$; if $x = z \,.\, y$, $z \in D''$, $y \in E$, $y \neq \emptyset$, put $S_x = g_z \,.\, T_y$. Put $\mathscr{S} = (S_x: x \in F)$. Clearly, $\mathscr{S}$ is a dyadic expansion of P, $\mathscr{S}'' = (g_z \,.\, T_y: z \in D'', y \in E'')$, and therefore $\mathscr{S}''$ refines $(T_y: y \in E'')$, hence also $\mathscr{U}$.

It is easy to see that $\Gamma_\tau(\mathscr{S})$ is equal to $\Gamma_\tau(P_z: z \in D) + \Sigma(p_y: y \in E')$, where $p_y = \Sigma(\Gamma_\tau(g_z \,.\, T_{y0}, g_z \,.\, T_{y1}): z \in D'')$. Clearly, $p_y \leqq \Sigma(\Gamma_\tau\text{-diam}\,(g_z \,.\, T_y): z \in D'')$. Since $g_z \,.\, T_y \leqq P_z$, we obtain $p_y \leqq \Sigma(\Gamma_\tau\text{-diam}\, P_z: z \in D'') = \Sigma\Gamma_\tau\text{-diam}\, \mathscr{P} < \varepsilon$ for any $y \in E'$. Since card $E' = m - 1$, we have $\Gamma_\tau(\mathscr{S}) < \Gamma_\tau(\mathscr{P}) + (m - 1)\,\varepsilon$, hence $\Gamma_\tau(\mathscr{S}) < a - m\varepsilon + (m - 1)\,\varepsilon < a$. Since $\mathscr{S}''$ refines $\mathscr{U}$ and $\mathscr{U}$ was an arbitrary partition of P, we have shown that $\mathscr{F}_{D}$-$\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \leqq a$. Hence by Theorem III (in Section 6),

$C_\tau(P) \leqq a$. Since $a > \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n)$ was arbitrary, we have proved that $C_\tau(P) \leqq \leqq \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n)$.

Now assume that all $\mathscr{P}_n$ are pure. Then for any pure partition $\mathscr{U}$ of P, the dyadic expansion $\mathscr{S}$ described above is pure. This implies $\mathscr{F}^*_{\mathrm{De}}\text{-}\underline{\lim}\, \Gamma_\tau(\mathscr{P}) \leqq a$, hence $C^*_\tau(P) \leqq a$, which proves (since $a > \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n)$ was arbitrary) that $C^*_\tau(P) \leqq \leqq \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n)$.

8.24. Proposition. *Let τ be a normal gauge functional. Let P be a W-space. Let $\mathscr{U}_n$, $n \in N$, be τ-admissible partitions of P. Assume that $\Sigma\Gamma_\tau$-diam $\mathscr{U}_n \to 0$ for $n \to \infty$. Then $C_\tau(P) \leqq \underline{\lim}\, C^*_\tau[\mathscr{U}_n]_\tau$. If, in addition, all $\mathscr{U}_n$ are pure, then $C^*_\tau(P) \leqq \leqq \underline{\lim}\, C^*_\tau[\mathscr{U}_n]_\tau$.*

Proof. By 4.28 there exist dyadic expansions $\mathscr{P}_n$ such that for every $n \in N$, $\mathscr{P}''_n$ is equal to $\mathscr{U}_n$ re-indexed and $\Gamma_\tau(\mathscr{P}_n) = C^*_\tau[\mathscr{U}_n]_\tau$. Since $\Sigma\Gamma_\tau$-diam $\mathscr{P}''_n \to 0$ we have, by 8.23, $C_\tau(P) \leqq \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n) = \underline{\lim}\, C^*_\tau[\mathscr{U}_n]_\tau$. If $\mathscr{U}_n$ are pure, then all $\mathscr{P}_n$ are pure, and we get $C^*_\tau(P) \leqq \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n) = \underline{\lim}\, C^*_\tau[\mathscr{U}_n]_\tau$.

Remark. A related but weaker assertion has been proved in [4], 3.11 for $\tau = r$.

8.25. Now we show (see 8.28) that if τ is a normal gauge functional, then neither $C^*_\tau(P)$ nor $C_\tau(P)$ can exceed wP . Ded (P). Then we prove, under assumptions considerably weaker than Ded $(P) < \infty$, the inequality $C_\tau(P) \leqq C^*_\tau(P)$. As an immediate consequence, we get (see 8.39) the inequality $C_\tau(P) \leqq C^*_\tau(P) < \infty$ for all De-bounded W-spaces and all normal gauge functionals τ.

8.26. Fact. *Let $\mathscr{P} = (P_x\colon x \in D)$ be a dyadic expansion of a W-space P. Then $\Gamma_\tau(\mathscr{P}) \leqq \Sigma(\Gamma_\tau\text{-diam } P_x\colon x \in D') \leqq wP\, \Sigma(\max\{d(P_x)\colon x \in \{0, 1\}^m \cap D'\}\colon \{0, 1\}^m \cap \cap D' \neq \emptyset)$.*

Proof. We have $\Gamma_\tau(\mathscr{P}) = \Sigma(\Gamma_\tau(P_{x0}, P_{x1})\colon x \in D') \leqq \Sigma(\Gamma_\tau\text{-diam } P_x\colon x \in D')$. By 8.22 we obtain $\Sigma(\Gamma_\tau\text{-diam } P_x\colon x \in D') \leqq \Sigma(wP_x \,.\, d(P_x)\colon x \in D')$, from which the assertion follows at once.

8.27. Fact. *Let $\mathscr{P} = (P_x\colon x \in D)$ be a dyadic expansion of a W-space $P = = \langle Q, \varrho, \mu\rangle$. Then there exists a pure dyadic expansion $\mathscr{S} = (S_x\colon x \in D)$ such that $d(S_x) \leqq d(P_x)$ for each $x \in D$.*

Proof. For any $z \in D''$ choose a function f_z such that $P_z = f_z \,.\, P$ and put $B_z = = \{q \in Q\colon f_z q > 0\}$. Choose a bijection $\psi\colon D'' \to \{0, \ldots, n\}$. For any $z \in D''$ put $E_z = B_z \setminus \bigcap(B_y\colon y \in D'', \psi y < \psi z)$. For any $x \in D$ put $S_x = \Sigma(E_z \,.\, P\colon z \in D'' \cap D(x))$. Clearly, $\mathscr{S} = (S_\alpha\colon x \in D)$ is a pure dyadic expansion of P. It is easy to show that $d(S_x) \leqq d(P_x)$ for each $x \in D$.

8.28. Proposition. *If τ is a normal gauge functional and P is a W-space, then $C^*_\tau(P) \leqq wP$. Ded (P), $C_\tau(P) \leqq wP$. Ded (P), hence $C^*_\tau(P)$ and $C_\tau(P)$ are finite whenever P is expansion-bounded.*

Proof. The assertion is evident if $wP = 0$ or Ded $(P) = \infty$. Hence we assume that $wP > 0$ and Ded $(P) < \infty$. Choose a number $b >$ Ded (P). Then there exist dyadic expansions $\mathscr{P}_n = (P_{n,x}\colon x \in D_n)$, $n = 1, 2, \ldots$, such that $d(P_{n,x}) < 1/n$ for all $x \in D''_n$

and $\Sigma(\max\{d(P_{n,x}): x \in \{0,1\}^k \cap D_n\} : \{0,1\}^k \cap D \neq \emptyset) \leqq b$. By 8.27 we can assume that $\mathscr{P}_n$ are pure. By 8.22 we have $\Sigma\Gamma_\tau$-diam $\mathscr{P}''_n < wP/n$. By 8.26, $\Gamma_\tau(\mathscr{P}_n) \leqq b \,.\, wP$. Hence, by 8.23, $C^*_\tau(P) \leqq b \,.\, wP$, $C_\tau(P) \leqq b \,.\, wP$. This proves the proposition, since $b > \mathrm{Ded}\,(P)$ was arbitrary.

8.29. Definition. A W-space P will be called *Σwd-partition-fine* if for any $\varepsilon > 0$ there is a partition $\mathscr{U} = (U_k: k \in K)$ of P such that $\Sigma(wU_k \,.\, d(U_k): k \in K) < \varepsilon$.

8.30. Fact. *Every partition-fine W-space is Σwd-partition-fine.*

Proof. We can assume $wP > 0$. Let $\varepsilon > 0$. There exists a partition $(U_k: k \in K)$ of P such that for all $k \in K$, $d(U_k) < \varepsilon/wP$. Clearly, $\Sigma(wU_k \,.\, d(U_k): k \in K) \leqq \leqq \Sigma(wU_k: k \in K)\,\varepsilon/wP = \varepsilon$.

Remark. There are Σwd-partition-fine W-spaces which are not partition-fine. An example: $P = \langle N, 1, \mu\rangle \in \mathfrak{W}$, where $\mu\{n\} > 0$ for infinitely many n.

8.31. Lemma. *Let P be a W-space. Let $(U_k: k = 0, \ldots, n)$ be a partition of P. Assume that $d(U_0) \leqq d(U_k)$ for $k = 1, \ldots, n$. Then there exists a partition $(V_k: k = 0, \ldots, n)$ of P such that V_0 is a pure subspace of P and $\Sigma(wV_k \,.\, d(V_k): k = = 0, \ldots, n) \leqq \Sigma(wU_k \,.\, d(U_k): k = 0, \ldots, n)$.*

Proof. Choose functions f_k, $k = 0, \ldots, n$, such that $U_k = f_k \,.\, P$, $0 \leqq f_k(q) \leqq 1$ for all $q \in Q$ and $k = 0, \ldots, n$, $\Sigma(f_k(q): k = 0, \ldots, n) = 1$ for all $q \in Q$. Put $g_0(q) = 1$ if $f_0(q) > 0$, $g_0(q) = 0$ if $f_0(q) = 0$. For $k = 1, .., n$ put $h_k = f_k g_0$, $g_k = f_k - h_k$. Clearly, $g_0 = f_0 + h_1 + \ldots + h_n$. Put $V_k = g_k \,.\, P$, $k = 0, \ldots, n$, and $T_k = h_k \,.\, P$, $k = 1, \ldots, n$. Then $V_0 = U_0 + T_1 + \ldots + T_n$, $d(V_0) = d(U_0)$, $d(V_k) \leqq d(U_k)$ for $k = 1, \ldots, n$. Since $wV_0 = wU_0 + wT_1 + \ldots + wT_n$, $wV_k = wU_k - wT_k$ for $k = = 1, \ldots, n$, we get $\Sigma(wV_k \,.\, d(V_k): k = 0, \ldots, n) \leqq (wU_0 + wT_1 + \ldots + wT_n)\, d(U_0) + + \Sigma(wV_k \,.\, d(U_k): k = 1, \ldots, n) = wU_0 \,.\, d(U_0) + \Sigma(wU_k \,.\, d(U_k): k = 1, \ldots, n)$.

8.32. Lemma. *Let $(U_k: k \in K)$ be a partition of a W-space P. Then there exists a pure partition $(V_k: k \in K)$ of P such that $\Sigma(wV_k \,.\, d(V_k): k \in K) \leqq \Sigma(wU_k \,.\, d(U_k): k \in K)$.*

Proof. The assertion follows at once from 8.31 by induction.

8.33. Proposition. *Let P be a bounded weakly Borel metric W-space. Assume that the reduced topological weight of P is not real-measurable. Then P is Σwd-partition-fine.*

Proof. Let $P = \langle Q, \varrho, \mu\rangle$. We can assume $wP > 0$, $d(P) > 0$. It follows from 7.28 that there exists a μ-measurable set Z such that $\mu(Q \setminus Z) = 0$ and $P \upharpoonright Z$ is second-countable. Let $\varepsilon > 0$ be given. Put $\delta = \varepsilon/4wP$. There exist balls $B_n = = \{x \in Z: \varrho(x, a_n) < \delta\}$ such that $\bigcup(B_n: n \in N) = Z$. Put $X_n = B_n \setminus \bigcup(B_i: i < n)$. Then X_n are disjoint, $\bigcup(X_n: n \in N) = Z$, $\operatorname{diam} X_n \leqq 2\delta$, $\Sigma(\mu X_n: n \in N) = wP$. Choose m such that $\Sigma(\mu X_n: n \geqq m) < \varepsilon/2d(P)$ and put $U_k = X_k \,.\, P$ for $k = 0, \ldots$ $\ldots, m - 1$, $U_m = (\bigcup(X_n: n \geqq m)) \,.\, P$. Clearly, $(U_0, \ldots, U_m)$ is a partition of P and $\Sigma(wU_k \,.\, d(U_k): k = 0, \ldots, m) < 2\delta \,.\, wP + \varepsilon/2 < \varepsilon$.

8.34. Proposition. *Let P be a bounded metric W-space. If the reduced topological weight of P is countable, then P is Σwd-partition-fine.*

Proof. See 7.2.4 and 8.33.

8.35. Definition. Let τ be a gauge functional. Let P be a W-space. If for any $\varepsilon > 0$ there exists a pure partition $\mathscr{U}$ of P such that $\Sigma(\Gamma_\tau\text{-diam } V_k: k \in K) < \varepsilon$ whenever $(V_k: k \in K)$ is a pure partition refining $\mathscr{U}$, then we will say that P is *strongly $\Sigma^*\Gamma_\tau$-partition-fine.*

8.36. Lemma. *Every Σwd-partition-fine W-space is strongly $\Sigma^*\Gamma_\tau$-partition-fine for every normal gauge functional τ.*

Proof. Let P be Σwd-partition-fine. Let $\varepsilon > 0$. Then there exists a partition $(U_k: k \in K)$ such that $\Sigma(wU_k \,.\, d(U_k): k \in K) < \varepsilon$. By 8.32 there exists a pure partition $\mathscr{V} = (V_k: k \in K)$ such that $\Sigma(wV_k \,.\, d(V_k): k \in K) < \varepsilon$. – Let $\mathscr{S} = (S_m: m \in M)$ be a pure partition refining $\mathscr{V}$. It is easy to see that $\Sigma(wS_m . d(S_m): m \in M) < \varepsilon$. Since, by 8.22, $\Gamma_\tau\text{-diam } S_m \leqq wS_m \,.\, d(S_m)$, we get $\Sigma(\Gamma_\tau\text{-diam } S_m: m \in M) < \varepsilon$. This proves the assertion.

Remark. For an example of a strongly $\Sigma^*\Gamma_r$-partition-fine space with is not Σwd-partition-fine, see 10.29.

8.37. Proposition. *Let τ be a normal gauge functional. If P is a strongly $\Sigma^*\Gamma_\tau$-partition-fine W-space, then $C_\tau(P) \leqq C^*_\tau(P)$.*

Proof. Since P is strongly $\Sigma^*\Gamma_\tau$-partition-fine, there exist pure partitions $\mathscr{U}_n = (U_{nk}: k \in K_n)$ such that, for $n = 1, 2, \ldots$ and any pure partition $\mathscr{V} = (V_m: m \in M)$ refining $\mathscr{U}_n$, we have $\Sigma(\Gamma_\tau\text{-diam } V_m: m \in M) < 1/n$. Put $b = C^*_\tau(P)$. We can assume $b < \infty$. Then, by Theorem III (in Section 6), there exist pure dyadic expansions $\mathscr{P}_n = (P_{nx}: x \in D_n)$ such that $\mathscr{P}''_n$ refines $\mathscr{U}_n$ and $\Gamma_\tau(\mathscr{P}_n) < b + 1/n$. Since $\mathscr{P}''_n$ refines $\mathscr{U}_n$, we have $\Sigma\Gamma_\tau\text{-diam } \mathscr{P}''_n = \Sigma(\Gamma_\tau\text{-diam } P_x: x \in D''_n) < 1/n$. Hence, by 8.23, $C_\tau(P) \leqq \leqq \underline{\lim}\, \Gamma_\tau(\mathscr{P}_n) \leqq b$.

8.38. Proposition. *Let τ be a normal gauge functional and let P be a W-space. Then $C_\tau(P) \leqq C^*_\tau(P)$ provided one of the following conditions is satisfied: (1) P is partition-fine, (2) P is bounded weakly Borel metric and the reduced topological weight of P is not real-measurable (this condition is satisfied, in particular, if P is a bounded metric W-space and the reduced topological weight of P is countable).*

Proof. See 8.37, 8.36, 8.30 and 8.33.

8.39. Proposition. *Let τ be a normal gauge functional and let P be a W-space. If P is* De-*bounded, then* $C_\tau(P) \leqq C^*_\tau(P) \leqq wP \,.\, \mathrm{Ded}\,(P) < \infty$.

Proof. See 8.14, 8.38 and 8.28.

8.40. Proposition. *Let τ be a normal gauge functional and let $\langle Q, \varrho, \mu\rangle$ be a W-space. If $\langle Q, \varrho\rangle$ is expansion -bounded, then* $C_\tau(P) \leqq C^*_\tau(P) \leqq wP \,.\, \text{De-diam}\, \langle Q, \varrho\rangle$.

Proof. See 8.16 and 8.39.

8.41. Fact. *Let* $\langle Q, \varrho\rangle$ *be a subspace of* $\boldsymbol{R}^n$. *Then* De-diam $\langle Q, \varrho\rangle \leqq$ $\leqq 2\,\Sigma\langle \text{diam}\,\{t \in \boldsymbol{R}: t = x_j$ *for some* $x = (x_i) \in Q\}: j < n\rangle \leqq 2n\,.\,\text{diam}\,Q$.

Proof. See 8.15 and 8.19.

8.42. Proposition. *Let* $P = \langle Q, \varrho, \mu\rangle$ *be a W-space and let* $\langle Q, \varrho\rangle$ *be a bounded subspace of* $\boldsymbol{R}^n$. *Then* $C_\tau(P) \leqq C^*_\tau(P) \leqq 2n\,.\,wP\,.\,\text{diam}\,Q$ *for any normal gauge functional* τ.

Proof. See 8.41 and 8.40.

8.43. Proposition. *Let* τ *be a normal gauge functional and let* $P = \langle Q, \varrho, \mu\rangle$ *be a W-space. If* $\langle Q, \varrho\rangle$ *is a bounded subspace of some* $\mathscr{L}_p(n)$, $n = 1, 2, \ldots, 1 \leqq$ $\leqq p \leqq \infty$, *then* $C^*_\tau(P)$ *and* $C_\tau(P)$ *are finite.*

Remark. This has been stated without proof in [4], 3.12, for the following special case: $\tau = r$, $p = 1$, $\langle Q, \varrho\rangle$ is bounded Lebesgue measurable and μ is the Lebesgue measure.

Proof. The assertion is an easy consequence of 8.42.

8.44. Proposition. *If there are no real-measurable cardinals, then* $C^*_\tau(P) \leqq C_\tau(P)$ *for any bounded weakly Borel metric W-space and any normal gauge functional* τ.

This is an immediate consequence of 8.38.

8.45. Proposition. *It is admissible to assume that* $C_\tau(P) \leqq C^*_\tau(P)$ *for any bounded weakly Borel metric W-space and any normal gauge functional* τ.

Proof. See 7.61 and 8.44.

9

The main results (see 9.4 and 9.36) of this section are as follows: if τ is a normal gauge functional, then C_τ is finitely continuous (see 2.13) and, for a fairly broad class of W-spaces P, $C_\tau \restriction \exp P$ and $C^*_\tau \restriction \exp P$ satisfy certain conditions of the Lipschitz type.

After giving some simple facts, we prove the key lemma 9.4. From this lemma, the proof of which is rather long, the main results follow in a number of relatively easy steps.

9.1. Notation. We denote (cf. 2.23.1) by V the function defined as follows: if $x, y \in \boldsymbol{R}$, $x > 0$, $y > 0$, then $V(x, y) = H(x, y)/xy$. – The letter V will sometimes be used with a different meaning (e.g., to denote a W-space) provided there is no danger of confusion.

9.2. Fact. *If* $n \in \boldsymbol{R}_+$, $n > 0$, $x_i \geqq ny_i > 0$, $i = 1, 2$, *then* $V(y_1, y_2) \geqq nV(x_1, x_2)$.

Proof. By 2.23.1, $V(x_1, x_2) \leqq V(ny_1, ny_2)$. By 2.4A, $V(ny_1, ny_2) = n^{-1}V(y_1, y_2)$.

9.3. Fact. *Let* $n \in \boldsymbol{R}_+$, $n \geqq 1$, $1 < y \leqq nx$. *Then* $H(1, x)/H(1, y) \geqq 1 - \log n/\log y$.

Proof. For any $z > 0$ we have $H(1, z) = f_z(1) - f_z(0)$ where $f_z(t) = (z + t)$.

. $\log(z + t)$. Since $f_z'(t) = \log \mathrm{e} + \log(z + t)$, there is, by the Cauchy Mean Value Theorem, a u, $0 < u < 1$, such that

$$H(1, x)/H(1, y) = \frac{\log \mathrm{e} + \log(x + u)}{\log \mathrm{e} + \log(y + u)}$$

and therefore $H(1, x)/H(1, y) = 1 - h$, where

$$h = \frac{\log(y + u) - \log(x + u)}{\log \mathrm{e} + \log(y + u)}.$$

Clearly, $h \leqq (\log(nx + u) - \log(x + u))/\log y = \log((nx + u)/(x + u))/\log y \leqq \leqq \log n/\log y$. This proves the assertion.

9.4. Lemma. *Let τ be a normal gauge functional. Let φ be a hypoentropy. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space and let $wP \leqq 1$, $d(P) \leqq 1$. Let m be a positive number and let $\varphi T \leqq m \,.\, wT$ for any $T \leqq P$. Let $\mathscr{P} = (P_x: x \in D)$ be a dyadic expansion of P. Let $S \leqq P$, $a = w(P - S)$. Let $b, t \in \boldsymbol{R}$, $b > 1$, $t > 1$, and put $u = = (b + 1)\, t/(b - 1)(t - 1)$. If either φ is τ-projective or φ is τ-semiprojective and $\mathscr{P}$ is pure, then $\varphi S \leqq u\, \Gamma_\tau(\mathscr{P}) + (1 + b)\, ma + bH(1, 2^t)\, a + \Sigma(\sup(\varphi U: U \leqq P_x)$: $x \in D'')$. If φ is τ-semiprojective and both $\mathscr{P}$ and S are pure, then the inequality holds with $\sup(\varphi U: U \leqq P_x)$ replaced by $\sup(\varphi U: U \leqq P_x, U$ pure$)$.*

Proof. I. By 1.35, there exist $\bar{\mu}$-measurable functions ξ, η_x, $x \in D$, such that $S = = \xi \,.\, P$, $P_x = \eta_x \,.\, P$ for each $x \in D$. Put $\xi_x = \xi\eta_x$, $S_x = \xi_x \,.\, P$. Then $\mathscr{S} = (S_x: x \in D)$ is a dyadic expansion of S, $S_x \leqq P_x$ for all $x \in D$, $(P_x - S_x: x \in D)$ is a dyadic expansion of $P - S$, and if $\mathscr{P}$ is pure, then so is $\mathscr{S}$. For any $x \in D$, put $p(x) = wP_x$, $s(x) = wS_x$, $\alpha(x) = p(x) - s(x)$. Put

(1) $Z = \{x \in D: b\, \alpha(x) > s(x)\}$.

Clearly,

(2) if $x0 \in Z$, $x1 \in Z$, then $x \in Z$; if $x0$ non $\in Z$, $x1$ non $\in Z$, then x non $\in Z$.

Let X be the set of all minimal (with respect to the order $\preceq$ introduced in 4.1) elements of Z. Clearly,

(3) $X \subset Z \subset D(X)$, and the sets $D(x)$, $x \in X$, are disjoint. It is also clear that

(4) if $X_0 \subset X$, $x \in D$, then for any dyadic expansion $(U_z: z \in D)$ of a W-space,

$$\Sigma(U_y: y \in X_0 \cap D(x)) = \Sigma(U_z: z \in D'' \cap D(D(x) \cap X_0)) \leqq U_x\,.$$

Put

(5) $Y = \{x \in X: p(x) \geqq 2\, s(x)\}$.

For $x \in D$ put $T_x = \Sigma(S_y: y \in Y \cap D(x))$; put $T = T_\varnothing$. By (4) we have $T_x \leqq S$. For any $x \in D$ put $\hat{S}_x = S_x - T_x$, $\sigma(x) = w\hat{S}_x$, and put $\hat{S} = S - T$. By 4.5, if $\mathscr{S}$ is pure, then all T_x, $\hat{S}_x$ are pure subspaces of S and $(\hat{S}, T)$ is a pure partition of S. If both $\mathscr{S}$ and S are pure, then $\hat{S}_x$ are pure subspaces of P. Since $T = \Sigma(S_y: y \in Y)$, we have $wT = \Sigma(wS_y: y \in Y)$ and therefore, by (5), $wT \leqq \Sigma(p(y) - s(y): y \in Y) \leqq \Sigma(p(x) - - s(x): x \in D'')$, which implies

(6) $wT \leqq a$.

II. Put $\hat{D} = D \setminus (D(X) \setminus X)$. It is easy to see that $\hat{D} \in \Delta$ and

(7) $\hat{D}' = D' \setminus D(X)$, $\hat{D}'' = X \cup (D'' \setminus D(X))$.

Put $\hat{\mathscr{S}} = (\hat{S}_x : x \in D)$. Clearly, $\hat{\mathscr{S}}$ is a dyadic expansion of $\hat{S}$ and if $\mathscr{S}$ is pure, then so is $\hat{\mathscr{S}}$.

III. Put

(8) $E_1 = \{x \in \hat{D}' : x0 \text{ non} \in Z,\ x1 \text{ non} \in Z\}$.

Since for any $x \in \hat{D}'$, $x \text{ non} \in D(X)$, hence, by (3), $x \text{ non} \in Z$, we have, by (2),

(9) if $x \in \hat{D}' \setminus E_1$, then there is exactly one $k \in \{0, 1\}$, denoted by k_x, such that $xk \in Z$.

Let $x \in \hat{D}' \setminus E_1$. Suppose $xk_x \in Z \setminus X$. Then there exists an $x' \in X$ such that $x' \prec xk$, and this implies $x' \precsim x$, $x \in D(X)$, which contradicts (7). Hence

(10) if $x \in \hat{D}' \setminus E_1$, then $xk_x \in X$.

For $k = 0, 1$, let $\bar{k}$ stand for $1 - k$. We put

(11) $E_2 = \{x \in \hat{D}' : xk_x \in Y\}$,

(12) $E_3 = \{x \in \hat{D}' : xk_x \in X \setminus Y,\ 2^t \sigma(xk_x) \geqq \sigma(x\bar{k}_x)\}$,

(13) $E_4 = \{x \in \hat{D}' : xk_x \in X \setminus Y,\ 2^t \sigma(xk_x) < \sigma(x\bar{k}_x)\}$.

Clearly, by (10) and (8) we have

(14) (E_1, E_2, E_3, E_4) is a partition of the set $\hat{D}'$.

We introduce the following abbreviations (for $y \in D$): $g(y) = H(p(y0), p(y1))$, $f(y) = H(\sigma(y0), \sigma(y1))$, $G(y) = g(y)\, \tau(P_{y0}, P_{y1})$, $F(y) = f(y)\, \tau(\hat{S}_{y0}, \hat{S}_{y1})$. We put $v = (b - 1)/(b + 1)$. Clearly,

(15) $\Gamma_\tau(\mathscr{P}) = \Sigma(G(y): y \in D')$,

(16) $\Gamma_\tau(\hat{\mathscr{S}}) = \Sigma(F(y): y \in \hat{D}')$.

IV. We shall need the following facts:

(17) for any $x \in D$, $s(x) - \sigma(x) \leqq \alpha(x)$,

(18) if $x \in D \setminus Z$, then $\sigma(x) \geqq v\, p(x)$,

(19) if $x \in X \setminus Y$, then $s(x) = \sigma(x)$, $p(x) < 2\, \sigma(x)$.

The assertion (17) is proved as follows. We have $s(x) - \sigma(x) = wT_x = \Sigma(wS_y : y \in Y \cap D(x)) = \Sigma(s(y): y \in Y \cap D(x))$. By (5) we obtain $wT_x \leqq \Sigma(p(y) - s(y): y \in Y \cap D(x)) = \Sigma(w(P_y - S_y): y \in Y \cap D(x))$. By (4) we have $\Sigma(w(P_y - S_y): y \in Y \cap D(x)) \leqq w(P_x - S_x) = \alpha(x)$, hence $wT_x \leqq \alpha(x)$.

To prove (18), observe that if $x \text{ non} \in Z$, then, by (1), $b\, \alpha(x) \leqq s(x)$. Together with $s(x) - \sigma(x) \leqq \alpha(x)$, this yields $\sigma(x) \geqq (1 - b^{-1})\, s(x)$, $p(x) \leqq (1 + b^{-1})\, s(x)$, hence $\sigma(x) \geqq v\, p(x)$.

Finally, if $x \in X \setminus Y$, then $Y \cap D(x) = \emptyset$, hence $wT_x = 0$, $S_x = \hat{S}_x$, $s(x) = \sigma(x)$, and, by (5), $p(x) < 2\, \sigma(x)$.

V. We are going to show that

(20) if $x \in E_1$, then $G(x) \geqq v\, F(x)$.

If $\sigma(x0) = 0$ or $\sigma(x1) = 0$, then $F(x) = 0$. Therefore we can assume that $\sigma(x0)$ and $\sigma(x1)$, hence also $p(x0)$ and $p(x1)$, are positive. We have

$F(x) = V(\sigma(x0), \sigma(x1))\, \sigma(x0)\, \sigma(x1)\, \tau(\hat{S}_{x0}, \hat{S}_{x1})$,

$G(x) = V(p(x0), p(x1))\, p(x0)\, p(x1)\, \tau(P_{x0}, P_{x1})$.

If $k = 0, 1$, then, by (8), xk non $\in Z$, hence, by (18), $\sigma(xk) \geqq vp(xk)$. This implies, by 9.2, $V(p(x0), p(x1)) \geqq vV(\sigma(x0), \sigma(x1))$, hence, by (NGF 2), $G(x) \geqq v\,F(x)$.
VI. Let $x \in E_2$. Then, with $k = k_x$, we have $xk \in Y$ and therefore (see (4)), $T_{xk} = S_{xk}$, $w\hat{S}_{xk} = 0$. Consequently,

(21) if $x \in E_2$, then $F(x) = 0$.

VII. Let $x \in E_3$. Then, with $k = k_x$, we have $xk \in X \setminus Y$, $2^t\,\sigma(xk) \geqq \sigma(x\bar{k})$ and therefore (since $\tau(U_1, U_2) \leqq d(P) \leqq 1$ for any $U_1 \leqq P$, $U_2 \leqq P$) $F(x) \leqq H(\sigma(xk), 2^t\,\sigma(xk)) = \sigma(xk)\,H(1, 2^t)$. Hence, $\Sigma(F(x)\colon x \in E_3) \leqq H(1, 2^t)\,\Sigma(\sigma(xk_x)\colon x \in E_3) \leqq$ $\leqq H(1, 2^t)\,\Sigma(\sigma(y)\colon y \in X \setminus Y)$. By (19) and (1), we get $\Sigma(F(x)\colon x \in E_3) \leqq bH(1, 2^t)\,.$ $.\,\Sigma(\alpha(y)\colon y \in X \setminus Y)$. By (4), with $X_0 = X \setminus Y$, $x = \emptyset \in D$, we have $\Sigma(w(P_y - S_y)\colon$ $y \in X \setminus Y) \leqq w(P - S) = a$. Since, for any $y \in D$, $\alpha(y) = w(P_y - S_y)$, we obtain $\Sigma(\alpha(y)\colon y \in X \setminus Y) \leqq a$ and therefore

(22) $\Sigma(F(x)\colon x \in E_3) \leqq bH(1, 2^t)\,a$.

VIII. Let $x \in E_4$. We are going to prove that $G(x) \geqq v(1 - t^{-1})\,F(x)$. Put $k = k_x$. By (13) and (19) we have

(23) $2^t\,\sigma(xk) < \sigma(x\bar{k})$, $p(x\bar{k}) < 2\,\sigma(xk)$.

Thus, $\sigma(xk)$ and $\sigma(x\bar{k})$, hence also $p(xk)$ and $p(x\bar{k})$ are positive. Put $p^* = p(x\bar{k})/p(xk)$, $\sigma^* = \sigma(x\bar{k})/\sigma(xk)$. By (23), we have

(24) $2^t < \sigma^* < 2p^*$.

By 9.3, the inequalities (24) imply

(25) $H(1, p^*)/H(1, \sigma^*) \geqq 1 - t^{-1}$.

Clearly, $V(p(x\bar{k}), p(xk))/V(\sigma(x\bar{k}), \sigma(xk)) = (H(1, p^*)/H(1, \sigma^*))\,\sigma(x\bar{k})/p(x\bar{k}))$. Since $x \in E_4$, we have $x\bar{k}$ non $\in Z$ by (14) and (9). Hence, by (18), $\sigma(x\bar{k}) \geqq vp(x\bar{k})$, and therefore, by (25), we get

(26) $V(p(x\bar{k}), p(xk))/V(\sigma(x\bar{k}), \sigma(xk)) \geqq v(1 - t^{-1})$.

It follows by (NGF 2) that

(27) if $x \in E_4$, then $G(x) \geqq v(1 - t^{-1})\,F(x)$.

IX. By (15), (16), (14), (20), (21), (22) and (27), and in view of $u^{-1} = v(1 - t^{-1})$, we obtain

(28) $\Gamma_\tau(\hat{\mathscr{S}}) \leqq u\,\Gamma_\tau(\mathscr{P}) + bH(1, 2^t)\,a$.

Since either φ is τ-projective or $\mathscr{S}$ is pure and φ is τ-semiprojective, we have by 4.13.2, $\varphi\hat{S} \leqq \Gamma_\tau(\hat{\mathscr{S}}) + \Sigma(\varphi\hat{S}_x\colon x \in D'')$. Since $\varphi U \leqq m\,.\,wU$ for any $U \leqq P$, we get $\Sigma(\varphi\hat{S}_x\colon x \in X) \leqq m\,\Sigma(w\hat{S}_x\colon x \in X) = m\,\Sigma(\sigma(x)\colon x \in X) \leqq m\,\Sigma(s(x)\colon x \in X)$, hence, by (1) and (4), $\Sigma(\varphi\hat{S}_x\colon x \in X) \leqq mb\,\Sigma(\alpha(x)\colon x \in X) \leqq mba$. Since, by (7), $\hat{D}'' = X \cup$ $\cup\,(D'' \setminus D(X))$, we obtain

(29) $\varphi\hat{S} \leqq \Gamma_\tau(\hat{\mathscr{S}}) + mba + \Sigma(\varphi\hat{S}_x\colon x \in D'' \setminus D(X))$.

Since either φ is τ-projective or φ is τ-semiprojective and $(\hat{S}, T)$ is a pure partition of S, we have $\varphi S \leqq \varphi\hat{S} + \varphi T + H(w\hat{S}, wT)\,\tau(\hat{S}, T)$. Since $\tau(\hat{S}, T) \leqq d(P) \leqq 1$ and $\varphi T \leqq m\,.\,wT$, we get by (6) and 2.4,

(30) $\varphi S \leqq \varphi\hat{S} + ma + H(1 - a, a)$.

From (28), (29) and (30), the assertion of the lemma follows at once, for, if both $\mathscr{P}$ and S are pure, then $\hat{S}_x$ are pure subspaces of P.

9.5. Fact. *If* $0 < a < 1$, *then* $H(a, 1 - a) < 2a^{2/3} + 2a$.

The proof consists in an easy, though somewhat lengthy calculation, and can be omitted.

9.6. Lemma. *Let* τ *be a normal gauge functional. Let* φ *be a hypoentropy. Let* P *be a W-space,* $wP \leqq 1$, $d(P) \leqq 1$. *Let* m *be a positive number and let* $\varphi T \leqq m \,.\, wT$ *for any* $T \leqq P$. *Let* $\mathscr{P} = (P_x : x \in D)$ *be a dyadic expansion of* P. *Let* $S \leqq P$, $S \neq P$. *Put* $a = w(P - S)$. *If either* (1) φ *is* τ-*projective or* (2) φ *is* τ-*semiprojective and* $\mathscr{P}$ *is pure, then* $\varphi(S) \leqq \Gamma_\tau(\mathscr{P}) + (3a^{1/3} + 2a^{2/3})\, \Gamma_\tau(\mathscr{P}) + a^{1/3} + (m + 6)\, a^{2/3} + (2m + 5)\, a + \Sigma(\sup(\varphi U: U \leqq P_x): x \in D'')$. *If* (3) φ *is* τ-*semiprojective and both* $\mathscr{P}$ *and* S *are pure, then the inequality holds with* $\sup(\varphi U: U \leqq P_x)$ *replaced by* $\sup(\varphi U: U \leqq P_x, U$ *pure*).

Proof. We consider only the case (1); the remaining cases are analogous. By 9.4, $\varphi S \leqq \Gamma_\tau(\mathscr{P}) + (u - 1)\, \Gamma_\tau(\mathscr{P}) + (1 + b)\, ma + H(a, 1 - a) + bH(1, 2^t)\, a + \Sigma(\sup(\varphi U: U \leqq P_x): x \in D'')$, where $u = (b + 1)\, t/(b - 1)(t - 1)$, and $b > 1$, $t > 1$ are arbitrary. Put $t = b = a^{-1/3} + 1$. Then

(1) $(u - 1)\, \Gamma_\tau(\mathscr{P}) = (3a^{1/3} + 2a^{2/3})\, \Gamma_\tau(\mathscr{P})$,

(2) $(1 + b)\, ma = ma^{2/3} + 2ma$.

By 9.5 we have

(3) $H(a, 1 - a) \leqq 2a^{2/3} + 2a$.

Since $H(1, x) = (x + 1)\log(x + 1) - x \log x$ is equal, by the Mean Value Theorem, to $\log e + \log(x + s)$, where $0 < s < 1$, we get $H(1, 2^t) \leqq \log(e\, 2^t + es)$, hence, due to $t \geqq 2$, $H(1, 2^t) < t + 2$, $bH(1, 2^t) < (a^{-1/3} + 1)(a^{-1/3} + 3)$, and therefore

(4) $bH(1, 2^t)\, a < a^{1/3} + 4a^{2/3} + 3a$.

Now, (1)–(4) imply the inequality asserted in the lemma.

9.7.1. Definition. Let P be a W-space and let φ be a non-negative functional, $\operatorname{dom} \varphi \supset \exp P$ (see 7.52). If for any $\varepsilon > 0$ there exists a partition (pure partition) $(U_k: k \in K)$ of P such that $\Sigma(\varphi U_k: k \in K) < \varepsilon$, then we will say that P is $\Sigma\varphi$-*partition-fine* (respectively, $\Sigma^*\varphi$-*partition-fine*). If for any $\varepsilon > 0$ there exists a partition (pure partition) $\mathscr{U}$ of P such that for any partition (pure partition) $\mathscr{V} = (V_k: k \in K)$ refining $\mathscr{U}$ we have $\Sigma(\varphi V_k: k \in K) < \varepsilon$, then we will say that P is *strongly* $\Sigma\varphi$-*partition-fine* (respectively, *strongly* $\Sigma^*\varphi$-*partition-fine*). Instead of "$\Sigma\varphi$-partition-fine", etc., we will often write "$\Sigma\varphi$-*fine*", etc.

9.7.2. Clearly, Σwd-partition-fine (see 8.29) W-spaces are exactly the $\Sigma\varphi$-partitione fine ones, where φ is the functional $P \mapsto wP \,.\, dP$, strongly $\Sigma\Gamma_\tau$-partition-fine (se 8.35) W-spaces are exactly the strongly $\Sigma\varphi$-partition-fine ones, where φ is the functional $P \mapsto \Gamma_\tau$-diam P, etc.

9.8. Lemma. *Let* τ *be a normal gauge functional. Let* $\varphi = C_\tau$ *or* $\varphi = C_\tau^*$. *Let* P *be a W-space,* $wP \leqq 1$, $d(P) \leqq 1$. *Let* m *be a positive number and let* $\varphi T \leqq m \,.\, wT$ *for any* $T \leqq P$. *Let* $S \leqq P$, $S \neq P$. *Put* $a = w(P - S)$. *If either* (1) $\varphi = C_\tau$ *and* P

is strongly $\Sigma\varphi$*-fine or* (2) $\varphi = C^*_\tau$, P *is strongly* $\Sigma^*\varphi$*-fine and* S *is pure, then* $\varphi S \leqq \leqq \varphi P + (3a^{1/3} + 2a^{2/3})\,\varphi P + a^{1/3} + (m + 6)\,a^{2/3} + m(2 + 5)\,a$.

Proof. We consider the case (1); the other case is quite analogous. Choose an $\varepsilon > 0$. Let $\mathscr{U}$ be a partition of P such that for any partition $\mathscr{V} = (V_m : m \in M)$ refining $\mathscr{U}$, $\Sigma(\varphi V_m : m \in M) < \varepsilon$. By Theorem III (in Section 6) there exists a dyadic expansion $\mathscr{P} = (P_x : x \in D)$ of P such that $\mathscr{P}''$ refines and $\Gamma_\tau(\mathscr{P}) < \varphi P + \varepsilon$. It is easy to see that $\Sigma(\sup(\varphi U : U \leqq P_x) : x \in D'') < \varepsilon$. By 9.6, $\varphi S < \varphi P + \varepsilon + (3a^{1/3} + 2a^{2/3})\,.\,(\varphi P + \varepsilon) + a^{1/3} + (m + 6)\,a^{2/3} + (2m + 5)\,a + \Sigma(\sup(\varphi U : U \leqq P_x) : x \in D'')$. Since $\varepsilon > 0$ was arbitrary, the lemma is proved.

9.9. Fact. *Let* τ *be a normal gauge functional. Let* P *be a W-space,* $wP \leqq 1$, $d(P) \leqq 1$. *Let* φ *be a hypoentropy. Let* $m \in \mathbf{R}$ *be positive and let* $\varphi T \leqq m\,.\,wT$ *for any* $T \leqq P$. *Let* $S \leqq P$, $a = w(P - S)$. *If either* φ *is* τ*-projective or* S *is pure and* φ *is* τ*-semiprojective, then* $\varphi P \leqq \varphi S + 2a^{2/3} + (m + 2)\,a$.

Proof. We have $\varphi P \leqq \varphi S + \varphi(P - S) + H(wS, w(P - S))\,\tau(S, P - S)$, hence $\varphi P \leqq \varphi S + ma + H(a, 1 - a)$. By 9.5, $H(a, 1 - a) \leqq 2a^{2/3} + 2a$.

9.10. Fact and notation. *Let* $S_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, *be a W-space. Assume that* $\langle Q, \varrho, \mu_1 + \mu_2 \rangle$ *is a W-space. Then there exists exactly one W-space* U *such that* (i) $U \leqq S_1$, $U \leqq S_2$, (ii) *if* $V \leqq S_1$, $V \leqq S_2$, *then* $V \leqq U$. – *The space* U *will be denoted by* $S_1 \wedge S_2$, *and the space* $S_1 + S_2 - (S_1 \wedge S_2)$ *will be denoted by* $S_1 \vee S_2$.

Proof. Put $\mu = \mu_1 + \mu_2$, $S = \langle Q, \varrho, \mu \rangle$. By 1.35 there exist $\bar{\mu}$-measurable functions f_i, $i = 1, 2$, such that $S_i = f_i\,.\,S$. Put $g = \min(f_1, f_2)$, $U = g\,.\,S$. It is easy to see that the space U satisfies (i) and (ii).

9.11. Fact. *Let* $P = \langle Q, \varrho, \mu \rangle$ *be a W-space,* $S_1 \leqq P$, $S_2 \leqq P$. *Let* $S_i = f_i\,.\,P$. *Then* $S_1 \wedge S_2 = \min(f_1, f_2)\,.\,P$, $S_1 \vee S_2 = \max(f_1, f_2)\,.\,P$. *If* S_1, S_2 *are pure subspaces,* $S_i = A_i\,.\,P$ *where* $A_i \subset Q$ *are* $\bar{\mu}$*-measurable, then* $S_1 \wedge S_2 = (A_1 \cap A_2)\,.\,.\,P$, $S_1 \vee S_2 = (A_1 \cup A_2)\,.\,P$.

9.12. Notation. If S_1, S_2 are W-spaces and there exists a W-space P such that $S_1 \leqq P$, $S_2 \leqq P$, then $w((S_1 \vee S_2) - (S_1 \wedge S_2))$ will be denoted by $\mathrm{md}(S_1, S_2)$ and will be occasionally called the *measure-distance* of S_1 and S_2.

9.13.1. Fact. *If* $S_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$ *are* FW*-spaces, then* $\mathrm{md}(S_1, S_2)$ *is equal to* $\mathrm{dist}(S_1, S_2)$, see 2.11.

9.13.2. Fact. *If* $P = \langle Q, \varrho, \mu \rangle$ *is a W-space,* $S_i \leqq P$, $S_i = f_i\,.\,P$, $i = 1, 2$, *then* $\mathrm{md}(S_1, S_2) = \int_Q |f_1 - f_2|\,\mathrm{d}\mu$.
This follows at once from 9.11.

9.14. Fact and notation. *If* P *is a W-space, then* $(S_1, S_2) \mapsto \mathrm{md}(S_1, S_2)$ *is a metric on* $\exp P$ (see 7.52). – This metric will be denoted by md_P or simply by md and, for any subset X of $\exp P$, X will also denote the metric space $\langle X, \mathrm{md} \restriction X \rangle$.

9.15. Lemma. *Let P be a W-space. Let φ be a non-negative functional,* $\operatorname{dom} \varphi \supset \supset \exp P$. *If P is strongly $\Sigma\varphi$-fine (strongly $\Sigma^*\varphi$-fine), then so is every $S \leqq P$ (every pure $S \leqq P$, respectively).*

Proof. Let P be strongly $\Sigma\varphi$-fine. Let $\varepsilon > 0$ be given. Let $\mathscr{U} = (U_k: k \in K)$ be a partition of P such that for any partition $\mathscr{V} = (V_m: m \in M)$ refining $\mathscr{U}$, $\Sigma(\varphi V_m: m \in M) < \varepsilon$. Choose functions f_k such that $U_k = f_k \,.\, P$ for all $k \in K$. For each $k \in K$ put $U_k^{(1)} = f_k \,.\, S$, $U_k^{(2)} = f_k \,.\, (P - S)$. Then $U_k^{(1)}, U_k^{(2)}$, $k \in K$, form a partition of P and $\mathscr{U}^{(1)} = (U_k^{(1)}: k \in K)$ is a partition of S. If a partition $\mathscr{T} = (T_j: j \in J)$ of S refines $\mathscr{U}^{(1)}$, then the spaces T_j, $j \in J$, and $U_k^{(2)}$, $k \in K$, form a partition of P refining $\mathscr{U}$, and therefore $\Sigma(\varphi T_j: j \in J) + \Sigma(\varphi U_k^{(2)}: k \in K) < \varepsilon$. This proves that S is strongly $\Sigma\varphi$-fine. — The other case is analogous.

9.16. Lemma. *Let τ be a normal gauge functional. Let $\varphi = C_\tau$ or $\varphi = C_\tau^*$. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space, $wP \leqq 1$, $d(P) \leqq 1$. Let $m \in \mathbf{R}$ be positive and let $\varphi T \leqq m \,.\, wT$ for all $T \leqq P$. Assume that $\varphi = C_\tau$ and P is strongly $\Sigma\varphi$-fine (or $\varphi = C_\tau^*$ and P is strongly $\Sigma^*\varphi$-fine). Let S_1, S_2 be subspaces (pure subspaces, respectively) of P. Put* $a = \operatorname{md}(S_1, S_2)$. *Then* $|\varphi S_1 - \varphi S_2| \leqq (3m + 1)\, a^{1/3} + (3m + 8)\, a^{2/3} + (3m + 7)\, a \leqq (9m + 16)\, a^{1/3}$.

Proof. Assume that $\varphi = C_\tau$ (the other case is analogous). Put $U = S_1 \wedge S_2$. By 9.11 and 9.13.2 we have $w(S_i - U) \leqq a$, $i = 1, 2$. By 9.15, S_1 is strongly $\sum\varphi$-fine. Since $\varphi S_1 \leqq m$, we have by 9.8,

$$\varphi U \leqq \varphi S_1 + (3a^{1/3} + 2a^{2/3})\, m + a^{1/3} + (m + 6)\, a^{2/3} + (2m + 5)\, a\,.$$

By 9.9 we get

$$\varphi S_2 \leqq \varphi U + 2a^{2/3} + (m + 2)\, a\,.$$

Hence

$$\varphi S_2 \leqq \varphi S_1 + (3m + 1)\, a^{1/3} + (3m + 8)\, a^{2/3} + (3m + 7)\, a\,.$$

This proves the lemma, since S_1 and S_2 can be interchanged.

9.17. Proposition. *Let τ be a normal gauge functional. Let $\varphi = C_\tau$ or $\varphi = C_\tau^*$. Let $P = \langle Q, \varrho, \mu \rangle$ be a bounded W-space. Let $m \in \mathbf{R}_+$ and let $\varphi T \leqq m \,.\, wT$ for all $T \leqq P$. Let $S_i = \langle Q, \varrho, v_i \rangle \leqq P$, $i = 1, 2$. If either (1) $\varphi = C_\tau$ and P is strongly $\Sigma\varphi$-fine, or (2) $\varphi = C_\tau^*$, P is strongly $\Sigma^*\varphi$-fine and S_i, $i = 1, 2$, are pure subspaces, then* $|\varphi S_1 - \varphi S_2| \leqq (9m + 16d(P))\,(wP)^{2/3}\,(\operatorname{md}(S_1, S_2))^{1/3}$.

Proof. Put $p = wP$, $t = d(P)$, $a = \operatorname{md}(S_1, S_2)$. By 8.7 we can assume $p > 0$, $t > 0$. Put $\hat{P} = \langle Q, t^{-1}\varrho, p^{-1}\mu \rangle$, $\hat{S}_i = \langle Q, t^{-1}\varrho, p^{-1}v_i \rangle$, $i = 1, 2$. Then $w\hat{P} \leqq 1$, $d(\hat{P}) \leqq 1$, $\operatorname{md}(\hat{S}_1, \hat{S}_2) = p^{-1}a$, and, for any $T \leqq \hat{P}$, $\varphi T \leqq (m/t) \,.\, wT$. By 9.16 we get $|\varphi\hat{S}_1 - \varphi\hat{S}_2| \leqq (9m/t + 16)\,(p^{-1}a)^{1/3}$. Clearly, $|\varphi S_1 - \varphi S_2| = pt|\varphi\hat{S}_1 - \varphi\hat{S}_2|$, hence $|\varphi S_1 - \varphi S_2| \leqq (9m + 16t)\, p(a/p)^{1/3}$.

9.18.1. We shall need the concept of a functional satisfying a Lipschitz type condition with respect to a non-negative functional ψ. Although in this section ψ will be either $P \mapsto wP$, defined on $\mathfrak{W}$, or some md_P (see 9.14), the definition is given in a fairly broad form (see also 9.46).

9.18.2. Definition. Let φ and ψ be functionals and let ψ be non-negative. Let $0 < p \leqq 1$ and let $0 < b \leqq \infty$. Let X be a class. If $0 < m < \infty$, $X \subset \operatorname{dom} \varphi \cap \cap \operatorname{dom} \psi$ and $|\varphi(x)| \leqq m|\psi(x)|^p$ whenever $x \in X$ and $\psi(x) \leqq b$, then we will say that φ *satisfies* $L(p, \psi; b, m)$ *on* X or that m is a *Lipschitz bound of order* p *for* φ *on* X *with respect to* ψ *(restricted by* b*)*. If φ satisfies $L(p, \psi; b, m)$ for some m, then we will say that φ *satisfies* $L(p, \psi; b, \cdot)$ on X or that φ is *Lipschitz of order* p *on* X *with respect to* ψ *(restricted by* b*)*. If φ satisfies $L(p, \psi; \infty, m)$, then we will also say that m is a ψ*-Lipschitz bound of order* p *for* φ *on* X, and if φ satisfies $L(p, \psi; \infty, \cdot)$ on X, we will also say that φ is ψ*-Lipschitz of order* p *on* X. – "On X" will be omitted if $X = \operatorname{dom} \varphi$, and "of order p" will be often omitted if $p = 1$.

9.19. Definition and conventions. Let $U = \langle Q, \varrho \rangle$ and $V = \langle T, \sigma \rangle$ be semimetric spaces and let $f: U \to V$ be a mapping. If the functional $(x, y) \mapsto \sigma(fx, fy)$ satisfies $L(p, \varrho; b, m)$, then we will say that f *satisfies* $L(p; b, m)$ or that m is a *Lipschitz bound of order* p *for* f*, with distance bound* b. If $(x, y) \mapsto \sigma(fx, fy)$ satisfies $L(p, \varrho; b, \cdot)$, we will say that f *satisfies* $L(p; b, \cdot)$ or that f *is Lipschitz of order* p*, with distance bound* b. In both cases, the words "with distance bound b" will be omitted if $b = \infty$; if $p = 1$, then we often omit "of order p". – We note that for metric spaces $\langle Q, \varrho \rangle$, $\langle T, \sigma \rangle$, this definition is equivalent to the usual one.

9.20. Proposition. *Let* τ *be a normal gauge functional. Let* P *be a bounded* W*-space. If* C_τ *is* w*-Lipschitz on* $\exp P$ *and* P *is strongly* ΣC_τ*-fine, then* $C_\tau \restriction \exp P$ *is Lipschitz of order* $1/3$. *If* C_τ^* *is* w*-Lipschitz on* $\exp P$ *and* P *is strongly* ΣC_τ^**-fine, then* $C_\tau^* \restriction \exp^* P$ *is Lipschitz of order* $1/3$.

Proof. It follows at once from 9.17.

9.21. Remarks. A) Proposition 9.20 does not assert anything concerning $C_\tau^* \restriction \exp P$. However, it will be proved below (9.26) that under the assumptions in 9.20, $C_\tau^* \restriction \exp P$ is Lipschitz of order $1/4$. – B) It seems that the Lipschitz orders $1/3$ (for $C_\iota \restriction \exp F$ and $C_\tau^* \restriction \exp^* P$) and $1/4$ (for $C_\tau \restriction \exp P$) are not the best possible. – C) Clearly, if, e.g., $P = \langle \{1, 2\}, 1, \mu \rangle$, where $\mu 1 = \mu 2 = 1$, then $C_r \restriction \exp P = C_\tau^* \restriction \exp P$ is not Lipschitz of order 1, since $H(1 - \varepsilon, \varepsilon)/\varepsilon \to \infty$ for $\varepsilon \to 0$. – D) If a W-space P and a normal gauge functional τ are given, then it is often rather difficult to check the assumptions in 9.20. Therefore we shall introduce (see 9.27) a property which implies, for any τ, the assumptions stated in 9.20, and is easier to check.

9.22. Lemma. *Let* τ *be a normal gauge functional. Let* $P = \langle Q, \varrho, \mu \rangle$ *be a* W*-space. Let* $0 < u < 1$. *Let* $S = \langle Q, \varrho, v \rangle$ *be a subspace of* P *and let* $v \geqq u\mu$. *Then* $C_\tau^*(P) \geqq \geqq u\, C_\tau^*(S)$, $C_\tau(P) \geqq u\, C_\tau(S)$.

Proof. We prove the second assertion only; the proof of the first is analogous.

Let $\varepsilon > 0$. Let g be a $\bar{\mu}$-measurable function such that $S = g \,.\, P$; clearly, $\bar{\mu}\{q \in Q: g(q) < u\} = 0$. Let $\mathscr{U} = (U_k: k \in K)$ be a partition of S. Let f_k be $\bar{\nu}$-measurable functions such that $U_k = f_k \,.\, S$. Since $\mu \geqq \nu \geqq u\mu$, f_k are $\bar{\mu}$-measurable. Put $\hat{\mathscr{U}} = (f_k \,.\, P: k \in K)$. Then there exists a dyadic expansion $\mathscr{P} = (P_x: x \in D)$ of P such that $\mathscr{P}''$ refines $\hat{\mathscr{U}}$ and $\Gamma_\tau(\mathscr{P}) < C_\tau(P) + \varepsilon$. Let h_x, $x \in D$, be $\bar{\mu}$-measurable functions such that $P_x = h_x \,.\, P$. Put $S_x = h_x \,.\, S$, $\mathscr{S} = (S_x: x \in D)$. Clearly, $\mathscr{S}$ is a dyadic expansion of S and $\mathscr{S}''$ refines $\mathscr{U}$. For each $x \in D'$ such that $wS_{x0} > 0$, $wS_{x1} > 0$, we have $\Gamma_\tau(P_{x0}, P_{x1}) = V(wP_{x0}, wP_{x1}) \,.\, wP_{x0} \,.\, wP_{x1} \,.\, \tau(P_{x0}, P_{x1})$, $\Gamma_\tau(S_{x0}, S_{x1}) = V(wS_{x0}, wS_{x1}) \,.\, wS_{x0} \,.\, wS_{x1} \,.\, \tau(S_{x0}, S_{x1})$. Since $S_y = g \,.\, P_y$ for all $y \in D$, we get $wS_{x0} \geqq u \,.\, wP_{x0}$, $wS_{x1} \geqq u \,.\, wP_{x1}$. Therefore, by 9.2 and (NGF 2), $\Gamma_\tau(P_{x0}, P_{x1}) \geqq u \,.\, \Gamma_\tau(S_{x0}, S_{x1})$. This implies $\Gamma_\tau(\mathscr{P}) \geqq u\, \Gamma_\tau(\mathscr{S})$, hence $u\, \Gamma_\tau(\mathscr{S}) < C_\tau(P) + \varepsilon$. We have shown that for any partition $\mathscr{U}$ of S there is a dyadic expansion $\mathscr{S}$ of S such that $\mathscr{S}''$ refines $\mathscr{U}$ and $u\, \Gamma_\tau(\mathscr{S}) < C_\tau(P) + \varepsilon$. It follows by Theorem III (in Section 6) that $u\, C_\tau(S) \leqq C_\tau(P) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the lemma is proved.

9.23. Lemma. *Let τ be a normal gauge functional. Let $\varphi = C_\tau$ or $\varphi = C_\tau^*$. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let $0 < u < 1$, $v = 1 - u$. Let $uP \leqq S \leqq P$. Then $\varphi S \geqq u \,.\, \varphi S$, $\varphi S \geqq u^2 \,.\, \varphi P$, and if $v \leqq \frac{1}{2}$, then $|\varphi P - \varphi S| \leqq 2v \,.\, \varphi P$, $|\varphi P - \varphi S| \leqq 6v \,.\, \varphi S$.*

Proof. By 9.22, $\varphi P \geqq u \,.\, \varphi S$. Put $T = u^{-1}S$. Then $P \leqq T$, $uT \leqq P$, hence, again by 9.22, $\varphi T \geqq u \,.\, \varphi P$ and therefore $\varphi S \geqq u^2 \,.\, \varphi P$. Thus, we have $u^{-1} \,.\, \varphi P \geqq \varphi S \geqq u^2 \,.\, \varphi P$, $u^{-2} \,.\, \varphi S \geqq \varphi P \geqq u \,.\, \varphi S$. Hence $|\varphi P - \varphi S| \leqq \max(u^{-1} - 1,\; 1 - u^2) \,.\, \varphi P$, $|\varphi P - \varphi S| \leqq \max(u^{-2} - 1, v) \,.\, \varphi S$. If $v \leqq \frac{1}{2}$, then $u^{-1} - 1 \leqq 2v$, $1 - u^2 < 2v$, $u^{-2} - 1 \leqq 6v$. This proves the lemma.

9.24. Lemma. *Let τ be a normal gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$ be a bounded W-space, $wP \leqq 1/16$. Let P be strongly ΣC_τ^*-fine. Let $m \in \mathbf{R}_+$ be a w-Lipschitz bound for C_τ^* on $\exp P$. If $S \leqq P$, then $|C_\tau^*(S) - C_\tau^*(P)| \leqq (18m(wP)^{2/3} + 32d(P)(wP)^{2/3} + 2m \,.\, wP)\, a^{1/4}$, where $a = \operatorname{md}(S, P) = w(P - S)$.*

Proof. Let f be a $\bar{\mu}$-measurable function such that $S = f \,.\, P$, $0 \leqq fq \leqq 1$ for all $q \in Q$. Put $v = a^{1/4}$, $u = 1 - v$, $X = \{q \in Q: fq \leqq u\}$, $Y = Q \setminus X$. Then $a = w(P - S) = \int_Q (1 - f)\, d\mu \geqq \int_X (1 - f)\, d\mu \geqq v\, \mu(X)$, hence $w(X \,.\, P) \leqq av^{-1} = a^{3/4}$ and therefore

(1) $\operatorname{md}(Y \,.\, P, P) \leqq a^{3/4}$, $\operatorname{md}(Y \,.\, S, S) \leqq a^{3/4}$.

Since $Y \,.\, P$ and $Y \,.\, S$ are pure subspaces of P and S, respectively, we obtain by 9.17 the following inequalities:

(2a) $|C_\tau^*(Y \,.\, P) - C_\tau^*(P))| \leqq (9m + 16d(P))\,(wP)^{2/3}\,(\operatorname{md}(Y \,.\, P, P))^{1/3}$,

(2b) $|C_\tau^*(Y \,.\, S) - C_\tau^*(S)| \leqq (9m + 16d(S))\,(wS)^{2/3}\,(\operatorname{md}(Y \,.\, S, S))^{1/3}$.

Clearly, $u \,.\, (Y \,.\, P) \leqq Y \,.\, S \leqq Y \,.\, P$. Since $u = 1 - v$, $v = a^{1/4} \leqq 1/2$, we get by 9.23

(3) $|C_\tau^*(Y \,.\, S) - C_\tau^*(Y \,.\, P)| \leqq 2a^{1/4} m \,.\, wP$.

Now, (1), (2a), (2b) and (3) imply the inequality asserted in the lemma.

9.25. Proposition. *Let τ be a normal gauge functional. Let P be a bounded W-space. Let P be strongly ΣC_τ^*-fine. Let $m \in \mathbf{R}_+$ be a w-Lipschitz bound for C_τ^* on* $\exp P$. *If $S_1 \leqq P$, $S_2 \leqq P$, then*

$$|C_\tau^*(S_1) - C_\tau^*(S_2)| \leqq F(m, d(P), wP)\,(\mathrm{md}(S_1, S_2))^{1/4},$$

where $F: \mathbf{R}_+^3 \to \mathbf{R}_+$ is defined as follows: $F(x, y, z) = 8(7x + 11y)\, z^{3/4}$.

Proof. We can assume $wP > 0$. Put $U = S_1 \wedge S_2$, $a = \mathrm{md}(S_1, S_2)$. If $wP = 1/16$, then by 9.24 with $1/16$ substituted for wP we get, for $i = 1, 2$.

(1) $|C_\tau^*(S_i) - C_\tau^*(U)| \leqq \frac{1}{2}(7m + 11d(P))\, a^{1/4}$.

Hence

(2) $|C_\tau^*(S_1) - C_\tau^*(S_2)| \leqq (7m + 11d(P))\, a^{1/4}$.

If $wP > 0$ is arbitrary, put $t = 1/16wP$. Since $w(tP) = 1/16$, we have by (2)

$$|C_\tau^*(tS_1) - C_\tau^*(tS_2)| \leqq (7m + 11d(P))\,(ta)^{1/4},$$

hence

$$|C_\tau^*(S_1) - C_\tau^*(S_2)| \leqq t^{-3/4}(7m + 11d(P))\, a^{1/4},$$

$$|C_\tau^*(S_1) - C_\tau^*(S_2)| \leqq F(m, d(P), wP)\, a^{1/4}.$$

9.26. Proposition. *Let τ be a normal gauge functional. Let P be a bounded W-space. If C_τ^* is w-Lipschitz on* $\exp P$ *and P is strongly ΣC_τ^*-fine, then $C_\tau^* \upharpoonright \exp P$ is Lipschitz of order* $1/4$.

Proof. See 9.25.

9.27. Definition. Let P be a W-space or a semimetric space. If for any $\varepsilon > 0$ there exists a partition $(U_k: k \in K)$ of P such that for all $k \in K$, Ded $(U_k) < \varepsilon$ (De-diam $U_k < \varepsilon$, respectively), then we will say that P is *De-partition-fine.* If, in addition, P is De-bounded (see 8.13), then we will call P *De-totally bounded* or *totally expansion-bounded.*

9.28. Fact. *Let $\langle Q, \varrho\rangle$ be a De-totally bounded (De-partition-fine) semimetric space. Then (1) any subspace of $\langle Q, \varrho\rangle$ is De-totally bounded (De-partition-fine, respectively), (2) if $\langle T, \sigma\rangle$ is a semimetric space, $f: Q \to T$ is a surjective mapping and $\sigma(fx, fy) \leqq \varrho(x, y)$ for any $x, y \in Q$, then $\langle T, \sigma\rangle$ is De-totally bounded (De-partition-fine, respectively).*

Proof. See 8.15.

9.29. Lemma. *Let P be a bounded semimetric space. If there exists a partition $\mathscr{U} = (U_k: k \in K)$ of P such that all U_k are expansion-bounded (De-partition-fine, totally expansion-bounded), then P is expansion-bounded (De-partition-fine, totally expansion-bounded).*

Proof. I. Let all U_k be De-bounded. It is enough to consider the case $\mathscr{U} = (U_0, U_1)$, since for an arbitrary $\mathscr{U}$, the assertion will follow by induction. Put $a = \mathrm{diam}\, P +$ De-diam U_0 + De-diam U_1. Let $b > a$. Let $\varepsilon > 0$. For $i = 0, 1$, there exists a dyadic expansion $\mathscr{P}_i = (P_{i_i x}: x \in D_i)$ of U_i such that diam $P_{i_i x} < \varepsilon$ if $x \in D_i''$ and

$\Sigma(\max\{\operatorname{diam} P_{i,x}: x \in \{0,1\}^m \cap D_i\}: \{0,1\}^m \cap D \neq \emptyset) <$ De-diam $U_i + (b-a)/2$. Let D consist of the void sequence $\emptyset$ and of all $(0) . x$, $x \in D_0$, $(1) . x$, $x \in D_1$. Clearly, $D \in \Delta$. Put $S_\emptyset = P$. If $z = (i) . x$, $i = 0, 1$, $x \in D_i$, put $S_z = P_{i,x}$. Clearly, $\mathscr{S} = (S_z: z \in D)$ is a dyadic expansion of P. It is easy to see that $\Sigma(\max\{\operatorname{diam} S_z: z \in \{0,1\}^m \cap D\}: \{0,1\}^m \cap D \neq \emptyset\} < \operatorname{diam} P + \sum_{i=0,1} \Sigma(\max\{\operatorname{diam} P_{i,x}: x \in \{0,1\}^m \cap D_i\}: \{0,1\}^m \cap D_i \neq \emptyset\} < \operatorname{diam} P +$ De-diam $U_0 +$ De-diam $U_1 + (b-a) = b$. Since $b > a$ was arbitrary, we have shown that De-diam $P \leqq a$. – II. It is easy to see that if U_k are De-partition-fine, then P is De-partition-fine. – III. If U_k are De-totally bounded, then P is De-bounded, by I, and De-partition-fine, by II.

9.30. Fact. *A metric space $S = \langle Q, \varrho \rangle$ is totally expansion-bounded if and only if it is De-partition-fine.*

Proof. Let S be De-partition-fine. By 8.10, S is bounded. Since S is De-partition-fine, there is a partition $(U_k: k \in K)$ such that De-diam $U_k < 1$ for each $k \in K$. Hence, by 9.29, S is De-bounded.

9.31. Lemma. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let $A \subset B \subset Q$, $B \in \operatorname{dom} \bar{\mu}$ and $\bar{\mu} B = \mu_e(A)$. Then* Ded $(B . P) \leqq$ De-diam $\langle A, \varrho \restriction A \rangle$.

Proof. Put $S = \langle A, \varrho \restriction A \rangle$. We can assume that De-diam $S = t < \infty$. Let $t' \in \boldsymbol{R}$, $t' > t$, and let $\varepsilon > 0$. Then there exists a dyadic expansion $(V_x: x \in D)$ of A such that (1) $\operatorname{diam} V_x < \varepsilon$ for each $x \in D''$, (2) $\Sigma(\max\{\operatorname{diam} V_x: x \in \{0,1\}^m \cap D\}: \{0,1\}^n \cap D \neq \emptyset) \leqq t'$. Choose sets $U_z \supset V_z$, $z \in D''$, such that $U_z \in \operatorname{dom} \mu$, $\mu U_z = \mu_e(V_z)$ for each $z \in D''$. For each $x \in D$ put $T_x = \bigcup(U_z: z \in D(x) \cap D'')$. Then $T_x \supset V_x$ and, by (*) in 8.16 (proof), $\mu T_x = \mu_e(V_x)$. Thus $T_\emptyset \supset A$, $\mu T_\emptyset = \mu_e(A) = \bar{\mu} B$, hence $T_\emptyset . P = B . P$. By 8.11 we have $d(T_x . P) \leqq \operatorname{diam} V_x$ for each $x \in D$. – Now let $f: D'' \to \{0, \ldots, n\}$ be a bijection, put $M_z = U_z \setminus \bigcup(U_y: f(y) < f(z))$ for each $z \in D''$ and put $E_x = \bigcup(M_z: z \in D(x) \cap D'')$ for each $x \in D$. Clearly, $T_x \supset E_x$ for each $x \in D$. Since $T_\emptyset = E_\emptyset$, we have $E_\emptyset . P = B . P$. We also see that $(E_x . P: x \in D)$ is a dyadic expansion of $B . P$ and that $d(E_x . P) \leqq d(T_x . P) \leqq \operatorname{diam} V_x$ for each $x \in D$. Hence $d(E_z . P) < \varepsilon$ for each $z \in D''$ and $\Sigma(\max\{d(E_x . P): x \in \{0,1\}^m \cap D\}: \{0,1\}^m \cap D \neq \emptyset) \leqq t'$. Since $t' > t$ and $\varepsilon > 0$ were arbitrary, we have shown that Ded $(B . P) \leqq$ De-diam S.

9.32. Proposition. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. If $\langle Q, \varrho \rangle$ is totally expansion-bounded (De-partition-fine), then so is P.*

Proof. I. Let $S = \langle Q, \varrho \rangle$ be De-partition-fine. Let $\varepsilon > 0$ be given. There exists a partition $(A_k: k \in K)$ of Q such that De-diam $(S \restriction A_k) < \varepsilon$ for each $k \in K$. Choose sets $B_k \supset A_k$ such that $\mu B_k = \mu_e(A_k)$ for each $k \in K$. By 9.31 we have Ded $(B_k . P) \leqq$ De-diam $(S \restriction A_k) < \varepsilon$ for each $k \in K$. Let $f: K \to \{0, \ldots, n\}$ be a bijection and put $E_k = B_k \setminus \bigcup(B_i: f(i) < f(k))$, $U_k = E_k . P$. Then $(U_k: k \in K)$ is a partition of P and Ded $U_k < \varepsilon$ for each $k \in K$. This proves that P is De-partition-fine.

II. If $\langle Q, \varrho \rangle$ is totally expansion-bounded, then it is expansion-bounded and therefore, by 8.16, P is expansion-bounded as well, hence, by I, it is totally expansion-bounded.

9.33. Proposition. *Let* $a = (a_i: i \in N) \in \mathscr{L}_\infty(N)$, $a \geqq 0$, $\Sigma(a_i: i \in N) < \infty$. *Then the subspace* $\{x: 0 \leqq x \leqq a\}$ *of* $\mathscr{L}_\infty(N)$ *is totally expansion-bounded.*

Proof. Clearly, we can assume that all a_i are positive. If $b = (b_i)$, $c = (c_i)$ are in $\mathscr{L}_\infty(N)$, $b_i < c_i$ for all $i \in N$ and $c_i - b_i \to 0$ for $i \to \infty$, put $T(b, c) = \{x = (x_i: i \in N) \in \mathscr{L}_\infty(N): b_i < x_i < c$ for all $i \in N\}$. Clearly, it is enough to show that $T(0, a)$ is De-totally bounded. By 8.20, $T(0, a)$ is De-bounded. Hence we have only to show that $T(0, a)$ is De-partition-fine. Let $\varepsilon > 0$. Choose a $p \in N$ such that $\Sigma(a_i: i \geqq p) < \varepsilon/4$. Choose an $m \in N$ such that $\Sigma(a_i: i < p)/m < \varepsilon/4$. If $\varkappa = (k_0, \ldots, k_{p-1}) \in \{0, \ldots, m - 1\}^p$, define $u(\varkappa)$ and $v(\varkappa)$ as follows: $u(\varkappa)(i) = k_i a_i/m$ if $i < p$, $u(\varkappa)(i) = 0$ if $i \geqq p$, $v(\varkappa)(i) = (k_i + 1) a_i/m$ if $i < p$, $v(\varkappa)(i) = a_i$ if $i \geqq p$. It is easy to see that $(T(u(\varkappa), v(\varkappa)): \varkappa \in \{0, \ldots, m - 1\}^p)$ is a partition of $T(0, a)$ and every $T(u(\varkappa), v(\varkappa))$ is isometric to $T(u(0), v(0))$. By 8.19 we have De-diam $T(u(0), v(0)) \leqq 2(\Sigma(a_i: i < p)/m + 2\,\Sigma(a_i: i \geqq p)) < \varepsilon$. This proves the proposition.

9.34. Remark. I do not know whether there are expansion-bounded metric spaces which are not totally expansion-bounded.

9.35. Proposition. *Every bounded subspace of* $\boldsymbol{R}^n$, $n = 1, 2, \ldots$, *is totally expansion-bounded.*

Proof. See 9.33.

9.36. Fact. *Let P be a De-totally bounded W-space. Let* τ *be a normal gauge functional. Let* $\varphi = C_\tau$ *or* $\varphi = C_\tau^*$. *Then* (1) φ *is w-Lipschitz on* exp P (*and* Ded (P) *is a w-Lipschitz bound for* φ *on* exp P); (2) P *is both strongly* $\Sigma\varphi$*-fine and strongly* $\Sigma^*\varphi$*-fine.*

Proof. The first assertion follows from 8.28 since, clearly, Ded $(T) \leqq$ Ded (P) for all $T \leqq P$. — We are going to prove that P is strongly ΣC_τ-fine. The proof of the remaining assertion is analogous. We can assume $wP > 0$. Let $\varepsilon > 0$. Then there exists a partition $\mathscr{U} = (U_k: k \in K)$ of P such that for any $k \in K$, Ded $(U_k) < \varepsilon/wP$. Let $(V_m: m \in M)$ be a artition refining $\mathscr{U}$. Let $(M_k: k \in K)$ be a partition of M such that $\Sigma(V_m: m \in M_k) = U_k$ for all $k \in K$. Then, by 8.28, for any $k \in K$, $m \in M_k$, $C_\tau(V_m) \leqq wV_m$. Ded $(V_m) \leqq wV_m$. Ded (U_k), hence $C_\tau(V_m) \leqq (\varepsilon/wP) \,.\, wV_m$ for any $m \in M$ and therefore $\Sigma(C_\tau(V_m): m \in M) \leqq (\varepsilon/wP) \,.\, wP = \varepsilon$.

9.37. Proposition. *Let* τ *be a normal gauge functional. Let* $P = \langle Q, \varrho, \mu \rangle$ *be a W-space. If P is totally expansion-bounded* (*in particular, if* $\langle Q, \varrho \rangle$ *is totally expansion-bounded*), *then* $C_\tau \restriction \exp P$ *and* $C_\tau^* \restriction \exp^* P$ *are Lipschitz of order* $1/3$, *and* $C_\tau^* \restriction \exp P$ *is Lipschitz of order* $1/4$.

Proof. It follows from 9.36, 9.20, 9.26 and 9.32.

Remark. As already mentioned, $1/3$ and $1/4$ are merely estimates of the Lipschitz order of $C_\tau \restriction \exp P$, etc., possibly rather rough. Therefore, we do not try to find tolerably good Lipschitz bounds (of order 1/3 or 1/4, respectively) for $C_\tau \restriction \exp P$, etc. Some bounds are given in 9.17 and 9.24.

9.38. We are now going to consider the functionals $C_\tau^* \restriction \mathfrak{W}_F(Q)$, $C_\tau \restriction \mathfrak{W}_F(Q)$ where τ is a normal gauge functional and Q is a non-void finite set. Various propositions concerning continuity and Lipschitz type properties of these functionals will be proved.

Let us recall, partly in a slightly modified form, some propositions from Section 5 (τ denotes a normal gauge functional, Q a non-void finite set). – The functionals $C_\tau^* \restriction \mathfrak{W}_F(Q)$ are continuous (see 5.7); every $C_\tau \restriction \mathfrak{W}(Q, \varrho, \cdot)$ (see 2.11) is continuous at every point $P = \langle Q, \varrho, \mu \rangle$ of the space $\mathfrak{W}(Q, \varrho, \cdot)$ such that $\mu q > 0$ for all $q \in Q$ (see 5.8); every $C_\tau^* \restriction \mathfrak{W}(Q, \cdot, \mu)$ satisfies $L(1; \infty, H(\mu))$ (this follows immediately from 5.1); every $C_\tau \restriction \mathfrak{W}(Q, \cdot, \mu)$ is continuous (see 5.4).

9.39. It will be proved (see 9.43 below) that the functionals $C_\tau \restriction \mathfrak{W}_F(Q)$ are continuous. It will also be shown that the functionals $C_\tau^* \restriction B$ where B is a bounded subspace of $\mathfrak{W}_F(Q)$ are Lipschitz of order 1/4 and the functionals $C_\tau \restriction B$ where is a bounded subspace of some $\mathfrak{W}(Q, \varrho, \cdot)$ are Lipschitz of order 1/3. However, I do not know whether the functionals of the form $C_\tau \restriction B$, where B is a bounded subspace of some $\mathfrak{W}(Q, \cdot, \mu)$, are Lipschitz (of some order). Therefore, we will also examine the question whether the functionals $C_\tau \restriction \mathfrak{W}_F(Q)$ are Lipschitz at some points of $\mathfrak{W}_F(Q)$ (the definition of "Lipschitz at a point" will be given in 9.46).

9.40. Lemma. *Let τ be a normal gauge functional. Let Q be a finite non-void set. Put $n = \operatorname{card} Q$. Let $P_i = \langle Q, \varrho, \mu_i \rangle \in \mathfrak{W}_F$, $i = 1, 2$. Then*

$$|C_\tau(P_1) - C_\tau(P_2)| \leqq m(\operatorname{dist}(P_1, P_2))^{1/3},$$

where $m = (9 \log n + 16) \operatorname{diam} \langle Q, \varrho \rangle (w(P_1 \vee P_2))^{2/3}$.

Proof. Put $t = \operatorname{diam} \langle Q, \varrho \rangle$. If $T = \langle Q, \varrho, \nu \rangle \in \mathfrak{W}_F$, then clearly $C_\tau(T) \leqq$ $\leqq C_\tau \langle Q, 1, \nu \rangle$, hence, by 3.9.2 and 3.20, we get $C_\tau(T) \leqq t C_E \langle Q, 1, \nu \rangle$. By Theorem II (in Section 6) we have $C_E \langle Q, 1, \nu \rangle = H(\nu)$; by 2.4 we have $H(\nu) \leqq (\log n)\, wT$. Therefore, $C_\tau(T) \leqq (t \log n)\, wT$. Clearly, every FW-space is strongly ΣC_τ-fine. By 9.17 we obtain

$$|C_\tau(P_1) - C_\tau(P_2)| \leqq (9t \log n + 16\, d(P))\, (w(P_1 \vee P_2))^{2/3} (\operatorname{md}(P_1, P_2))^{1/3}.$$

Since, by 9.13. 1, $\operatorname{md}(P_1, P_2) = \operatorname{dist}(P_1, P_2)$, the lemma is proved.

9.41. Fact. *Let Q be a non-void finite set. Let B be a bounded subspace of $\mathfrak{W}_F(Q)$. Then there exist numbers a, b such that* (1) *$wP \leqq a$ for every $P \in B$;* (2) *if $x \in Q$, $y \in Q$, $\langle Q, \varrho, \mu \rangle \in B$, then $\varrho(x, y) \leqq b$.*

9.42. Proposition. *Let τ be a normal gauge functional. Let Q be a finite non-void set. Let B be a bounded non-void subspace of $\mathfrak{W}(Q, \varrho, \cdot)$. Then $C_\tau \restriction B$ is Lipschitz of order* 1/3.

Proof. Let a, b satisfy (1), (2) in 9.41. Put $n = \text{card } Q$, $k = (9 \log n + 16b)(2a)^{2/3}$. Let $P_1, P_2 \in B$. Clarly, $w(P_1 \vee P_2) \leqq 2a$, diam $\{Q, \varrho\rangle \leqq b$. By 9.40 we get $|C_\tau(P_1) - C_\tau(P_2)| \leqq k(\text{dist}(P_1, P_2))^{1/3}$.

9.43. Proposition. *Let τ be a normal gauge functional. Let Q be a finite non-void set. Then the functional $C_\tau \upharpoonright \mathfrak{W}_F(Q)$ is continuous.*

Remark. For $\tau = r$ this has been stated in [4], 1.7, and [3], 3.5. A very short outline of proof is contained in [3], 4.1–4.9.

Proof. Let $P_0 = \langle Q, \varrho_0, \mu_0\rangle \in \mathfrak{W}c(Q)$. Let $\varepsilon > 0$. By 5.4, $C_\tau \upharpoonright \mathfrak{W}_F(Q, \cdot, \mu_0)$ is continuous, hence there exists an $\alpha > 0$ such that

(1) if $S = \langle Q, \varrho, \mu_0\rangle \in \mathfrak{W}_F$, dist $(S, P_0) < \alpha$, then $|C_\tau(S) - C_\tau(P_0)| < \varepsilon/2$.

Let B be the set of all $P \in \mathfrak{W}_F(Q)$ such that dist $(P, P_0) < \alpha$. Since B is bounded, by 9.40 and 9.41 there exists a number k such that

(2) if $S_i = \langle Q, \sigma, \nu_i\rangle \in B$, $i = 1, 2$, then $|C_\tau(S_1) - C_\tau(S_2)| \leqq k(\text{dist}(S_1, S_2))^{1/3}$.

Choose $\delta > 0$ such that $\delta < \alpha$, $k\delta^{1/3} < \varepsilon/2$. Let $P = \langle Q, \varrho, \mu\rangle \in \mathfrak{W}_F(Q)$, dist $(P, P_0) < \delta$. Put $P_1 = \langle Q, \varrho, \mu_0\rangle$. Clearly, dist $(P_1, P_0) \leqq$ dist $(P, P_0) < \delta$, hence, by (1), we have

(3) $|C_\tau(P_1) - C_\tau(P_0)| < \varepsilon/2$.

Since $P \in B$, $P_1 \in B$, we get by (2) the inequality

(4) $|C_\tau(P) - C_\tau(P_1)| \leqq k(\text{dist}(P, P_0))^{1/3} \leqq k\delta^{1/3} < \varepsilon/2$.

By (3) and (4) we have $|C_\tau(P) - C_\tau(P_0)| < \varepsilon$.

9.44. Lemma. *Let τ be a normal gauge functional. Let Q be a finite non-void set. Let B be a bounded non-void subspace of $\mathfrak{W}_F(Q)$. Then there exist $k', k'' \in \boldsymbol{R}_+$ such that*

(1) *if $P_i = \langle Q, \varrho, \nu_i\rangle \in B$, $i = 1, 2$, then $|C^*_\tau(P_1) - C^*_\tau(P_2)| \leqq k'(\text{dist}(P_1, P_2))^{1/4}$;*

(2) *if $S_i = \langle Q, \varrho_i, \nu\rangle \in B$, $i = 1, 2$, then $|C^*_\tau(S_1) - C^*_\tau(S_2)| \leqq k''$ dist (S_1, S_2).*

Proof. Let a, b satisfy (1) and (2) in 9.41. Put $n = \text{card } Q$, $m = b \log n$. Let F be the function introduced in 9.15. Put $k' = F(m, b, 2a)$, $k'' = \sup\{H(\mu): \langle Q, \varrho, \mu\rangle \in B\}$. Since B is bounded, $k'' < \infty$. We have $C^*_\tau(T) \leqq m \,.\, wT$ for every $T \in B$; this can be shown in the same way as the corresponding assertion in the proof of 9.40.

Let $P_i = \langle Q, \varrho, \nu_i\rangle \in B$, $i = 1, 2$. By 9.25 we get

$$|C^*_\tau(P_1) - C^*_\tau(P_2)| \leqq F(m, d(P_1 \vee P_2), w(P_1 \vee P_2))(\text{md}(P_1, P_2))^{1/4}.$$

Clearly, $d(P_1 \vee P_2) \leqq b$, $w(P_1 \vee P_2) \leqq 2a$. Since F is increasing (in each variable), we get

$$|C^*_\tau(P_1) - C^*_\tau(P_2)| \leqq F(m, b, 2a)(\text{md}(P_1, P_2))^{1/4}.$$

Let $S_i = \langle Q, \varrho_i, \nu\rangle \in B$, $i = 1, 2$. By 5.1 we have

$$|C^*_\tau(S_1) - C^*_\tau(S_2)| \leqq H(\nu)\,\text{dist}(S_1, S_2) \leqq k''\,\text{dist}(S_1, S_2).$$

9.45. Proposition. *Let τ be a normal gauge functional. Let Q be a finite non-void* *set. Let B be a bounded non-void subspace of* $\mathfrak{W}_F(Q)$. *Then* $C_\tau^* \upharpoonright B$ *is Lipschitz of order* $1/4$.

Proof. Let $k', k'' \in \boldsymbol{R}_+$ satisfy (1) and (2) from 9.44. Let b satisfy (3) from 9.41. Put $k = k' + k''b^{3/4}$. It is easy to see that for any $P_1, P_2 \in B$ we have $|C_\tau^*(P_1) - C_\tau^*(P_2)| \leqq k(\text{dist}(P_1, P_2))^{1/4}$.

9.46. Definition and convention. Let $U = \langle Q, \varrho \rangle$ and $V = \langle T, \sigma \rangle$ be semimetric spaces and let $f: U \to V$ be a mapping. Let $z \in Q$ and let ϱ_z denote the function $x \mapsto \varrho(x, z)$ defined on Q. — (1) If the function $x \mapsto \sigma(fx, fz)$ satisfies $L(p, \varrho_z; b, m)$ on Q (see 9.18.1), then we will say that f *satisfies* $L(p; b, m)$ *at* z or that m *is a Lipschitz bound of order* p *for* f *at* z, *with distance bound* b. — (2) If $x \mapsto \sigma(fx, fz)$ satisfies $L(p, \varrho_z; b, \cdot)$ on Q, then we will say that f *satisfies* $L(p; b, \cdot)$ *at* z *or that* f *is Lipschitz of order* p *at* z, *with distance bound* b. — (3) If for some b and some m, f satisfies $L(p; b, m)$ at z, then we will say that f *satisfies a Lipschitz condition of order* p *at* z. — (4) The words "with distance bound b" are omitted if $b = \infty$. — Note that by the convention just introduced, "f is Lipschitz of order p at z" means "f satisfies $L(p; \infty, m)$ for some m".

9.47. Proposition. *Let* τ *be an* NGF. *Let* $\varphi = C_\tau$ *or* $\varphi = C_\tau^*$. *Let* Q *be a finite non-void set. Let* $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_F(Q)$ *and assume* $\mu q > 0$ *for all* $q \in Q$. *Put* $t = \min\{\mu q: q \in Q\}$. *Then* $\varphi \upharpoonright \mathfrak{W}(Q, \varrho, \cdot)$ *satisfies* $L(1; t/4, 5\varphi P/t)$ at P.

Proof. Let $S = \langle Q, \varrho, \nu \rangle \in \mathfrak{W}_F(Q)$, $\text{dist}(S, P) = \varepsilon \leqq t/4$. It is easy to see that for all $q \in Q$, $\nu q \geqq (1 - \varepsilon/t)\,\mu q$, $(1 + \varepsilon/t)\,\mu q \geqq \nu q$, and therefore $S \geqq (1 - \varepsilon/t)\,P$, $(1 + \varepsilon/t)\,P \geqq S$. Put $P_1 = (1 + \varepsilon/t)\,P$. Put $u = (t - \varepsilon)/(t + \varepsilon)$, $v = 1 - u = 2\varepsilon/(t + \varepsilon)$. Clearly, $v < 1/2$. We have $uP_1 \leqq S \leqq P$, hence, by 9.23, $|\varphi S - \varphi P_1| \leqq 2v \,.\, \varphi P_1 \leqq 4\varepsilon t^{-1} \,.\, \varphi P$. Clearly, $\varphi P_1 - \varphi P = \varepsilon t^{-1} \,.\, \varphi P$. Hence $|\varphi S - \varphi P| \leqq (5\varphi P/t)\,\varepsilon$.

9.48. Proposition. *Let* φ *be a hypoentropy. Let* Q *be a finite non-void set. Let* $\mathfrak{W}_F(Q) \subset \text{dom}\,\varphi$. *Let* $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_F(Q)$ *and let* $\varrho(x, y) > 0$ *for all* $x, y \in Q$, $x \neq y$. *Put* $s = \min\{\varrho(x, y): x \in Q,\ y \in Q,\ x \neq y\}$. *Then* $\varphi \upharpoonright \mathfrak{W}(Q, \cdot, \mu)$ *satisfies* $L(1; s/2, 3\varphi P/s)$ *at* P.

Proof. Let $S = \langle Q, \sigma, \mu \rangle \in \mathfrak{W}_F(Q)$, $\text{dist}(S, P) = \varepsilon \leqq s/2$. Clearly, $\varrho \leqq \sigma + \varepsilon$, $\sigma \leqq \varrho + \varepsilon$, hence $\varrho \leqq (1 + \varepsilon/(s - \varepsilon))\,\sigma$, $\sigma \leqq (1 + \varepsilon/s)\,\varrho$, and therefore $\varphi P \leqq (1 + \varepsilon/(s - \varepsilon))\,\varphi S$, $\varphi S \leqq (1 + \varepsilon/s)\,\varphi P$, which implies $\varphi P - \varphi S \leqq (\varepsilon/(s - \varepsilon))\,\varphi S$, $\varphi S - \varphi P \leqq (\varepsilon/s)\,\varphi P$. Since $\varphi S \leqq (1 + \varepsilon/s)\,\varphi P$, we get $|\varphi P - \varphi S| \leqq \varepsilon(s + \varepsilon)\,s^{-1}(s - \varepsilon)^{-1}\,\varphi P$. Since $\varepsilon \leqq s/2$, we have $(s + \varepsilon)(s - \varepsilon)^{-1} \leqq 3$, hence $|\varphi P - \varphi S| \leqq (3\varphi P/s)\,\varepsilon$.

9.49. Proposition. *Let* τ *be a normal gauge functional. Let* Q *be a finite non-void set. Let* $P = \langle Q, \varrho, \mu \rangle \in \mathfrak{W}_F(Q)$. *Assume that* $\mu q > 0$ *for all* $q \in Q$ *and* $\varrho(x, y) > 0$ *for all* $x, y \in Q$, $x \neq y$. *Then* $C_\tau \upharpoonright \mathfrak{W}_F(Q)$ *satisfies a Lipschitz condition of order* 1 *at* P.

Proof. It is an easy consequence of 9.47 and 9.48.

9.50. Remark. It has been shown by I. Neumann in his Master Thesis (Charles University, 1984) that for any finite non-void set Q and any semimetric ϱ on Q, $C_\tau^* \upharpoonright B$ is Lipschitz of order p for any $p \in (0, 1)$ and any bounded subspace B of $\mathfrak{W}(Q, \varrho, \cdot)$. The proof is straightforward, though not quite simple, and involves only elementary calculus and basic properties of the functional H.

10

The section is organized as follows. First we present examples showing that (1) all $C_\tau^* \upharpoonright \mathfrak{W}_F$, where $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$, are distinct, (2) $C_r \upharpoonright \mathfrak{W}_F \neq C_E \upharpoonright \upharpoonright \mathfrak{W}_F$. Based on these facts, a proof is given (see 10.10) of the fact that there are exactly 2^ω finitely continuous extenced Shannon entropies on $\mathfrak{W}_F$. We also prove (see 10.12) that there are enormously many (as many as there are classes) extended (in the broad sense) Shannon entropies on $\mathfrak{W}$. Then we turn (see 10.17 and 10.18) to examples announced in or connected with Part I. Finally, we present a number of examples connected with Sections 7 through 9.

10.1. Fact. *Let $P = \langle Q, \varrho, \mu \rangle$ be an FW-space, $Q = \{0, 1, 2\}$, $\varrho(0, 1) = 1$, $\varrho(1, 2) = 2$, $\varrho(0, 2) = 3$, $\mu\{0\} = \mu\{1\} = \mu\{2\} = 1$. Then $C_{r(t)}^*(P)$, where we put $r(t) = r_t$, see 3.2, is a strictly increasing function for $t \in [1, \infty]$.*

Proof. If $1 \leqq t < \infty$, $\tau = r(t)$, then by 4.20, $C_\tau^*(P)$ is equal to the least of the numbers $2 + H(1, 2)(2^t + 3^t)^{1/t}/2^{1/t}$, $4 + H(1, 2)(1 + 3^t)^{1/t}/2^{1/t}$, $6 + H(1, 2) \,.\, (1 + 2^t)^{1/t}/2^{1/t}$. Now, it is well known that any function of the form $f(t) = (pa^t + qb^t)^{1/t}/(p + q)^{1/t}$, where p, q, a, b are positive, $a \neq b$, is strictly increasing for $1 \leqq t < \infty$. This proves the assertion for $t \in [1, \infty)$. Clearly, $C_{r(\infty)}^*(P) = 2 + 3H(1, 2) > C_{r(t)}^*(P)$ for all $t \in [1, \infty)$.

10.2. Fact. *Let $Q = \{0, 1, 2\}$. Let ϱ be the semimetric on Q such that $\varrho(0, 1) = \varrho(0, 2) = 0$, $\varrho(1, 2) = 1$. Let $0 < b < 1/2$. Let μ_b be the measure on Q such that $\mu_b\{0\} = a = 1 - 2b$, $\mu_b\{1\} = \mu_b\{2\} = b$. Put $S_b = \langle Q, \varrho, \mu_b \rangle$. Then* (1) $C_E^*(S_b) = H(b, 1 - b)$; (2) $C_{r(\infty)}^*(S_b) = 2b$; (3) if $1 \leqq t < \infty$, *then* $C_{r(t)}^*(S_b) = \min(2b, H(b, 1 - b)\, b^{1/t}(1 - b)^{-1/t})$, *hence, in particular*, (4) $C_{r(1)}^*(S_b) = H(b, 1 - b)\, b/(1 - b)$; (5) *for any t, $1 \leqq t < \infty$,* we have $C_{r(t)}(S_b) \leqq (4b^2)^{1/t}$, *hence, in particular*, (6) $C_{r(1)}(S_b) \leqq 4b^2 < C_{r(1)}^*(S_b)$, *and* (7) $C_{r(1)}(S_b) \leqq 4b^2 < C_E^*(S_b)$.

Proof. Using again 4.20, we get (8) $C_E^*(S_b) = \min(H(b, 1 - b), H(2b, 1 - 2b) + 2b)$, (9) $C_{r(\infty)}^*(S_b) = \min(2b, H(b, 1 - b))$, and (10) if $1 \leqq t < \infty$, then $C_{r(t)}^*(S_b) = \min(2b, H(b, 1 - b)\, b^{1/t}/(1 - b)^{1/t}$, which is the equality (3). It is easy to show that foı any x, $0 \leqq x \leqq 1/2$, we have (11) $2x \leqq H(x, 1 - x) \leqq H(2x, 1 - 2x) + 2x$, (12) $H(x, 1 - x)/(1 - x) \leqq 2$. The inequalities (8) and (11) imply (1); (9) and (11) imply (2); (3) and (12) imply (4).

Let $\mathcal{S} = (S_x \colon x \in D)$ be the dyadic expansion of S_b such that $D'' = \{0, 1\}^2$, $S_{00} = \frac{1}{2}\{0\} \,.\, S_b$, $S_{01} = \{1\} \,.\, S_b$, $S_{10} = \frac{1}{2}\{0\} \,.\, S_b$, $S_{11} = \{2\} \,.\, S_b$. Celarly, for $1 \leqq t < \infty$,

$\Gamma_{r(t)}(\mathscr{S}) = (4b^2)^{1/t}$, hence, by 4.24, $C_{r(t)}(S_b) \leqq (4b^2)^{1/t}$. By 2.16.1, $4x(1-x) < < H(x, 1-x)$ for any x, $0 < x < 1/2$. Hence $4b^2 < H(b, 1-b)\, b/(1-b)$, which proves (6). Since $b/(1-b) < 1$, we have $4b^2 < H(b, 1-b)$, which proves (7).

Remark. The spaces S_b, $0 < b < 1/2$, provide examples mentioned in [4], 3.7.

10.3. Examples in 10.1 and 10.2 show that (i) all $C_\tau^* \restriction \mathfrak{W}_F$, where $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$, are distinct, (ii) $C_r \restriction \mathfrak{W}_F \neq C_r^* \restriction \mathfrak{W}_F$. The following questions remain open: (1) Does there exist, for any normal gauge functional $\tau \geqq r$, an FW-space S such that $C_\tau(S) < C_\tau^*(S)$? It seems that the answer is affirmative if $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$. However, this question will not be examined here since only the fact that $C_r \restriction \mathfrak{W}_F \neq C_r^* \restriction \mathfrak{W}_F$ will be used in some of the proofs below. – (2) Does there exist, e.g. for $\tau = r$, a metric FW-space P such that $C_\tau(P) < C_\tau^*(P)$? A negative answer would mean, among other things, that $C_r(P)$, $P \in \mathfrak{W}_F \cap \mathfrak{W}_M$, could be calculated in finitely many steps. – (3) Are all $C_\tau \restriction \mathfrak{W}_F$, where $\tau = r_t$, $1 \leqq t \leqq \infty$, or $\tau = E$, distinct? In what follows, only a special case will be solved: we shall prove that $C_r \restriction \mathfrak{W}_F \neq C_E \restriction \mathfrak{W}_F$. To this end, we need some facts concerning C_E and C_E^*.

10.4. Lemma. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Let $(U_k : k \in K)$ be a partition of P. Then there exists a pure partition $(U_k^* : k \in K)$ such that $d(U_k^*) \leqq d(U_k)$ for all $k \in K$ and $H(wU_k^* : k \in K) \leqq H(wU_k : k \in K)$.*

Proof. If $\mathscr{T} = (T_m : m \in M)$ is a partition of P, put $s(\mathscr{T}) = \operatorname{card}\{(i, j) \in M \times M : i \neq j, w(T_i \wedge T_j) > 0\}$. We are going to show that $(*)$ for any partition $\mathscr{V} = (V_k : k \in K)$ of P such that $s(\mathscr{V}) > 0$, there exists a partition $\mathscr{V}' = (V_k' : k \in K)$ such that $s(\mathscr{V}') < s(\mathscr{V})$, $H(wV_k' : k \in K) < H(wV_k : k \in K)$ and $d(V_k') \leqq d(V_k)$ for each $k \in K$.

Choose $i, j \in K$ such that $i \neq j$, $w(V_i \wedge V_j) > 0$, $wV_i \geqq wV_j$. There exist $\bar{\mu}$-measurable functions f_i, f_j such that $V_i = f_i . P$, $V_j = f_j . P$. For any $q \in Q$, put $g_i(q) = f_i(q) + f_j(q)$ if $f_i(q) > 0$, and $g_i(q) = 0$ if $f_i(q) = 0$. Put $g_j = f_i + f_j - g_i$, $V_i' = g_i . P$, $V_j' = g_j . P$, $V_k' = V_k$ if $k \in K$, $k \neq i$, $k \neq j$. Clearly, $\mathscr{V}' = (V_k' : k \in K)$ is a partition of P. It is easy to see that $s(\mathscr{V}') < s(\mathscr{V})$, $d(V_k') \leqq d(V_k)$ for all $k \in K$. Clearly, $wV_i' > wV_i \geqq wV_j > wV_j'$, $w(V_i' + V_j') = w(V_i + V_j)$. Since $V_k' = V_k$ for $k \neq i$, $k \neq j$, we get $H(wV_k' : k \in K) < H(wV_k : k \in K)$.

The assertion $(*)$ implies that there exists a partition $\mathscr{U}^* = (U_k^* : k \in K)$ such that $s(\mathscr{U}^*) = 0$, $d(U_k^*) \leqq d(U_k)$ for all $k \in K$ and $H(wU_k^* : k \in K) \leqq H(wU_k : k \in K)$. Since, clearly, for any partition $\mathscr{T}$, $s(\mathscr{T}) = 0$ iff $\mathscr{T}$ is pure, the lemma is proved.

10.5. Lemma. *Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space. Assume that $[\mu \times \mu]\{(x, y) \in Q \times Q : 0 \neq \varrho(x, y) \neq 1\} = 0$. Let $\mathscr{P} = (P_x : x \in D)$ be a dyadic expansion of P. Let Z consist of all minimal elements of the set $\{t \in D : d(P_t) = 0$ or $t \in D''\}$. Then $(P_z : z \in Z)$ is a partition of P and $\Gamma_E(\mathscr{P}) = H(wP_z : z \in Z) \leqq H(wP_y : y \in D'')$.*

Proof. We can assume $d(P) > 0$. By the definition of Γ_E, we have $\Gamma_E(\mathscr{P}) = \Sigma(H(wP_{x0}, wP_{x1})\, d(P_x) : x \in D')$. Clearly, for any $x \in D$, we have either $d(P_x) = 1$

or $d(P_x) = 0$, and therefore (see 2.1) $\Gamma_E(\mathscr{P}) = \Sigma(L(wP_{x0}) + L(wP_{x1}) - L(wP_x))$: $x \in D'$, $d(P_x) > 0) = \Sigma(L(wP_z): z \in Z) - L(wP)$. Clearly, (i) for any $z \in Z$, $wP_z = \Sigma(wP_y: y \in D(z) \cap D'')$, (ii) for any $u, v \in Z$, $D(u) \cap D(v) = \emptyset$, and (iii) $\bigcup(D(z) \cap D'': z \in Z) = D''$. Hence, $(P_z: z \in Z)$ is a partition of P, $wP = \Sigma(wP_z: z \in Z)$, $\Sigma(L(wP_z): z \in Z) - L(wP) = H(wP_z: z \in Z)$ and $\Gamma_E(\mathscr{P}) = H(wP_z: z \in Z)$. By 4.12.1 we have $H(wP_z: z \in Z) + \Sigma(H(wP_y: y \in D(z) \cap D''): z \in Z) = H(wP_y: y \in D'')$.

10.6. Proposition. *Let $P = \langle Q, \varrho, \mu\rangle$ be a W-space. Assume that $[\mu \times \mu]\{(x, y) \in Q \times Q: 0 \neq \varrho(x, y) \neq 1\} = 0$ and that there exists a partition $\mathscr{U} = (U_1, \ldots, U_n)$ of P such that $d(U_i) = 0$ for $i = 1, \ldots, n$. Then $C_E(P) = C_E^*(P) =$
$= \inf\{H(wV_k: k \in K) : (V_k: k \in K) \in \mathrm{Pt}(P),\ \max(d(V_k): k \in K) = 0\} =$
$= \inf\{H(wV_k: k \in K) : (V_k: k \in K) \in \mathrm{Pt}^*(P),\ \max(d(V_k): k \in K) = 0\}$.*

Proof. Let h and h^* denote, respectively, the infima mentioned in the proposition. By 10.4 we get $h = h^*$. – Let $\varepsilon > 0$ and choose a partition $\mathscr{V} = (V_k: k \in K)$ such that $H(wV_k: k \in K) < h + \varepsilon$ and $d(V_k) = 0$ for all $k \in K$. By 4.6.2 there exists a dyadic expansion $\mathscr{P} = (P_x: x \in D)$ of P such that $\mathscr{P}''$ is equal to $\mathscr{V}$ re-indexed, hence $d(P_x) = 0$ for all $x \in D''$. By 10.5 we have $\Gamma_E(\mathscr{P}) \leqq H(wP_x: x \in D'') = H(wV_k: k \in K)$. By 4.12.4 and Theorem III, this proves $C_E(P) < h + \varepsilon$, hence, $\varepsilon > 0$ being arbitrary, we get $C_E(P) \leqq h$. In a similar way one proves $C_E^*(P) \leqq h^*$.

We are going to show that $C_E(P) \geqq h$ (the proof of $C_E^*(P) \geqq h^*$ is similar). Let $\mathscr{U} = (U_1, \ldots, U_n) \in \mathrm{Pt}(P)$, $d(U_i) = 0$ for each i. Let $\mathscr{P} = (P_x: x \in D)$ be a dyadic expansion of P such that $\mathscr{P}''$ refines $\mathscr{U}$. Let Z consist of all minimal elements of $\{t \in D: d(P_t) = 0 \text{ or } t \in D''\}$. By 10.5, $\Gamma_E(\mathscr{P}) = H(wP_z: z \in Z)$. Since $\mathscr{P}''$ refines $\mathscr{U}$, we have $d(P_x) = 0$ for all $x \in D''$, hence $d(P_z) = 0$ for all $z \in Z$. Since, by 10.5, $(P_z: z \in Z)$ is a partition of P, we get $\Gamma_E(\mathscr{P}) \geqq h$ and therefore $C_E(P) \geqq h$. This proves the proposition.

10.7. Proposition. *Let $P = \langle Q, \varrho, \mu\rangle$ be an FW-space. Assume that for any $x, y \in Q$, $\varrho(x, y) = 0$ or $\varrho(x, y) = 1$. Then $C_E(P) = C_E^*(P)$.*

This follows immediately from 10.6.

10.8. Fact. *Let $0 < b < 1/2$. Let S_b be the space described in 10.2. Then $C_r(S_b) < C_E(S_b)$.*

Proof. See 10.2 and 10.7.

10.9. Fact. *For $0 \leqq u \leqq 1$, $P \in \mathfrak{W}$, put $\varphi_u(P) = uC_{r(1)}(P) + (1 - u)\, C_E(P)$. Then every φ_u is a finitely continuous SCI-persistent extended Shannon entropy on $\mathfrak{W}$ and all $\varphi_u \upharpoonright \mathfrak{W}_F$ are distinct.*

Proof. Let $0 < b < 1/2$. Let S_b be the space described in 10.2. Then, by 10.8, $C_{r(1)}(S_b) < C_E(S_b)$ and therefore all $\varphi_u \upharpoonright \mathfrak{W}_F$, $0 \leqq u \leqq 1$, are distinct. By 2.28 every φ_u is an extended (b.s.) Shannon semientropy on $\mathfrak{W}$. Clearly, every φ_u is strongly regular (see 2.9), hence an extended (b.s.) Shannon entropy. By 9.43, $C_{r(1)}$ and C_E are finitely continuous, hence so is every φ_u. By 7.57, $C_{r(1)}$ and C_E are SCI-persistent, hence so is every φ_u.

10.10. Proposition. *There are exactly 2^ω finitely continuous SCI-persistent extended Shannon entropies on $\mathfrak{W}_F$ and also exactly 2^ω extended Shannon semi-entropies on $\mathfrak{W}_F$.*

Proof. See 7.46 and 10.9.

10.11. Proposition. *If A is a class of infinite sets, put, for any $P \in \mathfrak{W}$, (1) $\varphi_A(P) = C_r(P)$ if $|P|$ non $\in A$, (2) $\varphi_A(P) = C_E(P)$ if $|P| \in A$. Then (I) φ_A is a finitely continuous extended (b.s.) Shannon entropy on $\mathfrak{W}$ and $\varphi_A \restriction \mathfrak{W}_F = C_r \restriction \mathfrak{W}_F$, (II) if U and V are distinct classes of infinite sets, then $\varphi_U \neq \varphi_V$, (III) if A is a class of infinite sets, T_1 and T_2 are infinite sets, $T_1 \in A$, T_2 non $\in A$ and $\psi: T_1 \to T_2$ is a bijection, then there are W-spaces P_1 and P_2 such that $|P_i| = T_i$, $\psi: P_1 \to P_2$ is a PC-bijection, and $\varphi_A(P_1) \neq \varphi_A(P_2)$.*

Proof. The proof of (I) is easy and can be omitted. To prove (II) and (III), we show that (IV) if Q_0 and Q_1 are infinite sets, $\psi: Q_0 \to Q_1$ is a bijection, U and V are classes of infinite sets and Q_0 non $\in U$, $Q_1 \in V$, then there are W-space P_i, $i = 0, 1$, such that $|P_i| = Q_i$, $\psi: P_0 \to P_1$ is a PC-bijection and $\varphi_U(P_0) = C_r(P_0) \neq C_E(P_1) = \varphi_V(P_1)$. – In 10.2, we have exhibited FW-spaces $S_b = \langle \{0, 1, 2\}, \varrho_b, \mu_b \rangle$. By 10.8, if $0 < b < 1/2$, then $C_r(S_b) \neq C_E(S_b)$. Choose a fixed b, $0 < b < 1/2$. Choose an injective mapping $f_0: |S_b| \to Q_0$ and put $f_1 = \psi \circ f_0$. For $x, y \in Q_i$, let $\varrho_i(x, y) = \varrho_b(f_i^{-1}x, f_i^{-1}y)$ if $x, y \in f_i(|S_b|)$, $\varrho_i(x, y) = 0$ otherwise. For $X \subset Q_i$ put $\mu_i X = \mu_b(f_i^{-1}X)$. Then $P_i = \langle Q_i, \varrho_i, \mu_i \rangle$, $i = 0, 1$, are W-spaces, $\psi: P_0 \to P_1$ is a PC-bijection, and the mappings $f_i: |S_b| \to Q_i$ satisfy SCI. Hence, by 7.56, $C_r(P_0) = C_r(S_b)$, $C_E(P_0) = C_E(S_b)$. Since $C_r(S_b) \neq C_E(S_b)$, we get $\varphi_u(P_0) = C_r(P_0) \neq C_E(P_1) = \varphi_v(P_1)$.

We are going to prove (II). We can assume that there is an infinite set $Q \in V \setminus U$. Put $Q_0 = Q_1 = Q$ and let $\psi: Q_0 \to Q_1$ be the identity mapping. Then the spaces P_0 and P_1 from (IV) coincide and $\varphi_U(P_0) \neq \varphi_V(P_1)$.

To prove (III), it is enough to put in (IV), $U = V = A$.

10.12. Proposition. *There are exactly as many extended (b.s.) Shannon entropies on $\mathfrak{W}$ as there are classes.*

This follows easily from 10.11 (for the definition of "there are exactly as many ... as ...", see 7.41.4).

10.13. Fact. *Let P be a W-space and let S be an FW-space. Let $\psi: S \to P$ satisfy* SCI. *Then for any gauge functional τ, $C_\tau^*(S) = C_\tau^*(P)$, $C_\tau(S) = C_\tau(P)$.*

Proof. Put $\mathscr{U}_0 = (\{t\} . S: t \in |S|)$, $\mathscr{V}_0 = (\{\psi t\} . P: t \in |S|)$. Clearly, $\mathscr{U}_0$ and $\mathscr{V}_0$ are pure partitions of S and P, respectively. By 7.54 and 7.55, $\mathscr{U} = (U_k: k \in K)$ is a (pure) partition of S refining $\mathscr{U}_0$ iff $\mathscr{V} = (\psi U_k: k \in K)$ is a pure partition of P refining $\mathscr{V}_0$, and it is easy to show, using (GF6), that $[\mathscr{V}]_\tau = [\mathscr{U}]_\tau$. By the definition (3.17) of C_τ^* and C_τ we get $C_\tau^*(S) = C_\tau^*(P)$, $C_\tau(S) = C_\tau(P)$.

10.14. Fact. *Let A be a non-void class of infinite sets, and let φ_A be the extended*

($b.s.$) *Shannon entropy described in* 10.11. *Then for any gauge functional* τ, $\varphi_A \neq$ $\neq C_\tau$, $\varphi_A \neq C^*_\tau$.

Proof. By 10.8 there exists an FW-space S such that $C_r(S) \neq C_E(S)$. Clearly, there exist a W-space P and a mapping $\psi: S \to P$ such that $|P| = Q \in A$ and ψ satisfies SCI. We have $\varphi_A(S) = C_r(S)$ and $\varphi_A(P) = C_E(P)$, hence, by 10.13, $\varphi_A(P) = C_E(S)$ and therefore $\varphi_A(P) \neq \varphi_A(S)$. On the other hand, for any gauge functional τ, $C^*_\tau(P) =$ $= C^*_\tau(S)$, $C_\tau(P) = C_\tau(S)$ by 10.13.

10.15. Proposition. *For any class A of infinite sets and for any $P_1, P_2 \in \mathfrak{W}$ such that $P_1 + P_2 \leqq P$ for some $P \in \mathfrak{W}$ put $\eta_A(P_1, P_2) = r(P_1, P_2)$ if $|P_1|$ non $\in A$, $\eta_A(P_1, P_2) = E(P_1, P_2)$ if $|P_1| \in A$. Then* (I) *η_A is a gauge functional satisfying* (NGF2) *and* (NGF3), (II) *if U and V are distinct classes of infinite sets, then $\eta_U \neq \eta_V$.*

Proof. The proof of (I) consists in a straightforward verification of (GF1)–(GF7), (NGF2) and (NGF3), and can be omitted. To prove (II), choose a set $Q \in (U \setminus V) \cup$ $\cup (V \setminus U)$. We can assume $Q \in V \setminus U$. Choose distinct points $a_i \in Q$, $i = 0, 1, 2$. Put $\varrho(a_i, a_j) = |i - j|$. If $x, y \in Q$, $\{x, y\} \setminus \{a_0, a_1, a_2\} \neq \emptyset$, put $\varrho(x, y) = 0$. For $X \subset Q$ put $\mu X = \text{card}(X \cap \{a_0, a_1, a_2\})$. Then $P = \langle Q, \varrho, \mu \rangle$ is a W-space. Put $B_0 = \{a_0\}$, $B_1 = \{a_1, a_2\}$. Clearly, $\eta_U(B_0 . P_1,\ B_1 . P) = r(B_0 . P,\ B_1, P) = 3/2$, $\eta_V(B_0 . P,\ B_1 . P) = E(B_0 . P,\ B_1 . P) = 2$.

Remark. It can be shown that η_A, $A \neq \emptyset$, do not satisfy (NGF1). However, we shall not do it, since in 10.18.1 we exhibit a simple example of a gauge functional which satisfies (NGF2) and (NGF3), but does not satisfy (NGF1), even if restricted to the FW-spaces.

10.16. Proposition. *There are exactly as many gauge functionals as there are classes.*

Proof. See 10.15.

10.17. In 1.27 we have stated that there are W-spaces $\langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, such that dom $\mu_1 =$ dom μ_2 and $\langle Q, \varrho, \mu_1 + \mu_2 \rangle$ is not a W-space. In fact, there are even metric W-spaces of this kind, as the following example shows. – Let $Q = [-1, 1] \subset$ $\subset \boldsymbol{R}$, $A = [-1, 0]$, $B = (0, 1]$. Let λ be the Lebesgue measure on Q. Put $\mu_1 = A . \lambda$, $\mu_2 = B . \lambda$. Let $f: A \times B \to [1, 2]$ be an arbitrary non-measurable (with respect to $[\lambda \times \lambda]$) mapping. For $x, y \in Q$ put (1) $\varrho(x, y) = 0$ if $x = y$, (2) $\varrho(x, y) = 1$ if $x \neq y$ and $(x, y) \in (A \times A) \cup (B \times B)$, (2) $\varrho(x, y) = f(x, y)$ if $(x, y) \in A \times B$, (4) $\varrho(x, y) = f(y, x)$ if $(x, y) \in B \times A$. Clearly, ϱ is a metric on Q. – Let an open set $G \subset \boldsymbol{R}$ be given. Put $Z = \{(x, y) \in Q \times Q: \varrho(x, y) \in G\}$. Then $Z = X \cup Y$, where $Y \subset (Q \times Q) \setminus (A \times A)$ and X is equal either to $A \times A$ or to $\{(x, x): x \in A\}$. Consequently, Z is $[\mu_1 \times \mu_1]$-measurable. This proves that $\langle Q, \varrho, \mu_1 \rangle$ is a metric W-space. In the same way, $\langle Q, \varrho, \mu_2 \rangle$ is shown to be a metric W-space. – On the other hand, $\varrho: Q \times Q \to \boldsymbol{R}$ is not $[\lambda \times \lambda]$-measurable, for otherwise $\varrho: A \times B \to \boldsymbol{R}$, i.e., $f: A \times B \to \boldsymbol{R}$, would be $[\lambda \times \lambda]$-measurable as well.

10.18. We present examples (10.18.2–10.18.4) showing that a gauge functional can fail to satisfy any of the conditions (NGF1)–(NGF3), see Section 3. The examples are due to J. Hejcman.

10.18.1. Fact. *If* τ_1 *and* τ_2 *are gauge functionals,* $a_1, a_2 \in \boldsymbol{R}_+$, $a_1 + a_2 = 1$, *then* $a_1\tau_1 + a_2\tau_2$ *is a gauge functional. If, in addition,* τ_1 *and* τ_2 *satisfy* (NGFj) *with* $j = 2$ *or* $j = 3$, *then so does* $a_1\tau_1 + a_2\tau_2$.

10.18.2. *The gauge functional* $\frac{1}{2}(r + d)$ *does not satisfy* (NGF1).

Proof. Put $m = \frac{1}{2}(r + d)$. Let $Q = \{a, b, c, e\}$. Let $0 < \delta < 1$. Let ϱ be any metric on Q such that $\varrho(a, b) = \varrho(a, c) = \delta$, $\varrho(a, e) = 1$, and let $\mu a = \ldots = \mu e = 1$. Then $P = \langle Q, \varrho, \mu\rangle$ is an FW-space. Let $\mathscr{U} = (U_1, U_2, U_3) = (\{a\} . P, \{b\} . P, \{c, e\} . P)$, $S = [\mathscr{U}]_m = \langle K, \sigma, v\rangle$. Then $\sigma(1, 2) = m(U_1, U_2) = \delta$, $\sigma(1, 3) = m(U_1, U_3) = 3/4 + \delta/4$, $v1 = v2 = 1$, $v3 = 2$. Put $v_1 = \{1\} v$, $v_2 = \{2, 3\} v$, $S_i = \langle K, \sigma, v_i\rangle$, $i = 1, 2$. Since $r(S_1, S_2) = 1/2 + \delta/2$, $d(S_1, S_2) = 3/4 + \delta/4$, we have $m(S_1, S_2) = 5/8 + 3\delta/8$. – Let $a_{ik} = v_ik/vk$ for $i = 1, 2$, $k \in K$ (i.e., $k = 1, 2, 3$). Then $a_{11} = a_{22} = a_{23} = 1$, $a_{12} = a_{13} = a_{21} = 0$. We have $r(U_1, U_2 + U_3) = 1/3 + 2\delta/3$, $d(U_1, U_2 + U_3) = 1$, hence $m(U_1, U_2 + U_3) = 2/3 + \delta/3 > 5/8 + 3\delta/8 = m(S_1, S_2)$, and therefore (NGF1) is not satisfied.

10.18.3. *If* $0 < t < 1$, *then the gauge functional* r_t *does not satisfy* (NGF2).

Proof. Let $P = \langle\{a, b\}, 1, \mu\rangle$, where $\mu a = \mu b = 1$. Put $P_1 = \{a\} . P$, $P_2 = P$, $S_1 = P_1$, $S_2 = \{b\} . P$. Then for any $t > 0$, $r_t(S_1, S_2) = 1$, $r_t(P_1, P_2) = 2^{-1/t}$, $wS_1 . wS_2 . r_t(S_1, S_2) = 1$, $wP_1 . wP_2 . r_t(P_1, P_2) = 2^{1-1/t}$. Hence, for $t < 1$, (NGF2) is not satisfied.

10.18.4. *The gauge functional* $r_{1/2}$ *fails to satisfy* (NGF2) *and* (NGF3).

Proof. Let $P = \langle\{a, b, c\}, \varrho, \mu\rangle \in \mathfrak{W}_F$, $\varrho(a, b) = 0$, $\varrho(a, c) = 1$, $\mu a = \mu b = \mu c = 1$. Put $\mu_1 = \{a\} . \mu$, $\mu_2 = \{b, c\} . \mu$, $\sigma = \varrho + 1$, $P_i = \langle\{a, b, c\}, \varrho, \mu_i\rangle$, $S_i = \langle\{a, b, c\}, \sigma, \mu_i\rangle$. Then $r_{1/2}(P_1, P_2) = 1/4$, $r_{1/t}(S_1, S_2) = (1 + 2^{1/2})^2/4 > 1/4 + 1$. Hence (NGF3) is not satisfied.

10.19. We are going to construct the example announced in 7.35.1. We shall need some facts (10.19.2–10.19.4), all of which are known and easy to prove.

10.19.1. Notation. In 10.19.2 through 10.20, λ denotes the Lebesgue measure on $\boldsymbol{R}$ or on some Lebesgue measurable non-void subset of $\boldsymbol{R}$.

10.19.2. Fact. *Let* $a, b \in \boldsymbol{R}$, $a < b$. *Let* X *be a* λ*-measurable subset of the interval* $[a, b]$. *Let* k, p *be positive numbers. If* $\lambda X > k(b - a)$, $z \in \boldsymbol{R}$, $|z| \leqq p(b - a)$, *then* $\lambda\{x \in X: x + z \in X\} > (2k - 1 - p)(b - a)$.

10.19.3. Fact. *Let* X *and* Y *be* λ*-measurable subsets of* $\boldsymbol{R}$ *and let* $\lambda X > 0$, $\lambda Y > 0$. *Then there exists a non-void open set* $G \subset \boldsymbol{R}$ *and a positive number* $\varepsilon > 0$ *such that* $\lambda\{x \in X: x + z \in Y\} > \varepsilon$ *for every* $z \in G$.

Although this is well known, we give an outline of proof. Choose positive p and q

such that $q < 1$, $p < 4q - 3$. It is a well-known fact that there are $u \in X$ and $v \in Y$ such that

$$\lambda(X \cap [u - \delta/2, u + \delta/2]) > q\delta, \quad \lambda(Y \cap [v - \delta/2, v + \delta/2]) > q\delta$$

for all sufficiently small $\delta > 0$. Put $T = X \cap \{y - v + u : y \in Y\}$. Choose a sufficiently small $\delta > 0$ and put $S = T \cap [u - \delta/2, u + \delta/2]$. Clearly, $\lambda S > (2q - 1)\,\delta$. Hence, by 10.19.2, if $|z| < p\delta$, then $\lambda\{x \in S : x + z \in S\} > (4q - 3 - p)\,\delta$ and therefore $\lambda\{x \in X : x + v - u + z \in Y\} > (4q - 3 - p)\,\delta$.

10.19.4. Fact. *There exist λ-measurable disjoint sets $A_0 \subset \boldsymbol{R}$, $A_1 \subset \boldsymbol{R}$ such that $\lambda(A_i \cap G) > 0$ for $i = 0, 1$ and each non-void open $G \subset \boldsymbol{R}$.*

An outline of proof follows. Let $\mathscr{F}$ consist of all nowhere dense closed sets $X \subset \boldsymbol{R}$ such that $\lambda X > 0$. Let $G_k \subset \boldsymbol{R}$, $k \in \boldsymbol{N}$, be open non-void sets such that $\{G_k : k \in \boldsymbol{N}\}$ is an open base. Assume that for some $n \in \boldsymbol{N}$ we have chosen sets $A_i(k) \in \mathscr{F}$, $k < n$, $i = 0, 1$, such that $A_i(k) \subset G_k$ for each $k < n$, $i = 0, 1$, and $A_i(h) \cap A_j(k) = \emptyset$ whenever $(i, h) \neq (j, k)$. Clearly, there are disjoint sets $A_0(n) \in \mathscr{F}$, $A_1(n) \in \mathscr{F}$ such that $A_0(n) \cup A_1(n) \subset G_n \setminus \bigcup(A_j(k) : k < n, j = 0, 1)$.

10.20. Example. Let Q be the interval $[0, 1]$ and let λ be the Lebesgue measure on Q. Let A_0, A_1 be sets with the properties described in 10.19.4. Let B consist of all $x \in \boldsymbol{R}_+ \cap A_0$ and all x such that $-x \in \boldsymbol{R}_+ \cap A_0$. Let Z consist of all $(x, y) \in Q \times Q$ such that $x - y \in B$. For $x, y \in Q$ put $\varrho(x, y) = 0$ if $x = y$ or $(x, y) \in Z$, and put $\varrho(x, y) = 1$ if $x \neq y$, (x, y) non $\in Z$. Then (1) $P = \langle Q, \varrho, \lambda \rangle$ is a W-space, (2) $d(P) > 0$, (3) if $X \subset Q$, $Y \subset Q$ are λ-measurable, $\lambda X > 0$, $\lambda Y > 0$, then $[\lambda \times \lambda] \,.\, \{(x, y) \in X \times Y : \varrho(x, y) = 0\} > 0$ (hence $\inf\{\varrho(x, y) : x \in X, y \in Y\} = 0$).

Clearly, ϱ is a semimetric. Since B is Lebesgue measurable, the set $\{(x, y) \in \boldsymbol{R}^2 : x - y \in B\}$ is also Lebesgue measurable (as a subset of $\boldsymbol{R}^2$) and therefore Z is $[\lambda \times \lambda]$-measurable. This proves the first assertion.

It is easy to see that $\lambda(B \cap [-1, 1]) < 2$, hence $[\lambda \times \lambda](Z) < 1$. This implies $[\lambda \times \lambda](Q \times Q \setminus Z) > 0$ and proves the second assertion.

We are going to prove (3). By 10.19.3 there exists a non-void open set $G \subset \boldsymbol{R}$ and an $\varepsilon > 0$ such that for any $z \in G$, $\lambda\{x \in X : x - y = z$ for some $y \in Y\} > \varepsilon$. Since the Lebesgue measure of $G \cap B$ is positive, this implies $[\lambda \times \lambda]\{(x, y) \in X \times Y : x - y \in G \cap B\} > 0$, hence $[\lambda \times \lambda]\{(x, y) \in X \times Y : (x, y) \in Z\} > 0$.

10.21. The following example shows that the inequalities in 8.6 can fail to hold for a gauge functional τ not satisfying $\tau \geqq r$. – Let $\tau = r_{1/2}$. Let P be the space described in 10.18.4. Clearly, $C^*_\tau(P) \leqq H(1, 2)\,\tau(\{a\} \,.\, P, \{b, c\} \,.\, P) + H(1, 1) \,.\, \varrho(b, c) = H(1/2)/4 + 2\varrho(b, c)$. On the other hand, $\hat{r}(P) = 2 + 2\varrho(b, c)$, $2\hat{r}(P)/wP = (4 + 4\varrho(b, c))/3$. Since $H(1, 2)/4 < 1$, we have $C^*_\tau(P) < 2\hat{r}(P)/wP$ provided $\varrho(b, c)$ is sufficiently small.

10.22. We are going to exhibit an example of a metric W-space P such that $\Sigma\Gamma_r$-diam $\mathscr{U} = \infty$ for any partition $\mathscr{U}$ of P. – For $m, n \in \boldsymbol{N}$, put $\varrho(m, n) = |4^m - 4^n|$, For $X \subset N$, put $\mu(X) = \Sigma(2^{-n}: n \in X)$. Then $P = \langle N, \varrho, \mu \rangle$ is a metric W-space. Let $\mathscr{U} = (U_1, \ldots, U_p)$ be a partition of P, $U_i = \langle N, \varrho, \mu_i \rangle$. Then, for some $k = 1, \ldots, p$, the set $A_k = \{x \otimes N: \mu_k\{x\} \geqq p^{-1}\mu\{x\}\}$ is infinite. Clearly, if $m \in A_k$, $n \in A_k$, $m < n$, then $\Gamma_r(\{m\} . U_k, \{n\} . U_k) \geqq p^{-1}H(2^{-m}, 2^{-n}) |4^m - 4^n| > p^{-1}H(2^{-n}, 2^{-n}) . 3 . 4^{n-1} > 3 . 2^{n-1}p^{-1}$. Since A_k is infinite, this implies Γ_r-diam $U_k = \infty$.

10.23. We present an example, due to J. Hejcman, of a W-space with the properties mentioned in 8.11.4 (Remark). – Put $Q = [0, 1]$; let λ denote the Lebesgue measure on Q; for any $X \subset Q$ let $\lambda_e(X)$ denote the outer Lebesgue measure of X. For any $u \in \boldsymbol{R}$ put $Z_u = \{y \in \boldsymbol{R}: y - u$ is rational$\}$. Let $L \subset Q$ be a set such that card $(L \cap Z_u) = 1$ for each $u \in \boldsymbol{R}$. For $x, y \in Q$ put $\varrho(x, y) = 1$ if $x \neq y$, $x - y$ is rational and x or y is in L, and $\varrho(x, y) = 0$ in all remaining cases. It is easy to see that $[\lambda \times \lambda] . \{(x, y) \in Q \times Q: \varrho(x, y) = 1\} = 0$. Hence $P = \langle Q, \varrho, \lambda \rangle$ is a W-space and $d(P) = 0$. Clearly, diam $L = 0$, diam $(Q \setminus L) = 0$, hence $\langle Q, \varrho \rangle$ is totally bounded.

We are going to show that $(*)$ if $T \in$ dom λ, $\lambda_e(T \cap L) > 0$, then diam $T = 1$. Clearly, there is a number $a < 1$ such that $\lambda_e\{x \in T \cap L: x < a\} > 0$. Put $A = \{x \in T \cap L: x < a\}$ and choose a set $B \in$ dom λ such that $T \cap [0, a) \supset B \supset A$, $\lambda B = \lambda_e(A)$. Let S consist of all rational numbers $s \in [0, 1 - a)$. For any $s \in S$, put $B_s = \{x + s: x \in B\}$. Clearly, $\bigcap(B_s: s \in S) \subset Q$ and $\lambda B_s = \lambda B$ for all $s \in S$. Since $\lambda B > 0$, this implies that for some $s \in S$, $t \in S$, $s < t$, we have $\lambda(B_s \cap B_t) > 0$, hence $\lambda(B \cap B_{t-s}) > 0$. Since $\lambda B = \lambda_e(A)$, we get $\lambda_e(A \cap B_{t-s}) > 0$. Choose a point $x \in A \cap B_{t-s}$ and put $y = x - (t - s)$. Then $x \in T \cap L$, $x \in B_{t-s}$, $x = y + (t - s)$, hence $y \in B \subset T$. Since $x - y$ is rational, we have $\varrho(x, y) = 1$. This proves that diam $T = 1$.

Clearly, $(*)$ implies that for any $(T_n: n \in N)$ such that T_n are λ-measurable and $\bigcup T_n = Q$, we have diam $T_n = 1$ for some n.

10.24. We are going to present (see 10.28 below) the example announced in 8.12. We shall need some simple facts.

10.25. Fact. *Let $P_i = \langle Q, \varrho, \mu_i \rangle$, $i = 1, 2$, be subspaces of a W-space P. Let $a \in \boldsymbol{R}_+$ and assume that $[\mu_1 \times \mu_2] \{(x, y) \in Q \times Q: \varrho(x, y) \neq a\} = 0$ and $wP_i > 0$, $d(P_i) \leqq a$, $i = 1, 2$. Then $\tau(P_1, P_2) = a$ for any gauge functional τ.*

Proof. See (GF6).

10.26. Lemma. *Let τ be a gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space and let $S = \langle T, \sigma, \nu \rangle$ be an FW-space. Let $f: Q \to T$ be a surjective mapping such that $f^{-1}(t) \in$ dom $\bar{\mu}$ and $\bar{\mu}(f^{-1}(t)) = \nu t$ for any $t \in T$, and $\sigma(fq_1, fq_2) = \varrho(q_1, q_2)$ for any $q_1, q_2 \in Q$. Let K_t, $t \in T$, be finite non-void sets, let b_{tk}, $t \in T$, $k \in K_t$, be non-negative numbers and let $\Sigma(b_{tk}: k \in K_t) = 1$ for each $t \in T$. Put $K = \{(t, k): t \in T, k \in K_t\}$, $\mathscr{U} = (b_{tk}\{t\} . S: (t, k) \in K)$. Let $\mathscr{V}$ be a partition of P refining the partition $(b_{tk}f^{-1}(t) . P: (t, k) \in K)$. Then $C_\tau(S) \leqq C^*_\tau[\mathscr{V}]_\tau \leqq C^*_\tau[\mathscr{U}]_\tau$.*

Proof. Let $\mathscr{V} = (V_m: m \in M)$. Let $(M_{tk}: (t, k) \in K)$ be a partition of M such that

$\Sigma(V_m: m \in M_{tk}) = b_{tk} f^{-1}(t) . P$ for all $t \in T$, $k \in K_t$. Let g be the mapping of M onto K such that $g(m) = (t, k)$ if $m \in M_{tk}$. By using 10.25 it is easy to see that $g: [\mathscr{V}]_\tau \to [\mathscr{U}]_\tau$ is conservative. Hence, by 3.21, $C_\tau^*[\mathscr{V}]_\tau \leqq C_\tau^*[\mathscr{U}]_\tau$. By 3.24 and 3.26 we have $C_\tau[\mathscr{V}]_\tau = C_\tau[\mathscr{U}]_\tau$, $C_\tau[\mathscr{V}]_\tau \leqq C_\tau^*[\mathscr{V}]_\tau$. Let h be the mapping of K onto T such that $h(t, k) = t$. By 10.25, h is conservative and therefore, by 3.24, $C_\tau[\mathscr{U}]_\tau = C_\tau(S)$. This implies $C_\tau^*[\mathscr{V}]_\tau \geqq C_\tau(S)$.

10.27. Proposition. *Let τ be a gauge functional. Let $P = \langle Q, \varrho, \mu \rangle$ be a W-space and let $S = \langle T, \sigma, \nu \rangle$ be an FW-space. Assume that there exists a surjective mapping $f: Q \to T$ such that $f^{-1}(t) \in \operatorname{dom} \bar{\mu}$ and $\bar{\mu}(f^{-1}(t)) = \nu t$ for any $t \in T$, and $\sigma(fq_1, fq_2) = \varrho(q_1, q_2)$ for any $q_1, q_2 \in Q$. Then $C_\tau^*(P) \leqq C_\tau^*(S)$, $C_\tau(P) = C_\tau(S)$.*

Proof. Put $\mathscr{U}_0 = (\{t\} . S: t \in T)$, $\mathscr{V}_0 = (f^{-1}(t) . P: t \in T)$. By 10.46, $C_\tau(S) \leqq C_\tau^*[\mathscr{V}]_\tau \leqq C_\tau^*[\mathscr{U}_0]_\tau$ for any $\mathscr{V} \in \operatorname{Pt}(P)$ finer than $\mathscr{V}_0$. Since, by 3.21, $C_\tau^*[\mathscr{U}_0]_\tau = C_\tau^*(S)$, we get (see 3.17 and 3.15) $C_\tau(S) \leqq C_\tau(P) \leqq C_\tau^*(S)$, $C_\tau(S) \leqq C_\tau^*(P) \leqq C_\tau^*(S)$. If $C_\tau(S) = \infty$, this proves the proposition. If $C_\tau(S) < \infty$, let $p > C_\tau(S)$. By 3.17 and 3.15 there exists a partition $\mathscr{U}_1$ of S finer than $\mathscr{U}_0$ and such that $C_\tau^*[\mathscr{U}_1]_\tau < p$. We can assume that $\mathscr{U}_1$ is of the form $\mathscr{U}_1 = (b_{tk}\{t\} . S: (t, k) \in K)$, where $b_{tk} \geqq 0$, $\Sigma(b_{tk}: (t, k) \in K) = 1$ for each $t \in T$. Put $\mathscr{V}_1 = (b_{tk} f^{-1}(t) . P) : (t, k) \in K)$. By 10.26 we have $C_\tau^*[\mathscr{V}]_\tau \leqq C_\tau^*[\mathscr{U}_1]_\tau$, hence $C_\tau^*[\mathscr{V}]_\tau < p$ for any $\mathscr{V} \in \operatorname{Pt}(P)$ finer than $\mathscr{V}_1$. This implies $C_\tau(P) \leqq p$ and proves the proposition, since $p > C_\tau(S)$ was arbitrary.

10.28. For any $k \in N$ put $m(k) = 5^k$, $n(k) = 2^{m(k)}$. Let X consist of all (i, j), $i \in N$, $j = 0, \ldots, n(i)$. Define a metric ϱ on X by $\varrho((i, j), (h, k)) = \max(2^{-i}, 2^{-h})$ for $(i, j) \neq (h, k)$. Let μ be a measure on X such that $\operatorname{dom} \mu = \exp X$, $\mu\{(i, j)\} = 2^{-i}/n(i)$. Clearly, $P = \langle X, \varrho, \mu \rangle$ is a countable metric W-space. It is easy to see that P is totally bounded.

We are going to show that $C_r(P) = \infty$. Let $p \in N$, $p \geqq 1$. Put $X_p = \{(p, j): j = 0, 1, \ldots, n(p)\}$, let μ_p be a measure on X_p such that $\mu_p\{(p, j)\} = \mu\{(p, j)\}$ for $j = 1, \ldots, n(p)$ and $\mu_p X = \mu X$, and put $S_p = \langle X_p, 2^{-p}, \mu_p \rangle$. Let f_p be the mapping of X onto X_p defined by $f_p(p, j) = (p, j)$ and $f_p(i, j) = (p, 0)$ if $i \neq p$. For $x, y \in Y$ put $\varrho_p(x, y) = 2^{-p}$ if $f_p(x) \neq f_p(y)$, $\varrho_p(x, y) = 0$ if $f_p(x) = f_p(y)$. Clearly, $\langle X, \varrho_p, \mu \rangle$ is a W-space and $\varrho_p \leqq \varrho$. It is easy to see that $f_p: \langle X, \varrho_p, \mu \rangle \to S_p$ has the properties described in 10.27. Hence, by 10.27, $C_r\langle X, \varrho_p, \mu \rangle = C_r(S_p)$ and therefore, due to $\varrho_p \leqq \varrho$, we have $C_r(P) \geqq C_r(S_p)$.

Clearly, $C_r(S_p) = 2^{-p} H(\mu_p) = 2^{-p}\, \Sigma(\mu_p\{(p, j)\} \log \mu_p\{(p, j)\} : 0 \leqq j \leqq n(p)) + 2^{-p} \mu X \log \mu X$. Since $\mu_p\{(p, j)\} = 2^{-p}/n(p)$ for $j > 0$ and $1 < \mu_p\{(p, 0)\} < \mu X$, we get $C_r(S_p) > 2^{-p} . 2^{-p} \log(2^p\, n(p)) > 4^{-p} . 5^p$ and therefore $C_r(P) > (5/4)^p$. Since $p = 1, 2, \ldots$ was arbitrary, we have shown that $C_r(P) = \infty$.

10.29. We present an example announced in 8.36, of a W-space P such that (I) P is a metric W-space, (II) P is strongly $\Sigma^*\Gamma_r$-partition-fine, (III) P is not Σwd-partition-fine, (IV) $C_r^*(P) < \infty$. — Let $P = \langle N, \varrho, \mu \rangle$, where $\varrho(m, n) = |m - n|$, $\operatorname{dom} \mu =$

$= \exp N$, $\mu\{n\} = 2^{-n}$. We are going to show that P has the properties (I)–(IV). This will be done in several steps.

10.29.1. Evidently, P is a metric W-space. Clearly, if $S = \langle N, \varrho, v \rangle \leqq P$ and $v(n) > 0$ for infinitely many n, then $d(S) = \infty$. Hence, P is not Σ wd-partition-fine.

10.29.2. Fact. *For* $n \in N$ *let* $0 \leqq x_n \leqq 2^{-n}$. *Let* $u \in N$, $2^{-u} \leqq s$, *where* $s = \Sigma(x_n : n \in N)$. *Then* $\Sigma(nx_n : n \in N) \leqq (u + 2)\, s$.

Proof. It is easy to see that $\Sigma(nx_n : n \in N) = \Sigma(nx_n : n \leqq u) + \Sigma(nx_n : n > u) \leqq u\, \Sigma(x_n : n \leqq u) + u\, \Sigma(x_n : n > u) + \Sigma((n - u)\, x_n : n > u) \leqq us + 2^{-u}\, \Sigma(k\, 2^{-k} : k \in N) = us + 2^{-u+1} \leqq (u + 2)\, s$.

10.29.3. Fact. *Let* $U \leqq P$, $V \leqq P$. *Let* $u \in N$, $v \in N$, $u \leqq v$, *and let* $2^{-u} \leqq wU \leqq 2^{-u+1}$, $2^{-v} \leqq wV \leqq 2^{-v+1}$. *Then*

(1) $\hat{r}(U, V) \leqq (u + v + 4)\, wU \,.\, wV$,

(2) $r(U, V) \leqq u + v + 4$,

(3) $\Gamma_r(U, V) < 2(v - u + 4)(v + u + 4)\, 2^{-v}$.

Proof. Let $U = \langle N, \varrho, \xi \rangle$, $V = \langle N, \varrho, \eta \rangle$, $x_n = \xi\{n\}$, $y_n = \eta\{n\}$. Then $\hat{r}(U, V) = \Sigma(|m - n|\, x_m y_n : m \in N,\ n \in N) \leqq \Sigma(m x_m y_n : m \in N,\ n \in N) + \Sigma(n x_m y_n : m \in N, n \in N) = wV \,.\, \Sigma(m x_m : m \in N) + wU \,.\, \Sigma(n y_n : n \in N)$. By 10.29.2 we get $\hat{r}(U, V) \leqq wV \,.\, (u + 2)\, \Sigma(x_m : m \in N) + wU \,.\, (v + 2)\, \Sigma(y_n : n \in N) = (u + v + 4) \,.\, wU \,.\, wV$. This proves (1) and (2). – Clearly, $H(wU, wV) \leqq 2H(2^{-u}, 2^{-v})$. It is easy to see that if $0 < x \leqq 1$, then $H(1, x) < -x \log x + 4x$. Hence $H(2^{-u}, 2^{-v}) = 2^{-u} H(1, 2^{u-v}) < 2^{-u} \,.\, ((v - u)\, 2^{u-v} + 4 \,.\, 2^{u-v}) = (v - u + 4)\, 2^{-v}$ and therefore $H(wU, wV) < 2(v - u + 4)\, 2^{-v}$. By (2), this proves the inequality (3).

10.29.4. Fact. *Let* $U \leqq P$, $V \leqq P$. *Let* $k \in N$ *and let* $wU < 2^{-k+1}$, $wV < 2^{-k+1}$. *Then* $\Gamma_r(U, V) < 16(k + 2)\, 2^{-k}$.

Proof. We can assume $0 < wV \leqq wU$. Choose $u, v \in N$ such that $2^{-u} \leqq wU < 2^{-u+1}$, $2^{-v} \leqq wV < 2^{-v+1}$. Clearly, $k \leqq u \leqq v$. Hence, by 10.29.3, $\Gamma_r(U, V) < f(u, v)$, where $f(u, v) = 2(u + v + 4)(v - u + 4)\, 2^{-v}$. Clearly, we always have $f(u, v) \geqq f(u, v + 1)$ and $f(u, u) \geqq f(u + 1, u + 1)$. Hence $\Gamma_r(U, V) < f(k, k) = 16(k + 2)\, 2^{-k}$.

10.29.5. Fact. *For any* $k \in N$ *put* $P_k = \{n : n \geqq k\} \,.\, P$. *Then* Γ_r-diam $P_k \leqq 16(k + 2)\, 2^{-k}$.

Proof. See 10.29.4.

10.29.6. Fact. *Let* $\emptyset \neq X \subset N$. *Then* Γ_r-diam $(X \,.\, P) \leqq 16(k + 2)\, 2^{-k}$, *where* $k = \min X$.

Proof. See 10.29.5.

10.29.7. Fact. *Let* $k \in N$. *Let* $(U_j : j \in J)$ *be a pure partition of* P_k. *Then* $\Sigma(\Gamma_r\text{-diam } U_j : j \in J) < \Sigma(16(n + 2)\, 2^{-n} : n \geqq k)$.

Proof. Let $U_j = X_j \,.\, P$. We can assume that X_j are non-void. Put $p(j) = \min X_j$. By 10.29.6, Γ_r-diam $U_j \leqq 16(p(j) + 2)\, 2^{-p(j)}$. Since $(U_j: j \in J)$ is a pure partition, $X_i \cap X_j = \emptyset$ for any distinct $i, j \in J$. Hence all $p(j)$, $j \in J$, are distinct, and therefore $\Sigma(\Gamma_r\text{-diam } U_j: j \in J) < \Sigma(16(n + 2)\, 2^{-n}: n \geqq k)$.

10.29.8. Fact. *The space P is strongly $\Sigma^*\Gamma_r$-partition-fine.*

Proof. Let $\varepsilon > 0$. Since $\Sigma((n + 2)\, 2^{-n}: n \in N) < \infty$, there exists a $k \in N$ such that $\Sigma(16(n + 2)\, 2^{-n}: n \geqq k) < \varepsilon$. Then by 10.29.7, for any pure partition $(V_m: m \in M)$ refining the partition $(\{0\} \,.\, P, \ldots, \{k - 1\} \,.\, P, P_k)$ we have $\Sigma(\Gamma_r\text{-diam } V_m: m \in M) < < \varepsilon$.

10.29.9. Fact. $C_r^*(P) \leqq 8$.

Proof. We are going to compute $\Gamma_r(\{k - 1\} \,.\, P, P_k)$ for any $k = 1, 2, \ldots$. Clearly, $r(\{k - 1\} \,.\, P, P_k) = \Sigma(n2^{-k-n+1}: n = 1, 2, \ldots)/\Sigma(2^{-n}: n \geqq k)$. Hence $r(\{k - 1\} \,.\, P, P_k) = 2^{-k+1}\, \Sigma(n2^{-n}: n \in N)/2^{-k+1} = 2$. Since $w(\{k - 1\} \,.\, P) = = w(P_k) = 2^{-k+1}$, we get $\Gamma_r(\{k - 1\} \,.\, P, P_k) = 2^{-k+3}$. – Now choose an $m = = 1, 2, 3, \ldots$ and let D consist of all $(a_i: i < j) \in \{0, 1\}^j$ such that $j \leqq m$ and $a_i = 1$ whenever $i + 1 < j$ (thus, e.g., if $m = 3$, then $D = \{\emptyset, (0), (1), (10), (11), (110), (111)\}$). Put $S_\emptyset = P$. If $x = (a_i: i < j) \in D$, $j > 0$, put $S_x = \{j - 1\} \,.\, P$ if $a_{j-1} = 0$, $S_x = P_j$ if $a_{j-1} = 1$. Clearly, $\mathscr{S} = (S_x: x \in D)$ is a pure dyadic expansion of P. Since $(S_x: x \in D'')$ is equal to $(\{0\} \,.\, P, \ldots, \{m - 1\} \,.\, P, P_n)$ re-indexed, we have $\Sigma(\Gamma_r\text{-diam } S_x: x \in D'') = \Gamma_r\text{-diam } P_m \leqq 16(m + 2)\, 2^{-m}$ by 10.29.6. Clearly, $\Gamma_r(\mathscr{S}) = = \Sigma(\Gamma_r(\{k\} \,.\, P, P_{k+1}): k < m)$, hence $\Gamma_r(\mathscr{S}) = \Sigma(2^{-k+2}: k < m) \leqq 8$. By 8.23, this proves $C_r^*(P) \leqq 8$.

11

As in Part I, we sum up the main results in the form of a number of propositions, some of which are labelled as theorems (as rule, the more important ones). The theorems are numbered consecutively through Part I and II.

Proposition 11.1. *The folloving properties of a (finite) measure on a set Q are equivalent:* (1) $\langle Q, 1, \mu\rangle$ *is a W-space,* (2) *there is a metric ϱ on Q such that $\langle Q, \varrho, \mu\rangle$ is a W-space,* (3) *there is a countable set $A \subset Q$ such that $\mu \upharpoonright (Q \setminus A)$ is Dabroux and every $\{a\} \subset A$ is μ-measurable.*

This is 7.6, slightly re-formulated.

Proposition 11.2. *Let $P = \langle Q, \varrho, \mu\rangle$ be a metric W-space. Let V consist of all $x \in Q$ such that $\bar{\mu}X > 0$ whenever X is a $\bar{\mu}$-measurable neighborhood of x. Then V is closed $\bar{\mu}$-measurable and $P \upharpoonright V$ is second-countable. If, in addition, P is weakly Borel, then either $\bar{\mu}(Q \setminus V) = 0$ or the topological weight of P is real-measurable.*

Proof. See 7.23 and 7.28.

Proposition 11.3. *If τ is a normal gauge functional and P is a totally expansion-*

bounded W-space, then the functions $S \mapsto C_\tau(S)$ *and* $S \mapsto C_\tau^*(S)$ *defined for all* $S \leqq P$, *as well as* $S \mapsto C_\tau^*(S)$ *restricted to pure subspaces* $S \leqq P$, *are Lipschitz of order, respectively,* 1/3, 1/4 *and* 1/3, *in all cases with respect to the measure-distance* $\mathrm{md}_p(S_1, S_2)$.

Proof. See 9.37.

Remark. Though this proposition is fairly important, it is not labelled as a theorem since it is an open question whethe the orders 1/3 and 1/4 can be replaced by some (or, perhaps, any) p from the interval (1/3, 1) or, respectively, (1/4, 1).

Proposition 11.4. *There are at most* $\exp^{(4)}\omega$ *persistent (in the broad sense) extended Shannon semientropies on the class of all weakly Borel metric spaces with the topological weight not real-measurable. It is admissible to assume that this assertion remains true if the condition on the topological weight is omitted.*

See 7.57 and 7.62. For the meaning of the expression "it is admissible to assume ..." see 7.59.

Theorem V. *If* P *is a W-space of positive diameter, then* $C_\tau(P)$ *and* $C_\tau^*(P)$ *are positive for every gauge functional* τ *satisfying* $\tau \geqq r$.

Theorem VI. *If* τ *is a normal gauge functional and* P *is a semimetrized probability space, then neither* $C_\tau(P)$ *nor* $C_\tau^*(P)$ *exceed* Ded (P), *the expansion-diameter of* P.

Proof. See 8.28.

Theorem VII. *If* P *is an expansion-bounded W-space (in particular, if* $P = \langle Q, \varrho, \mu \rangle$ *and* $\langle Q, \varrho \rangle$ *is a bounded subspace of some* $\mathscr{L}_p(n)$*), then* $C_\tau(P)$ *and* $C_\tau^*(P)$ *are finite for every normal gauge functional* τ.

Proof. See 8.40 and 8.43.

Theorem VIII. *If* P *is either a totally bounded W-space or a second-countable bounded metric W-space, then* $C_\tau(P) \leqq C_\tau^*(P)$ *for every normal gauge functional* τ. *It is admissible to assume that* $C_\tau(P) \leqq C_\tau^*(P)$ *for every bounded weakly Borel metric W-space* P *and every gauge functional* τ.

Proof. See 8.38 and 8.45.

Theorem IX. *On* $\mathfrak{W}_F$, *the class of all FW-spaces, there are exactly* 2^ω *finitely continuous extended Shannon entropies and also* 2^ω *extended Shannon semi-entropies.*

Proof. See 10.10.

Theorem X. *For any normal gauge functional* τ, *the extended Shannon entropy* C_τ *is finitely continuous.*

Proof. See 9.24.

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