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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 4, 555–564

Persistent URL: http://dml.cz/dmlcz/102050

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

GROUPOIDS WITH NON-ASSOCIATIVE TRIPLES ON THE DIAGONAL

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(Received June 25, 1984)

Given a non-empty set S, every binary operation . on S divides S^3 into two disjoint sets: that of associative triples (i.e. $x . yz = xy . z$) and the complement. On the contrary, given $T \subseteq S^3$, we may ask if there is a binary operation on S whose set of non-associative triples coincides with T. It is known (see [1]) that for S finite such an operation exists whenever $\operatorname{card}(T) \leqq (\operatorname{card}(S) - 2)/4$. On the other hand, there is no such an operation if T is the diagonal of S^3 (i.e. $T = \{(x, x, x); x \in S\}$). The class $\mathscr{S}$ of groupoids whose non-associative triples belong to the diagonal seems therefore to be worth of study.

In this paper we describe the variety generated by $\mathscr{S}$ and give an estimate of $\operatorname{card}(T)$. Moreover, we show that an "almost free" groupoid $E(X, s) \in \mathscr{S}$ can be connected (in a natural way) with every non-empty partially ordered set (X, s).

1. INTRODUCTION

Let $\mathscr{S}$ denote the class of all groupoids G such that $a . bc \neq ab . c$ implies $a = b = c$ for any $a, b, c \in G$. For $G \in \mathscr{S}$ let $K = K(G) = \{a \in G;\ a . aa \neq aa . a\}$ and $L = L(G) = \{a \in G;\ a . aa = aa . a\}$, so that $G = K \cup L$ and $K \cap L = \emptyset$. Moreover, define a relation $r = r(G)$ by $(a, b) \in r$ iff $a, b \in G$ and either $a = b$ or $b = ab \in K$.

1.1. Proposition. (i) *The class $\mathscr{S}$ is closed under subgroupoids and nomomorphic images.*

(ii) *If $G \in \mathscr{S}$, then $G \times G \in \mathscr{S}$ iff G is associative.*

(iii) *If $G \in \mathscr{S}$ is not associative, then G is neither cancellative nor divisible.*

Proof. The assertions (i) and (ii) are easy and (iii) is clear from [2], [3] and [4].

1.2. Lemma. *Let $G \in \mathscr{S}$. Then*

(i) *For all $a, b \in G$, either $ab \in L$ or $a = ab$ or $b = ab$.*

(ii) *For every $a \in G$, $a^2 \in L$.*

(iii) *For all $a \in K$ and $b \in G$, $a = ab$ iff $a = ba$.*

(iv) *If* $a, b, c, d \in G$, $a = ab \in K$ *and* $b = cd$, *then* $a = ac = ca = ad = da$ *and* $c \neq a \neq d$.

Proof. (i) Let $ab = c$, $a \neq c \neq b$. Then $cc \,.\, c = (c \,.\, ab)\, c = (ca \,.\, b)\, c = ca \,.\, bc = c(a \,.\, bc) = c(ab \,.\, c) = c \,.\, cc$ and $c \in L$.

(ii) is an easy consequence of (i).

(iii) Let $ab = a \neq ba$. Then $aa \,.\, a = (a \,.\, ab)\, a = (aa \,.\, b)\, a = aa \,.\, ba = a(a \,.\, ba) = a(ab \,.\, a) = a \,.\, aa$, a contradiction.

(iv) By (iii), $ab = a = ba$, and hence $b \neq a^2$. Consequently, either $a \neq c$ or $a \neq d$. If $a = d$, then $a \neq c$, $c \,.\, a^2 = ca \,.\, a = cd \,.\, a = ba = a \notin L$, $c \,.\, a^2 \neq c$, $c \,.\, a^2 = a^2$ by (i), $a = a^2$, a contradiction. Thus $a \neq d$ and, similarly, $a \neq c$. However, $ac \,.\, d = a = c \,.\, da$, and therefore $ac = a = da$ by (i). The rest follows from (iii).

1.3. Corollary. *If* $G \in \mathcal{S}$, *then* L *is a subgroupoid of* G *and* card $(L) \geqq 2$ *provided* G *is nontrivial.*

1.4. Lemma. *Let* A *be a generator set of a groupoid* $G \in \mathcal{S}$. *Then*:

(i) $K \subseteq A$.

(ii) *If* $ab \in L$ *for all* $a, b \in A$ *then* $cd \in L$ *for all* $c, d \in G$.

Proof. (i) is an easy consequence of 1.2(i).

(ii) Suppose that $cd \in K$ for some $c, d \in G$. In virtue of 1.2(i) we can restrict ourselves to the case $cd = c$. Let W be the absolutely free groupoid over A (A is non-empty by (i)) and let f be the homomorphism of W onto G such that $f(a) = a$ for every $a \in A$. There is a term $t \in W$ with $f(t) = d$ and we can assume that d is chosen in such a way that the length $l(t)$ of t is minimal. Since $c \in A$, $d \notin A$, and hence $t \notin A$. We have $t = pq$ for some $p, q \in W$ and $d = uv$ where $u = f(p)$ and $v = f(q)$. Now, $c = cu$ by 1.2(iv) and $l(p) < l(t)$, a contradiction.

1.5. Lemma. *Let* $G \in \mathcal{S}$, $a \in G$ *and* $T(a) = \{b \in G;\ b = ab = ba\}$. *If the set* $T(a)$ *is non-empty, then it is a subgroupoid of* G.

Proof is easy.

1.6. Lemma. *Let* $G \in \mathcal{S}$. *Then the relation* r *is a partial ordering.*

Proof. By 1.2 (i), (iii), r is antisymmetric. If $(a, b), (b, c) \in r$, $a \neq b \neq c$, then $ac = a \,.\, bc = ab \,.\, c = bc = c$ and $(a, c) \in r$.

1.7. Lemma. *Let* $G \in \mathcal{S}$ *and* $Z(a) = \{b \in G;\ (b, a) \in r,\ b \neq a\}$. *Then*:

(i) *For any* $a \in G$, $Z(a)$ *is either empty or a subgroupoid of* G.

(ii) *If* A *is a generator set of* G *and* $Z(a)$ *is non-empty, then* $Z(a)$ *is generated by the set* $A \cap Z(a)$.

Proof is easy (use 1.2(iv)).

1.8. Lemma. *Let* $G \in \mathcal{S}$, *let* A, B *be non-empty subsets of* G *and let* C, D *be the subgroupoids generated by* A, B, *respectively. Suppose that* $(b, a) \in r$ *and* $b \neq a$

whenever $a \in A$, $b \in B$. *Then* $cd = dc = c$ *for any* $c \in C$, $d \in D$. *Moreover*, $\text{card}\,(C \cap D) \leqq 1$.

Proof. By 1.7(i), $D \subseteq Z(a)$ for every $a \in A$. Hence, by 1.5, $C \subseteq T(d)$ for every $d \in D$.

1.9. Lemma. *Let* $G \in \mathscr{S}$, $a \in K$ *and let* H *be the subgroupoid generated by* a. *Then* $H \cap Z(a) = \emptyset$.

Proof. If $b \in H \cap Z(a)$, then $a = ab = ba \in K$, which contradicts 1.4(ii).

1.10. Lemma. *Let* $G, H \in \mathscr{S}$ *and let* f *be a homomorphism of* G *into* H. *If* $a, b \in G$, $a \neq b$ *and* $f(a) \in K(H)$, *then* $f(a) \neq f(b)$.

Proof is obvious.

2. REGULAR GROUPOIDS

Let $G \in \mathscr{S}$. We shall say that G is regular if $ab \in L$ for all $a, b \in G$.

2.1. Proposition. *Let* $G \in \mathscr{S}$.
(i) *If* G *can be generated by the empty set (i.e.* G *contains no proper subgroupoid), then* G *is associative.*
(ii) *If* G *can be generated by a one-element set* $\{a\}$, *then* $K \subseteq \{a\}$ *and* G *is regular.*

Proof. Apply 1.2(ii) and 1.4.

Now, let $\mathscr{R}$ denote the variety of groupoids satisfying the following identities: $xy \,.\, uv = (xy \,.\, u)\, v$, $xy \,.\, uv = x(y \,.\, uv)$ and $(x \,.\, yu)\, v = x(yu \,.\, v)$.

2.2. Lemma. *Let* W *be the absolutely free groupoid over a non-empty set* X *and* $r, s \in W$, $l(r) \geqq 5$. *Then the identity* $r = s$ *is satisfied in* $\mathscr{R}$ *iff it is satisfied in every semigroup.*

Proof. Only the converse implication requires a proof. First, observe that for every $t \in W$ with $l(t) \geqq 4$ there exist a variable $x \in X$ and a term $q \in W$ such that $t = xq$ is satisfied in $\mathscr{R}$. On the other hand, $x(y(uv \,.\, z)) = (xy)\,(uv \,.\, z) = (xy \,.\, uv)\, z = ((xy \,.\, u)\, v)\, z = (xy \,.\, u)\,(vz) = (xy)\,(u \,.\, vz) = x(y(u \,.\, vz))$ holds in $\mathscr{R}$ and the rest is clear.

2.3. Lemma. *Let* F *be a free groupoid in* $\mathscr{R}$ *and let* $\alpha \geqq 1$ *be the rank of* F. *Then there exist two congruences* p *and* q *of* F *such that* F/p *is free in* $\mathscr{R}$ *and of* rank 1, F/q *is a free semigroup of rank* α *and* $p \cap q = \text{id}_F$.

Proof is easy.

2.4. Corollary. *Let* F *be a free* $\mathscr{R}$*-groupoid of rank* 1. *Then the variety* $\mathscr{R}$ *is generated by* F *and by the variety of semigroups.*

2.5. Proposition. (i) *A groupoid* $G \in \mathscr{S}$ *is regular iff* $G \in \mathscr{R}$.
(ii) *If* $G \in \mathscr{S}$ *can be generated by at most one element, then* $G \in \mathscr{R}$.
(iii) *If* $G \in \mathscr{R}$ *can be generated by at most one element, then* $G \in \mathscr{S}$.

Proof. (i) If G is regular, then clearly $G \in \mathscr{R}$. Now, suppose that $ab \in K$ for some $a, b \in G$. Then we can assume $ab = a$ and $ab \,.\, aa = a \,.\, aa \neq aa \,.\, a = (ab \,.\, a)\, a$, so that $G \notin \mathscr{R}$.
(ii) Follows from (i) and 2.1(ii).
(iii) It is easy to see that every free $\mathscr{R}$-groupoid of rank 1 is contained in $\mathscr{S}$.

2.6. Corollary. *$\mathscr{R}$ is just the variety generated by all regular groupoids from $\mathscr{S}$.*

2.7. Lemma. *Let $G \in \mathscr{S}$ be a non-associative groupoid which is generated by an element a. Then $K = \{a\}$, the elements $a, aa, a \,.\, aa, aa \,.\, a$ are pairwise different and* card $(G) \geqq 4$.

Proof. By 2.1(ii), $K = \{a\}$. If $aa = a \,.\, aa$, then $aa = a(a(a.aa)) = (a(a.aa))\, a = aa \,.\, a$, a contradiction. The rest is clear.

2.8. Example. Consider the following groupoid $G(*) = \{0, 1, 2, 3\}$: $a * b = 0$ if $a, b \in G$, $(a, b) \neq (1, 1), (2, 1)$; $1 * 1 = 2$ and $2 * 1 = 3$. Then $G(*) \in \mathscr{S}$ and $G(*)$ is not associative. Moreover, if $H \in \mathscr{S}$, H is not associative and card $(H) = 4$, then H is either isomorphic or antiisomorphic to $G(*)$.

2.9. Corollary. *Let $G \in \mathscr{S}$ be non-associative. Then* card $(L) \geqq 3$.

2.10. Corollary. *Let $G \in \mathscr{S}$ and $a \in G$. Then $aa, a \,.\, aa, aa \,.\, a, \ldots \in L$.*

2.11. Lemma. *Let W be the absolutely free groupoid over a one-element set $\{x\}$ and f a homomorphism of W into $G \in \mathscr{S}$.*
(i) If $y = f(x)$, then $y \,.\, (yy \,.\, y) = (y \,.\, yy) \,.\, y$ and $y \,.\, (y \,.\, yy) = yy \,.\, yy = (yy \,.\, y)y$.
(ii) If $p, q \in W$ and $l(p) = l(q) \geqq 5$, then $f(p) = f(q)$.

Proof. Use 2.5(ii) and 2.2 for the subgroupoid of G generated by y.

3. AUXILIARY RESULTS

Let X be a non-empty set with a partial ordering s, W the absolutely free groupoid over X and S the free semigroup over X. Denote by g the unique surjective homomorphism of W onto S such that $g(x) = x$ for every $x \in X$.

Let $t \in W$, $2 \leqq l(t) = n$ and $1 \leqq i \leqq n$. We shall define a term $d(t, i)$ by induction on n as follows: Let $t = pq$, $p, q \in W$. If $i = 1$ and $p \in X$, then $d(t, i) = q$. If $1 \leqq i \leqq l(p)$ and $2 \leqq l(p)$, then $d(t, i) = d(p, i)\, q$. If $l(p) + 1 \leqq i$ and $2 \leqq l(q)$, then $d(t, i) = p d(q, i - l(p))$. If $i = n$ and $q \in X$, then $d(t, i) = p$.

3.1. Lemma. *Let $t \in W$, $3 \leqq l(t)$ and $1 \leqq i < j \leqq l(t)$. Then $d(d(t, j), i) = d(d(t, i), j - 1)$.*

Proof is obvious.

Now, if $t \in W$ and M is a proper non-empty subset of $\{1, 2, \ldots, l(t)\}$, we can define a term $d(t, M)$ by 3.1. Further, we put $d(t, \emptyset) = t$.

Let $t \in W$, $l(t) = n$ and $g(t) = x_1 x_2 \dots x_n$. Define a relation s_t on the set $\{1, 2, \dots, n\}$ as follows: If $1 \leqq i \leqq n$ then $(i, i) \in s_t$. If $1 \leqq i < j \leqq n$, then $(i, j) \in s_t$ iff $(x_i, x_j) \in s$, $(x_{i+1}, x_j) \in s, \dots, (x_{j-1}, x_j) \in s$ and $x_i \neq x_j$, $x_{i+1} \neq x_j, \dots, x_{j-1} \neq x_j$. If $1 \leqq i < j \leqq n$, then $(j, i) \in s_t$ iff $(x_{i+1}, x_i) \in s$, $(x_{i+2}, x_i) \in s, \dots, (x_j, x_i) \in s$ and $x_{i+1} \neq x_i$, $x_{i+2} \neq x_i, \dots, x_j \neq x_i$. It is easy to see that s_t is a partial ordering of the set $\{1, 2, \dots, n\}$. We denote by $M(t)$ the set of all maximal elements of $\{1, 2, \dots, n\}$ (in the ordering s_t) and put $N(t) = \{1, 2, \dots, n\} - M(t)$. Further, define a relation s'_t on $\{1, 2, \dots, n\}$ by $(i, j) \in s'_t$ iff $(i, j) \in s_t$ and $|i - j| \leqq 1$.

A term $t \in W$ is said to be *s-irreducible* iff $s'_t = \text{id}$. This is clearly equivalent to the fact that $N(t) = \emptyset$, i.e. $s_t = \text{id}$.

Now, define a relation α on W by $(p, q) \in \alpha$ iff $p = d(q, i)$ for some $(i, j) \in s'_q$, $i \neq j$. Let β be the least equivalence containing α. Then, as one can easily see, β is a congruence of the absolutely free groupoid W.

3.2. Lemma. *Let $p, q \in W$ and $(p, q) \in \beta$. Then $t = d(p, N(p)) = d(q, N(q))$ is s-irreducible and $(q, t) \in \beta$.*

Proof. Without loss of generality, we may assume that $(p, q) \in \alpha$, i.e. $p = d(q, i)$ for some $(i, j) \in s'_q$. Suppose that $g(q) = x_1 \dots x_n$, $n = l(q)$. The rest of the proof is divided into two parts.

(i) Denote by f the bijection of the set $\{1, 2, \dots, i-1, i+1, \dots, n\}$ onto the set $\{1, 2, \dots, n-1\}$ defined by $f(k) = k$ for $1 \leqq k \leqq i-1$ and $f(k) = k - 1$ for $i + 1 \leqq k \leqq n$. We shall show that $N(p) = f(N(q) - \{i\})$. Indeed, let $(f(k), f(m)) \in s_p$ and put $I = \{r; k \leqq r \leqq m$ or $m \leqq r \leqq k\}$. If $i \notin I$ then $(k, m) \in s_q$. If $i \in I$ then $j \in I$ as well, and hence $(f(j), f(m)) \in s_p$, $(x_j, x_m) \in s$ and $(x_i, x_m) \in s$, $x_i \neq x_m$. Hence we get $(i, m) \in s_q$ and then $(k, m) \in s_q$. The proof of the other inclusion is immediate.

(ii) From (i) we conclude that $d(p, N(p)) = d(q, N(q)) = t$ and $r = \operatorname{card}(N(q)) = \operatorname{card}(N(p)) + 1$. There is a sequence $q = q_r, q_{r-1}, \dots, q_1, q_0$ of terms such that $(q_{r-1}, q_r) \in \alpha$, $(q_{r-2}, q_{r-1}) \in \alpha, \dots, (q_0, q_1) \in \alpha$ and $\operatorname{card}(N(q_k)) = k$ for any $0 \leqq k \leqq r$. This implies that $t = q_0 = d(q, N(q))$ is s-irreducible and $(q, t) \in \beta$.

3.3. Lemma. *Every block of β contains just one s-irreducible term.*

Proof. If $p, q \in W$ are s-irreducible terms such that $(p, q) \in \beta$, then $p = d(p, N(p)) = d(q, N(q)) = q$ by 3.2.

Taking into account 3.3, we can view the set $F(X, s)$ of all s-irreducible terms in a natural way as a groupoid isomorphic to the factorgroupoid W/β.

Finally, define an equivalence γ on $F(X, s)$ by $(xx . xx, x . (x . xx) \in \gamma$, $(xx . xx, (xx . x) x) \in \gamma$, $(x(xx . x), (x . xx) x) \in \gamma$ for every $x \in X$ and $(p, q) \in \gamma$ whenever $p, q \in F(X, s)$, $g(p) = g(q)$ and either $l(p) \geqq 5$ or p contains at least two different variables. Then γ is a congruence of the groupoid $F(X, s)$ and we denote by $E(X, s)$ the corresponding factorgroupoid.

Let X be a non-empty set, W the absolutely free groupoid over X, S the free semigroup over X and $g: W \to S$, $g(x) = x$ for every $x \in X$. Further, let f be a homomorphism of W into $G \in \mathcal{S}$.

4.1. Lemma. *Let $t \in W$, $g(t) = x_1 \dots x_n$. Then $f(t) \in K(G)$ iff there is $1 \leqq k \leqq n$ such that $f(x_i) \neq f(x_k)$, $(f(x_i), f(x_k)) \in r(G)$ for any $1 \leqq i \leqq n$, $k \neq i$. Moreover, if $f(t) \in K(G)$ then $f(t) = f(x_k)$.*

Proof. The case $n = 1$ is trivial. If $t = pq$, then by 1.2(i) $f(t) \in K(G)$ implies either $f(p) = f(t)$, or $f(q) = f(t)$. Using induction, we get the lemma from 1.2(iii) and (iv).

4.2. Lemma. *Let $t \in W$, $g(t) = x_1 \dots x_n$, $n \geqq 2$, $1 \leqq i, j \leqq n$, $j = i + 1$ (or $j = i - 1$) such that $f(x_i) f(x_j) = f(x_i)$ ($f(x_j) f(x_i) = f(x_i)$, respectively). Then $f(t) = f(d(t, j))$.*

Proof. Assume $j = i + 1$, the other case is similar. We shall proceed by induction on $n = l(t)$; there is nothing to prove for $n = 2$. If $t = pq$ and $l(p) \neq i$, the induction hypothesis can be used for p or q. Suppose $g(p) = x_1 \dots x_i$, $g(q) = x_{i+1} \dots x_n$. Then either $i > 1$ or $i + 1 < n$.

(i) $i > 1$ and $p = uv$. Put $a = f(d(q, 1))$. If $f(u) f(v) . f(q) = f(u) . f(v) f(q)$, then $f(vq) = f(d(vq, l(v) + 1)) = f(v)\, a$ (which holds by the induction hypothesis) implies that $f(t) = f(u) . f(v)\, a = f(u) f(v) . a = f(d(t, j))$ whenever $a \in L(G)$ or $a = f(q)$ or $f(v) \neq a$. However, if $f(q) \neq a = f(v) \in K(G)$, then $(f(x_i), a) \in r(G)$ by 4.1 and $f(x_i) f(x_j) = f(x_i)$ yields $(f(x_j), a) \in r(G)$, hence $a = f(q)$ by 4.1, a contradiction. On the other hand, if $f(u) f(v) . f(q) \neq f(u) . f(v) f(q)$, then $f(u) = f(v) = f(q) = d \in K(G)$ and $(f(x_i), d) \in r(G)$, $(f(x_j), d) \in r(G)$ by 4.1. Since $f(x_i) f(x_j) = f(x_i)$, we get $f(x_j) \neq d$, and therefore $f(q) = a$ by 4.1.

(ii) $n > i + 1$ and $q = uv$. The proof can be done in a similar way as in (i).

4.3. Lemma. *Let $p, q \in W$, $g(p) = x_1 \dots x_n = g(q)$. Then either $f(p) = f(q)$, or there is $1 \leqq k \leqq n$ such that $(f(x_i), f(x_k)) \in r(G)$ for any $1 \leqq i \leqq n$ and $\operatorname{card}\{1 \leqq i \leqq n;\ f(x_i) = f(x_k)\} \in \{3, 4\}$. Moreover, if $f(p) \neq f(q)$, then $f(p) = f(d(p, M))$ and $f(q) = f(d(q, M))$ for $M = \{1 \leqq i \leqq n;\ f(x_i) \neq f(x_k)\}$.*

Proof. There is nothing to prove for $n = 1, 2$; we shall use induction for $n > 2$.

(i) Suppose there are $1 \leqq i, j \leqq n$ satisfying the hypothesis of 4.2. Then the induction hypothesis can be used for $d(p, j)$, $d(q, j)$ and since $(f(x_i), f(x_k)) \in r(G)$ implies $(f(x_j), f(x_k)) \in r(G)$, $f(x_j) \neq f(x_k)$ for any $1 \leqq k \leqq n$, $j \neq k$, we get our lemma in this case from the induction hypothesis and 4.2.

(ii) We can now assume that $f(x_i) \neq f(x_i) f(x_j) \neq f(x_j)$ for any $1 \leqq i, j \leqq n$, $|i - j| = 1$. If $f(x_i) = f(x_j)$ for every $1 \leqq i, j \leqq n$, then 2.11(ii) may be used. Suppose there is $f(x_i) \neq f(x_1)$ for some $1 \leqq i \leqq n$ and let $t_2 = x_2(x_3 \dots (x_{n-1} x_n) \dots)$,

$t_1 = x_1 t_2$. If $p = p_1 p_2 \,.\, p_3$, then by 4.1 our assumption yields $f(p) = f(p_1 \,.\, p_2 p_3)$, and hence there is $u \in W$ with $f(p) = f(x_1 \,.\, u)$, $g(u) = x_2 \ldots x_n$. If there is $x_2 \neq x_j$ for some $2 < j \leqq n$, then $f(u) = f(t_2)$ by the induction hypothesis and hence $f(p) = f(t_1)$. Let $x_j = x_2$ for any $2 \leqq j \leqq n$ and denote $a = f(x_1)$, $b = f(x_2)$. Since $a \neq ab \neq b$, we have $ab \in L(G)$ by 1.2(i) and the subgroupoid of G generated by a, b is therefore regular by 1.4(ii). Using 2.5(i) and 2.2 we get $f(p) = f(t_1)$ for $n \geqq 5$. For $3 \leqq n \leqq 4$ we have either $u = t_2$, or $u = bb \,.\, b$. In the latter case we have $f(p) = a(bb \,.\, b) = (a \,.\, bb) \,.\, b = (ab \,.\, b)\, b = ab \,.\, bb = a(b \,.\, bb) = f(t_1)$. Therefore we get $f(p) = f(t_1)$ in all cases and since $f(q) = f(t_1)$ holds as well, we have $f(p) = f(q)$.

5. ALMOST FREE GROUPOIDS

5.1. Proposition. *Let X be a non-empty set partially ordered by s and let $E = E(X, s)$ (see Section 3). Then $E \in \mathscr{S}$, $K(E) = X$ and for $x, y \in X$, $xy = y$ iff $x \neq y$ and $(x, y) \in s$. Hence $s = r(E) \mid X$.*

Proof is easy.

Let (A, u) and (B, v) be two partially ordered sets. A homomorphism f of A into B is said to be an *immersion*, if f induces an isomorphism of (A, u) onto $(f(A), v \mid f(A))$.

5.2. Proposition. *Let $G, H \in \mathscr{S}$ and let f be a homomorphism of G into H, $A = f^{-1}(K(H))$, $u = r(G) \mid A$ and $v = r(H) \mid K(H)$. If A is not empty, then $A \subseteq K(G)$ and $f \mid A$ is an immersion of (A, u) into $(K(H), v)$.*

Proof. Obviously $f(L(G)) \subseteq L(H)$, therefore $A \subseteq K(G)$. Suppose that A is non-empty, then $f \mid A$ is injective by 1.10. If $a, b \in A$ and $ab = b$, then $f(a) f(b) = f(b)$, and hence $f \mid A$ is a homomorphism of (A, u) into $(K(H), v)$. If $a, b \in A$, $(a, b) \notin r(G)$, then $ab \in L(G)$, $f(a) f(b) \in L(H)$, $(f(a), f(b)) \notin r(H)$, and hence $f \mid A$ is an immersion.

5.3. Corollary. *Let $G, H \in \mathscr{S}$ and let f be a surjective homomorphism of G onto H, $u = r(G) \mid K(G)$ and $v = r(H) \mid K(H)$. Then there exists an immersion of the partially ordered set $(K(H), v)$ into $(K(G), u)$.*

5.4. Corollary. *Let $G \in \mathscr{S}$ be generated by $K(G)$, let $u = r(G) \mid K(G)$, let X be a non-empty set partially ordered by s and let $f\colon E(X, s) \to G$ be a homomorphism with $f(X) = K(G)$. Then f is surjective and $(K(G), u)$ is isomorphic to (X, s).*

5.5. Proposition. *Let X be a non-empty set partially ordered by s and let h be a mapping of X into $G \in \mathscr{S}$. Then h can be extended into a (unique) homomorphism $f\colon E(X, s) \to G$ if and only if the following conditions are satisfied:*

(a) *whenever $(x, y) \in s$, $x \neq y$, then $h(x)\, h(y) = h(y) = h(y)\, h(x)$,*

(b) *whenever $(x, y) \in s$ and $h(y) \in K(G)$, then $h(x)\, h(y) \neq h(y) \neq h(x)$.*

Proof. In virtue of 5.1, 1.10 and 1.2(i) only the converse implication requires a proof. Denote by W the absolutely free groupoid over the set X, by S the free semigroup over X and let $k\colon W \to E(X, s)$, $j\colon W \to G$, $g\colon W \to S$ be such that $k(x) = g(x) = x$ and $j(x) = h(x)$.

(i) If $p, q \in W$ and $(p, q) \in \alpha$ (see Section 3 for definition), then $p = d(q, i)$, $g(q) = x_1 \ldots x_n$, $1 \leqq i \leqq n$ and $(x_i, y) \in s$, where $y = x_{i+1}$, $i < n$ or $y = x_{i-1}$, $i > 1$. By (a) $h(x_i)\, h(y) = h(y) = h(y)\, h(x_i)$ and hence $j(p) = j(q)$ by 4.2. Therefore $j(p) = j(q)$ for any $p, q \in W$, $(p, q) \in \beta$.

(ii) Let $p, q \in F(X, s)$, $(p, q) \in \gamma$ and $j(p) \neq j(q)$. Then $g(p) = g(q) = x_1 \ldots x_n$ and we put $V = \{x_1, \ldots, x_n\}$. If $x = y$ for all $x, y \in V$, then $j(p) = j(q)$ by 2.11. Thus there is $x, y \in V$ such that $(x, y) \in s$ and by 4.3 we can assume that $j(y) = h(y) \in K(G)$ and $(j(x), j(y)) \in r(G)$. However, this contradicts (b). Therefore $j(p) = j(q)$ for any $p, q \in F(X, s)$, $(p, q) \in \gamma$.

(iii) Combining (i) and (ii) we conclude that $\operatorname{Ker}(k) \subseteq \operatorname{Ker}(j)$ and hence we can put $f(k(p)) = j(p)$ for any $p \in W$.

5.6. Corollary. *Let $G \in \mathcal{S}$ be a groupoid generated by a non-empty set A and let $s = r(G) \mid A$. Then there exists a unique surjective homomorphism f of $E(A, s)$ onto G such that $f \mid A = \mathrm{id}_A$.*

5.7. Proposition. *Let X be a non-empty set partially ordered by s, $G \in \mathcal{S}$ and let $f\colon G \to E(X, s)$ be such that $f(K(G)) = X$. Then f is an isomorphism iff G is generated by $K(G)$.*

Proof. Suppose that G is generated by $K(G)$; the other implication is obvious. By 5.2 $f \mid K(G)$ bijects onto X and by 5.5 there is a homomorphism $h\colon E(X, s) \to G$ such that $h(f(a)) = a$ for any $a \in K(G)$. Then $f(h(x)) = x$ for any $x \in X$ and fh is the identity mapping of $E(X, s)$ by 5.6. Since $h(X) = K(G)$ generates G, h is surjective and therefore $f = h^{-1}$.

6. EQUATIONS

Let $X = \{y_1, y_2, \ldots\}$ be a countable infinite set of variables and let W be the absolutely free groupoid over X. Define an endomorphism e of W by $e(y_i) = y_1$ for every $i \geqq 1$.

Now, let $t \in W$ and $g(t) = x_1 x_2 \ldots x_n$, $n \geqq 1$, $x_1, \ldots, x_n \in X$. Then $\operatorname{var}(t) = \{x_1, \ldots, x_n\}$ and for any proper subset V of $\operatorname{var}(t)$ we put $v(V) = \{i;\ 1 \leqq i \leqq n,\ x_i \in V\}$. Moreover, put $e_V(t) = e(d(t, v(V)))$.

Define sets $\mathcal{E}$ and $\mathcal{F}$ of identities as follows: The identities $t = t$, $(xx \,.\, x)\, x = xx \,.\, xx$, $x(x \,.\, xx) = xx \,.\, xx$ and $(x \,.\, xx)\, x = x(xx \,.\, x)$, where $x = y_1$ and $t \in W$, belong to $\mathcal{E}$. If $p, q \in W$, $l(p) \geqq 5$ and $g(p) = g(q)$, then the identity $p = q$ belongs to $\mathcal{E}$. Finally, if $u, w \in W$, then $u = w$ belongs to $\mathcal{F}$ iff $g(u) = g(w)$ and $e_V(u) = e_V(w)$ belongs to $\mathcal{E}$ for any proper subset V of $\operatorname{var}(u)$.

6.1. Lemma. *Let $G \in \mathcal{S}$ and let $p, q \in W$ be such that $p = q$ belongs to $\mathcal{F}$. Then G satisfies the identity $p = q$.*

Proof. Let $f: W \to G$ be a homomorphism and assume $f(p) \neq f(q)$. We have $g(p) = g(q)$ and by 4.3 there is $V \subseteq \operatorname{var}(p) = \operatorname{var}(q)$ such that $f(p) = f(d(p, v(V)))$, $f(q) = f(d(q, v(V)))$ and $f(x) = f(y)$ for any $x, y \in \operatorname{var}(p) - V$. Since $e_V(p) = e_V(q)$, we get $f(p) = f(q)$ from 2.11, a contradiction.

6.2. Lemma. *Let $Y = \{a, b\}$ be a two-elelement set partially ordered by s, $(a, b) \in s$. Let h be a homomorphism of W into $E(Y, s)$, $h(X) \subseteq Y$, $h(y_1) = b$. Then for $t \in W$, $V = \{x \in \operatorname{var}(t);\ h(x) = a\}$ either $V = \operatorname{var}(t)$, or $V \neq \operatorname{var}(t)$ and $h(t) = h(e_V(t))$.*

Proof. If $x \in V$, $y \in \operatorname{var}(t) - V$, then $h(x)\, h(y) = ab = b = h(y)$. Therefoere we can (repeatedly) use 4.2.

6.3. Lemma. *Let $p, q \in W$ be such that every groupoid from $\mathcal{S}$ satisfies $p = q$. Then the identity $p = q$ belongs to $\mathcal{F}$.*

Proof. Suppose, on the contrary, that $p = q$ is not contained in $\mathcal{F}$. We may assume $y_1 \notin \operatorname{var}(p)$. However, every semigroup satisfies $p = q$, hence $g(p) = g(q)$ and therefore the identity $e_V(p) = e_V(q)$ does not belong to $\mathcal{E}$ for some proper subset V of $\operatorname{var}(p) = \operatorname{var}(q)$. Now, let $Y = \{a, b\}$ be a two-element set partially ordered by s, $(a, b) \in s$, and let h be the homomorphism from W onto $E(Y, s)$ such that $h(x) = a$ for $x \in V$ and $h(x) = b$ for $x \in X - V$. Then $h(p) \neq h(q)$ by 6.2, a contradiction.

6.5. Corollary. *The variety $\mathcal{T}$ generated by $\mathcal{S}$ is just the variety of groupoids satisfying all the identities from $\mathcal{F}$.*

6.5. Corollary. *Let $Y = \{a, b\}$ be a two-element set partially ordered by s, $(a, b) \in s$. Then the variety $\mathcal{T}$ is generated by the groupoid $E(Y, s)$.*

It seems to be an open problem whether the variety $\mathcal{T}$ is finitely based.

7. FURTHER RESULTS

7.1. Lemma. *Let $G \in \mathcal{S}$ be such that $a^2 = b^2$ for all $a, b \in K = K(G)$. Then $ab \neq ca$ for all $a, b, c \in K$, $b \neq a \neq c$.*

Proof. Assume the contrary. Then $a \,.\, aa = a \,.\, bb = ab \,.\, b = ca \,.\, b = c \,.\, ab = c \,.\, ca = cc \,.\, a = aa \,.\, a$, a contradiction.

7.2. Lemma. *Let $G \in \mathcal{S}$ be such that $a^2 = b^2$ for all $a, b \in K$. Then $cd \in L = L(G)$ for all $c, d \in K$.*

Proof. If $cd \notin L$ then, by 1.2(i), we can assume that $c = cd \in K$. By 1.2(iii), $c = dc$, a contradiction with 7.1.

7.3. Lemma. *Let $G \in \mathcal{S}$ be a finite groupoid such that $a^2 = b^2$ for all $a, b \in K = K(G)$. Put $J(G) = \{ab;\ a, b \in K,\ a \neq b\}$ and suppose that $J(G)$ is non-empty.*

Then there exists a subgroupoid H of G such that $2\,\mathrm{card}\,(K(H)) \geqq \mathrm{card}\,(K)$ *and* $\mathrm{card}\,(J(G)) \geqq \mathrm{card}\,(J(H)) + 1$.

Proof. Let $x \in J(G)$, $A = \{a \in K;\ ab = x$ for some $a \neq b \in K\}$ and $B = \{b \in K;\ ab = x$ for some $b \neq a \in K\}$. By 7.1, $A \cap B = \emptyset$ and we can assume without loss of generality that $\mathrm{card}\,(B) \leqq \mathrm{card}\,(K)/2$. Now, denote by H the subgroupoid generated by $K - B$. Then $K(H) = K - B$ and $x \notin J(H)$.

7.4. Lemma. *Let $G \in \mathscr{S}$ be a finite groupoid such that $a^2 = b^2$ for all $a, b \in K = K(G)$. Suppose that* $\mathrm{card}\,(K) \geqq 2^m$ *for some $m \geqq 0$. Then* $\mathrm{card}\,(J(G)) \geqq m$.

Proof. The result follows from 7.2 and 7.3 by induction on m.

7.5. Lemma. *Let $G \in \mathscr{S}$ be a finite non-associative groupoid. Define an equivalence s on $K = K(G)$ by $(a, b) \in s$ iff $a^2 = b^2$ and denote by n the number of blocks of s. Let m be an integer such that* $\mathrm{card}\,(K) \geqq n\,.\,2^m$. *Then* $\mathrm{card}\,(L(G)) \geqq \max\,(n, m)$.

Proof. Let A be a block of s with the maximum $\mathrm{card}\,(A)$ and let H be the subgroupoid of G generated by A. Then $K(H) = A$, $\mathrm{card}\,(A) \geqq \mathrm{card}\,(K)/n$ and $\mathrm{card}\,(L(H)) \geqq m$ by 7.4. On the other hand, $\mathrm{card}\,(L(G)) \geqq n$ by 1.2(ii)

7.6. Corollary. *For every positive integer n there exists an integer m such that* $\mathrm{card}\,(L(G)) \geqq n$ *whenever $G \in \mathscr{S}$ and* $\mathrm{card}\,(G) \geqq m$.

7.7. Example. Let n be a positive integer and let $G = \{a_1, \ldots, a_n, b_1, \ldots, b_n, c, d\}$ be a set containing $2n + 2$ elements. Define a multiplication on G by $a_i\,.\,a_i = b_i$, $b_i a_i = c$ and $xy = d$ in all the remaining cases. Then $G \in \mathscr{S}$ and $\mathrm{card}\,(K(G)) = n$, $\mathrm{card}\,(L(G)) = n + 2$.

7.8. Remark. For a positive integer n, let $\alpha(n) = \min\,(\mathrm{card}\,(L(G));\ G \in \mathscr{S}, \mathrm{card}\,(K(G)) = n)$. By 7.7, $\alpha(n) \leqq n + 2$ and we have $\alpha(1) = 3$ (see 2.8). On the other hand, by 7.6, the values $\alpha(n)$ are not bounded.

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