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TWO PROOFS OF TRANSCENDENCY OF π AND e

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In 1873 Hermite [2] succeeded in proving the transcendency of e while in 1882 Lindemann [3] proved the transcendency of π. The present paper follows the papers [4] and [5]. The transcendency of π and e^a, where a is a nonzero real algebraic number, is proved here by two similar methods. In order to understand the proofs it is not necessary to have the knowledge of the theory of complex functions but it suffices to know Rolle's theorem, the Euler function and some properties of algebraic integers.

Lemma 1. *Let $V, p_0, p_1, \ldots, p_V$ be natural numbers and let $x_0, \ldots, x_V$ be real numbers ($x_i = x_j$ if and only if $i = j$). Put*

$$N = \sum_{i=0}^{V} (p_i + 1) - 1\,,$$

$$Q_j(z) = \prod_{\substack{k=0\\k\neq j}}^{V} (z - x_k)^{p_k+1} \quad \text{for} \quad j = 0, 1, \ldots, V.$$

Let us assume that f has N continuous derivatives on

$$\langle \min_{0\leq k\leq V} x_k, \max_{0\leq k\leq V} x_k \rangle\,.$$

Then there is

$$y \in \langle \min_{0\leq k\leq V} x_k, \max_{0\leq k\leq V} x_k \rangle$$

such that

$$(1) \qquad J(p_0, \ldots, p_V, f(x), V) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{m!\,(p_k - m)!} \left[\frac{1}{Q_k(z)}\right]^{(m)}_{z=x_k} = \frac{f^{(N)}(y)}{N!}\,.$$

Proof. Put

$$P(x) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k - m)!} \sum_{s=0}^{m} \frac{1}{(m - s)!} \left[\frac{1}{Q_k(z)}\right]^{(m-s)}_{z=x_k} Q_k(x)\,(x - x_k)^{p_k-s}\,.$$

For $j = 0, \ldots, V$, $q = 0, \ldots, p_j$ we have

$$P^{(q)}(x_j) = \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k - m)!} \sum_{s=0}^{m} \frac{1}{(m - s)!} \left[\frac{1}{Q_k(z)}\right]^{(m-s)}_{z=x_k} \left[Q_k(x)\,(x - x_k)^{p_k-s}\right]^{(q)}_{x=x_j} =$$

$$= \sum_{m=0}^{p_j} \frac{f^{(p_j-m)}(x_j)}{(p_j-m)!} \sum_{s=0}^{m} \frac{1}{(m-s)!} \left[\frac{1}{Q_j(z)}\right]_{z=x_j}^{(m-s)} \sum_{i=0}^{q} \binom{q}{i} [(x-x_j)^{p_j-s}]_{x=x_j}^{(i)} Q_j^{(q-i)}(x_j) =$$

$$= \sum_{m=p_j-q}^{p_j} \frac{f^{(p_j-m)}(x_j)}{(p_j-m)!} \sum_{s=p_j-q}^{m} \frac{1}{(m-s)!} \left[\frac{1}{Q_j(z)}\right]_{z=x_j}^{(m-s)} \binom{q}{p_j-s} (p_j-s)!\, Q_j^{(q-p_j+s)}(x_j) =$$

$$= \sum_{m=p_j-q}^{p_j} \frac{f^{(p_j-m)}(x_j)\, q!}{(p_j-m)!\,(m+q-p_j)!} \sum_{s=p_j-q}^{m} \binom{m-p_j+q}{q-p_j+s} \left[\frac{1}{Q_j(z)}\right]_{z=x_j}^{(m-s)} Q_j^{(q-p_j+s)}(x_j) =$$

$$= f^{(q)}(x_j)\,.$$

By Rolle's theorem there is

$$y \in \langle \min_{0 \leqq k \leqq V} x_k, \ \max_{0 \leqq k \leqq V} x_k \rangle$$

such that

$$f^{(N)}(y) - P^{(N)}(y) = 0\,.$$

Hence

$$f^{(N)}(y) = P^{(N)}(y) =$$

$$= \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k-m)!} \sum_{s=0}^{m} \frac{1}{(m-s)!} \left[\frac{1}{Q_k(z)}\right]_{z=x_k}^{(m-s)} [Q_k(x)(x-x_k)^{p_k-s}]_{x=y}^{(N)} =$$

$$= \sum_{k=0}^{V} \sum_{m=0}^{p_k} \frac{f^{(p_k-m)}(x_k)}{(p_k-m)!\, m!} \left[\frac{1}{Q_k(z)}\right]_{z=x_k}^{(m)} N!\,,$$

which implies (1).

Lemma 2. *Let V, $p_0, \ldots, p_V$, $Q_j(z)$ be the same as in Lemma 1. Put $p_0 = p_1 = \ldots$ $\ldots = p_V = n$, $x_k = k$, $k = 0, \ldots, V$. Then there is c depending only on V and such that*

$$c^n \geqq \left| \frac{(V!)^{2nV+V}}{m!} \left[\frac{1}{Q_k(z)}\right]_{z=k}^{(m)} \right| \in Z \quad \textit{for} \quad m = 0, \ldots, n\,, \tag{2}$$

where Z denotes the set of all integers.

Proof.

$$\frac{(V!)^{2nV+V}}{m!} \left[\frac{1}{Q_k(z)}\right]_{z=k}^{(m)} =$$

$$= \sum_{\substack{\sum_{i=0, i \neq k}^{V} m_i = m}} \prod_{s \neq k} \frac{(-1)^{m_s}(n+m_s)!}{n!\, m_s!} \frac{(V!)^{2n+1}}{(k-s)^{n+1+m_s}} =$$

$$= \sum_{\substack{\sum_{i=0, i \neq k}^{V} m_i = m}} \prod_{s \neq k} (-1)^{m_s} \binom{n+m_s}{n} \frac{(V!)^{2n+1}}{(k-s)^{n+1+m_s}}\,.$$

Hence (2) follows.

Lemma 3. *Let V, n, $V \neq 0$ be natural numbers and let $f(x)$ have $(V+1)n+V$ continuous derivatives. Put*

$$I_n(f(x), V) = \frac{(V!)^{2nV+V}}{(n!)^{V+1}} \int_0^1 \dots \int_0^1 f^{((V+1)n+V)}(Q(x_1, \dots, x_V)) \prod_{s=1}^{V} (x_s^{sn+s-1}(1-x_s)^n \, dx_s)$$

where

$$Q(x_1, \dots, x_V) = \sum_{i=1}^{V} \prod_{j=i}^{V} x_i .$$

Then the identity

$$(3) \qquad I_n(f(x), V) = \sum_{j=0}^{V} \sum_{k=0}^{n} \frac{1}{k!} A_{kj} f^{(k)}(j)$$

holds, where

$$(4) \qquad A_{kj} \text{ are integers and } |A_{kj}| \leqq c^n$$

and c depends only on V.

Proof. The proof proceeds by induction on V.

1. For $V = 1$ we have

$$I_n(f(x), 1) = \int_0^1 \frac{x_1^n(1-x_1)^n}{(n!)^2} f^{(2n+1)}(x_1) \, dx_1 =$$

$$= \sum_{k=0}^{n} \frac{(2n-k)!}{(n!)^2} \binom{n}{k} ((-1)^{n+k} f^{(k)}(1) + (-1)^{n+1} f^{(k)}(0)) =$$

$$= \sum_{j=0}^{1} \sum_{k=0}^{n} \frac{1}{k!} A_{kj} f^k(j) ,$$

where A_{kj} satisfy the condition (4).

2. Suppose (3), (4) are valid for $(V-1)$, we will prove that they hold for V. We have

$$I_n(f(x), V) = \frac{(V!)^{2n+1}}{n!} \int_0^1 (1-x_V)^n \, V^{2n(V-1)+V-1} I_n(f^{(n+1)}(x_V(x+1)), V-1) \, dx_V =$$

$$= \sum_{j=0}^{V-1} \sum_{k=0}^{n} \frac{A_{kj}(V!)^{2n+1}}{n! \, k!} \int_0^1 x_V^k (1-x_V)^n f^{(n+k+1)}(x_V(j+1)) \, dx_V ,$$

where A_{kj} satisfy the condition (4). Put

$$P(x) = x^k(1-x)^n .$$

Then the repeated integration by parts yields

$$(5) \qquad I_n(f(x), V) = \sum_{j=0}^{V-1} \sum_{k=0}^{n} \frac{A_{kj}}{n! \, k!} \left(\frac{V!}{j+1} \right)^{2n+1} B_{kj} ,$$

where

$$B_{kj} = \sum_{s=0}^{n+k} (-1)^s (j+1)^{n+k-s} (P^{(s)}(1) f^{(n+k-s)}(j+1) - P^{(s)}(0) f^{(n+k-s)}(0)) .$$

Now it is easy to see that

$$P^{(s)}(1) = s!\,(-1)^n \binom{k}{s-n} \quad \text{for} \quad n \leqq s \leqq n+k\,,$$

$$P^{(s)}(1) = 0 \quad \text{otherwise}\,;$$

$$P^{(s)}(0) = s!\,(-1)^{s-n+k} \binom{n}{s-k} \quad \text{for} \quad k \leqq s \leqq n+k\,,$$

$$P^{(s)}(0) = 0 \quad \text{otherwise}\,.$$

The substitution $s = n + k - s$ yields

$$\text{(6)} \qquad \frac{A_{kj}}{n!\,k!} \left(\frac{V!}{j+1}\right)^{2n+1} B_{kj} = \sum_{s=0}^{n} \frac{1}{s!} \left(A_{kjs} f^{(s)}(j+1) + B_{kjs} f^{(s)}(0)\right),$$

where A_{kjs}, B_{kjs} satisfy the condition (4). By (5), (6) we obtain (3) together with the condition (4).

Lemma 4. *Introduce N, V, n, $f(x)$, $I_n(f(x), V)$ as in Lemma 3. Let*

$$\max_{x \in \langle 0, V \rangle} |f^{(N)}(x)| \leqq c^n\,.$$

Then the inequality

$$|I_n(f(x), V)| \leqq \frac{c_1^n}{(n!)^{V+1}}$$

holds with c_1 depending only on V.

Proof. Lemma 3 implies

$$|I_n(f(x), V)| \leqq \frac{(V!)^{2nV+V}}{(n!)^{V+1}} \max_{x \in \langle 0, V \rangle} |f^{(N)}(x)| \leqq \frac{c_1^n}{(n!)^{V+1}}\,.$$

Lemma 5. *Let*

$$\text{(7)} \qquad B_n = \sum_{j=1}^{V} \sum_{t=0}^{n} \sum_{s=0}^{n} A_{tjs} A_1^j A_2^t A_3^s\,,$$

where A_1, A_2, A_3 are algebraic numbers depending only on V, B_n is also an algebraic number of degree V and

$$\text{(8)} \qquad |B_n| < \frac{c_1^n}{(n!)^V}\,,$$

A_{jts} are integers and

$$|A_{jts}| \leqq c_2^n\, n!\,,$$

where c_2 depends only on V. Then there is n_0 such that $B_n = 0$ for every $n > n_0$.

Proof. Put

$$\text{(9)} \qquad D_n = K^{2n+V} B_n = \sum_{j=1}^{V} \sum_{t=0}^{n} \sum_{s=0}^{n} A'_{jts} (KA_1)^j (KA_2)^t (KA_3)^s\,,$$

where K is a nonzero integer such that the numbers KA_1, KA_2, KA_3 are algebraic integers. Then the conjugates of D_n satisfy

$$D'_n = \sum_{j=1}^{V} \sum_{t=0}^{n} \sum_{s=0}^{n} A'_{jts} E_1^j E_2^t E_3^s ,$$

where E_i is the conjugate of KA_i for $i = 1, 2, 3$. Consequently,

$$|D_n| \leqq \sum_{j=1}^{V} \sum_{t=0}^{n} \sum_{s=0}^{n} A'_{jts} |E_1|^j |E_2|^t |E_3|^s \leqq c_2^n \, n! , \tag{10}$$

where c_2 depends only on V.

In virtue of (7), (8), (9) and (10), for the norm of D_n we obtain the inequalities

$$N(D_n) \leqq (c_2^n \, n!)^{V-1} \, |K|^{2n+V} \frac{c_1^n}{(n!)^V} \leqq \frac{c_3^n}{n!} .$$

Hence $N(D_n) = 0$ for every $n > n_0$, which implies $B_n = 0$.

Theorem 1. *Let a be a nonzero real algebraic number. Then e^a is transcendental.*

Proof 1. Suppose a and e^a are algebraic numbers. Denote by V the degree of the field $Q(a, \mathrm{e}^a)$. In Lemma 1 put $p_0 = p_1 = \ldots = p_V = n$, $x_k = k$ for $k = 0, \ldots, V$, $f(x) = \mathrm{e}^{ax}$ and in Lemma 5 put

$$B_n = J(n, \ldots, n, \mathrm{e}^{ax}, V) \, n! \, (V!)^{2nV+V} .$$

By Lemmas 2, 5 it is easy to see that $B_n = 0$ for every $n > n_0$. However, this is impossible because by Lemma 1

$$B_n = \frac{\mathrm{e}^{ay} a^N (V!)^{2nV+V}}{N!} \neq 0 \quad y \in \langle 0, V \rangle .$$

Proof 2. Suppose a and e^a are algebraic numbers. Denote by V the degree of the field $Q(a, \mathrm{e}^a)$. In Lemma 3 put $f(x) = \mathrm{e}^{ax}$ and in Lemma 5 put $B_n = n! \, I_n(\mathrm{e}^{ax}, V)$. Lemmas 3, 4, 5 imply that $B_n = 0$ for every $n > n_0$. However, this is impossible, because the function which is integrated in Lemma 3 is almost everywhere positive (negative).

Theorem 2. *The number π is transcendental.*

Proof. Suppose π is an algebraic number. Denote by M the degree of the field $Q(\pi)$. Let V be a number for which the inequality

$$\frac{4\varphi(2V)}{V} \leqq M^{-1}$$

holds. (φ is the Euler function and the inequality evidently holds if V has a large enough number of different prime divisors). Hence the degree of the field

$$Q\left(\pi, \cos\frac{\pi}{4V}, \sin\frac{\pi}{4V}\right)$$

is less than or equal to V.

Now the proof can be completed in two different ways.

The first way:

In Lemma 1 put $p_0 = p_1 = \ldots = p_V = n$, $x_k = k + 1$ for $k = 0, 1, \ldots, V$,

$$f(x) = \sin\frac{\pi}{4V}x\,,$$

and in Lemma 5 put

$$B_n = J(n, \ldots, n, f(x), V)\, n!\, (V!)^{2nV+V}\,.$$

By Lemmas 1, 2 it is easy to see that B_n satisfies the conditions of Lemma 5. Hence by Lemma 5, $B_n = 0$ for $n > n_0$. However, this is impossible because by Lemma 1

$$B_n = \frac{\left(\sin\left(\frac{\pi}{4V}x\right)\right)^{(N)}}{N!}(V!)^{2nV+V} \neq 0\,, \quad x \in \langle 1, V+1\rangle\,.$$

The second way:

In Lemma 3 put

$$f(x) = \sin\frac{\pi}{4V}x$$

and in Lemma 5 put

$$B_n = n!\, I_n(f(x), V)\,.$$

Now it is easy to see that B_n satisfies the conditions of Lemma 5. Hence by Lemma 5, $B_n = 0$ for $n > n_0$. However, this is impossible, because the function which is integrated in Lemma 3 is almost everywhere positive (negative).

Remark 1. We could prove Theorem 2 by Lemma 3 if we put

$$f(x) = e^{(\pi/V)ix}$$

(it is necessary to prove Lemma 3 for complex functions) and then continue as in the proof of Theorem 1.

Remark 2. Ju. V. Nesterenko presented similar proofs in [4]. He integrated the function over a simplex and obtained an identity similar to (3).

Remark 3. Identity (1) is a consequence of the Hermite identity which can be found e.g. in [1].

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