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# ON OSCILLATORY SOLUTIONS OF THE SYSTEM OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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A great number of papers is devoted to the investigation of the so-called proper solutions of non-linear ordinary differential equations or their systems (without or with deviating arguments) – see [2, 4, 5] and the bibliography therein. In these papers conditions are established under which each proper solution is either oscillatory or monotonically tends to infinity or to zero. The present paper generalizes and extends some results of [1 – 5] to the solutions of the following system of differential equations:

(1) $$y_i' = f_i(t, y_1(\sigma_{i,1}(t)), \ldots, y_n(\sigma_{i,n}(t)))\,, \quad i \in N_n\,, \quad n \geqq 2$$

where $N_n = \{1, 2, \ldots, n\}$, $f_i$: $D \to R$, $D = R_+ \times R^n$ satisfy the local Carathéodory conditions, there exist numbers $\nu_i \in \{0, 1\}$ such that

(2) $$(-1)^{\nu_i} f_i(t, x_1, \ldots, x_n) \operatorname{sgn} x_{i+1} \geqq 0 \quad \text{on} \quad D\,, \quad x_{n+1} = x_1\,, \quad i \in N_n$$

hold and $\sigma_{i,j}$: $R_+ \to R_+$ are continuous, $\lim_{t\to\infty} \sigma_{i,i+1}(t) = \infty$.

Here $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. Denote $\nu = \sum_{i=1}^{n} \nu_i$, let $C^0(D_1)$ be the set of all continuous functions on $D_1$, $L(R_+)$ the set of all functions that are summable on each finite segment of $R_+$.

By a *proper solution* $y$ of (1) we shall mean a vector function $y = \{y_i\}$, $i \in N_n$ such that its components are absolutely continuous on each segment of $R_+$, satisfy (1) for almost all $t \in R_+$ and

(3) $$\sup\{\sum_{i=1}^{n} |y_i(s)| : t \leqq s\} > 0 \quad \text{for} \quad t \in R_+\,.$$

A proper solution is called *oscillatory* if each of its components has a sequence of zeros tending to $\infty$.

The system (1) is said *to have Property* A if each of its proper solutions for $n$ even is oscillatory and for $n$ odd is either oscillatory or there exist numbers $\delta_i \in \{0, 1\}$ and $c \in R_+$ such that

(4) $$\lim_{t\to\infty} y_i(t) = 0\,, \quad (-1)^{\delta_i} y_i(t) \geqq 0\,, \quad t \in [c, \infty)\,, \quad i \in N_n\,.$$

The system (1) is said *to have Property* B if each of its proper solutions for $n$ even is either oscillatory or satisfies (4) or

$$\lim_{t\to\infty} |y_i(t)| = \infty\,, \quad i \in N_n\,, \tag{5}$$

and for $n$ odd is either oscillatory or satisfies (5).

**Lemma 1.** *Let $y$ be an arbitrary proper solution of* (1) *which is not oscillatory. Then there exist a number $\tau \in R_+$ and the sequence $\{\alpha_i\}$, $i \in N_n$, $\alpha_i \in \{0, 1\}$ such that*

$$(-1)^{\alpha_i} y_i(t) \geqq 0\,, \quad t \in [\tau, \infty)\,, \quad (-1)^{\alpha_{i+1}+\nu_i} y_i'(t) \geqq 0 \tag{6}$$

*for almost all $t \in [\tau, \infty)$.*

Proof. First we prove the relation

$$\begin{gathered} i \in N_n\,, \quad \alpha_i \in \{0, 1\}\,, \quad (-1)^{\alpha_i} y_i(t) \geqq 0\,, \quad t \in [\tau, \infty) \Rightarrow \exists \tau_1 \in [\tau, \infty)\,, \\ \exists \alpha_{i-1} \in \{0, 1\}\,, \quad (-1)^{\alpha_{i-1}} y_{i-1}(t) \geqq 0\,, \quad (-1)^{\alpha_i+\nu_{i-1}} y_{i-1}'(t) \geqq 0 \\ (\text{for almost all } t)\,, \quad t \in [\tau_1, \infty)\,. \end{gathered} \tag{7}$$

Thus suppose that $(-1)^{\alpha_i} y_i(t) \geqq 0$ on $[\tau, \infty)$. As $\lim_{t\to\infty} \sigma_{i-1,i}(t) = \infty$, there exists a number $\tau_2, \tau_2 \geqq \tau$ such that $\sigma_{i-1,i}(t) \geqq \tau$, $t \in [\tau_2, \infty)$ and according to (2)

$$\begin{gathered} (-1)^{\alpha_i+\nu_{i-1}} y_{i-1}'(t) = \\ = (-1)^{\nu_{i-1}} f_{i-1}(t, y_1(\sigma_{i-1,n}(t)), \ldots, y_n(\sigma_{i-1,n}(t))) \operatorname{sgn} y_i(\sigma_{i-1,i}(t)) \geqq 0\,, \quad t \in [\tau_2, \infty) \end{gathered}$$

holds. Thus $y_{i-1}$ is monotone on $[\tau_2, \infty)$, there exists a number $\tau_1 \in [\tau_2, \infty)$ such that $y_{i-1}$ does not change the sign on $[\tau_1, \infty)$ and (7) is true.

As according to the assumptions of the lemma $y$ is not oscillatory there exist $j \in N_n$ and $\tau_4 \in [0, \infty)$ such that $y_j(t) \neq 0$ on $[\tau_4, \infty)$. From this and from (7) we can conclude that the statement of lemma is proved.

**Lemma 2.** *Let an arbitrary proper solution of* (1) *be either oscillatory or satisfy* (4) *or* (5). *If $\nu$ is odd (even), then the system* (1) *has Property* A (B).

Proof. Let $y$ be an arbitrary proper solution which is not oscillatory. Then according to Lemma 1 the relation (6) holds and with respect to the assumptions of the lemma we have only two possibilities. Either $y_i(t)\, y_i'(t) \leqq 0$, $i \in N_n$ or $y_i(t)\, y_i'(t) \geqq 0$, $i \in N_n$ holds. From this and from (6) either

$$Z_i = \alpha_i + \alpha_{i+1} + \nu_i \quad \text{is odd}\,, \quad i \in N_n \tag{8}$$

or

$$Z_i = \alpha_i + \alpha_{i+1} + \nu_i \quad \text{is even}\,, \quad i \in N_n\,. \tag{9}$$

As $Z = \sum_{i=1}^{n} Z_i = 2 \sum_{i=1}^{n} \alpha_i + \nu$, the relation

$$\nu \text{ is odd (even)} \Leftrightarrow Z \text{ is odd (even)} \tag{10}$$

holds. Further, if $n$ is even, then in both cases (8) and (9), $Z$ is even, too. Thus, according to (10) the conclusion of the lemma is proved for $n$ even. Let $n$ be odd. According to (10) the case (8) ((9)) is impossible if $v$ is even (odd). The lemma is proved.

**Theorem 1.** *Let there exist continuous functions* $g_i: R_+^2 \to R_+$, $i \in N_n$ *such that* $g_i(x_i, 0) \equiv 0$, $g_i(x_i, x_{i+1}) > 0$ for $x_{i+1} > 0$, $x_i \geqq 0$ *and* $g_i$ *are non-decreasing with respect to the second argument. Let there exist non-negative functions* $a_i \in L(R_+)$ *such that* $\int_0^\infty a_i(t)\,dt = \infty$ *and*

$$|f_i(t, x, \ldots, x_n)| \geqq a_i(t)\, g_i(|x_i|, |x_{i+1}|)\,, \quad i \in N_n\,, \quad x_{n+1} = x_1$$

*in D. If v is odd (even), then the system* (1) *has Property* A (*Property* B).

Proof. Let $y$ be an arbitrary proper solution of (1) which is not oscillatory. Then according to Lemma 1 the relation (6) holds. First we prove the following implication by way of contradiction:

$$(11) \qquad i \in N_n\,, \quad |y_i(t)| \leqq M < \infty\,, \quad t \in R_+ \Rightarrow \lim_{t\to\infty} y_{i+1}(t) = 0\,.$$

Thus suppose that $|y_i(t)| \leqq M$ and $\lim_{t\to\infty} |y_{i+1}(t)| = c$, $c > 0$. According to (6) this limit exists and

$$(12) \qquad |y_{i+1}(t)| \geqq c_1 = \min(c, |y_{i+1}(\tau)|) > 0\,, \quad t \in [\tau, \infty)$$

holds. As $\lim_{t\to\infty} \sigma_{i,i+1}(t) = \infty$, there exists a number $\tau_1$, $\tau_1 \geqq \tau$ satisfying the relation $\sigma_{i,i+1}(t) \geqq \tau$ on $[\tau_1, \infty)$ and thus with respect to (12), $|y_{i+1}(\sigma_{i,i+1}(t))| \geqq c_1$, $t \in [\tau_1, \infty)$ is valid. From this and from the assumptions of the theorem we have

$$M \geqq \big|\, |y_i(\infty)| - |y_i(\tau_1)| \,\big| = \int_{\tau_1}^\infty |y_i'(t)|\,dt = \int_{\tau_1}^\infty |f_i(t, y_1(\sigma_{i,1}(t)), \ldots, y_n(\sigma_{i,n}(t)))|\,dt \geqq$$

$$\geqq \int_{\tau_1}^\infty a_i(t)\, g_i(|y_i(\sigma_{i,i}(t))|, |y_{i+1}(\sigma_{i,i+1}(t))|)\,dt \geqq$$

$$\geqq \int_{\tau_1}^\infty a_i(t)\, g_i(|y_i(\sigma_{i,i}(t))|, c_1)\,dt \geqq \min_{0\leqq s\leqq M} g_i(s, c_1) \int_{\tau_1}^\infty a_i(t)\,dt = \infty\,.$$

The contradiction obtained proves the relation (11). It follows from (6) and (11) that there are only two possibilities:

I. $\lim_{t\to\infty} |y_i(t)| = \infty$, $i \in N_n$,

II. $\lim_{t\to\infty} y_i(t) = 0$, $(-1)^{\alpha_i} y_i(t) \geqq 0$, $t \in [\tau, \infty)$, $i \in N_n$.

Thus the assumptions of Lemma 2 are fulfilled and the theorem is proved.

**Theorem 2.** *Let y be a proper solution of* (1) *which is not oscillatory. Let* $\sigma_{i,i}(t) \leqq t$, $\lim_{t\to\infty} \sigma_{i,i}(t) = \infty$, $i \in N_n$ *and let there exist continuous functions* $a: R_+ \to R_+$,

$g_i\colon R_+ \to (0, \infty)$ *such that* $\int_0^\infty a_i(t)\,\mathrm{d}t < \infty$,

$$|f_i(t, x_1, \ldots, x_n)| \leqq a_i(t)\, g_i(|x_{i+1}|)\,(1 + |x_i|)\,, \quad i \in N_n \text{ on } D\,.$$

*Then* $\lim_{t\to\infty} |y_i(t)| = c_i$ *and either* $c_i = \infty$, $i \in N_n$ *or* $c_i < \infty$, $i \in N_n$.

Proof. According to Lemma 1 the inequalities (6) are valid and hence $\lim_{t\to\infty} |y_i(t)|$ exists. First we prove the validity of the implication

$$(13) \qquad i \in N_n,\ y_{i+1} \text{ is bounded on } R_+ \Rightarrow y_i \text{ is bounded on } R_+\,.$$

On the contrary, suppose that $|y_{i+1}(t)| \leqq M$, $t \in R_+$ and

$$(14) \qquad \lim_{t\to\infty} |y_i(t)| = \infty\,.$$

According to (6), $|y_i|$ is non-decreasing on some $[\tau^*, \infty)$, $\tau^* \geqq \tau$. From this and from the assumptions of the theorem we have

$$|y_i(t)| - |y_i(t_0)| \leqq \int_{t_0}^t |y_i'(t)|\,\mathrm{d}t \leqq \int_{t_0}^t |f_i(t, y_1(\sigma_{i,1}(t)), \ldots, y_n(\sigma_{i,n}(t)))|\,\mathrm{d}t \leqq$$

$$\leqq \int_{t_0}^t a_i(t)\, g_i(|y_{i+1}(\sigma_{i,i+1}(t))|)\,(1 + |y_i(\sigma_{i,i}(t))|)\,\mathrm{d}t \leqq M_1 \int_{t_0}^t a_i(t)\,|y_i(t)|\,\mathrm{d}t\,,$$

$M_1 = 2 \max_{0 \leqq t \leqq M} g_i(t) < \infty$. Here $t_0 \geqq \tau$ is a suitable number with the properties $\sigma_{i,i}(t) \geqq \tau$, $1 \leqq |y_i(t)|$, $t \in [t_0, \infty)$. Hence according to Gronwall's inequality

$$|y_i(t)| \leqq |y_i(t_0)| \exp\left\{M_1 \int_{t_0}^t a_i(t)\,\mathrm{d}t\right\}, \quad t \in [t_0, \infty)$$

which contradicts (14) and the assumption $\int_0^\infty a_i(t)\,\mathrm{d}t < \infty$. Thus (13) is valid and the statement of the theorem follows from this relation. The theorem is proved.

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