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T-ALGEBRAS OF THE MONAD L-FUZZ

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The main purpose of this article is the characterization of T-algebras (A, h) of the monad L-Fuzz, which is connected with the fuzzification of mathematical objects, especially automata, applying a brouwerian lattice L [1]. The power set monad is a submonad of L-Fuzz and, as is well known, its T-algebras are precisely the complete sup-semi-lattices [2]. In the case of T-algebras of L-Fuzz the set A will also have the complete lattice structure making it possible to construct a Galois correspondence (h, g) between TA and the dual A^d of A. The above mentioned characterization will be performed by the statement of four independent conditions on g to be the residuated map of the morphism h of a T-algebra for L-Fuzz.

In Section 1, basic facts on the Kleisli and Eilenberg-Moore constructions are summarized and the monad L-Fuzz is constituted. Notations from the category theory not defined here may be found in [3]. For lattice theoretical facts see [4]. Section 2 starts with a partial order on the underlying set A of the T-algebra (A, h), which is shown to be a complete lattice order. Having introduced the Galois correspondence (h, g) additional properties of g are established, a suitable selection of which will be characteristic, as pointed out in the main result 2.13, 2.14. Section 3 studies the independence of the characteristic conditions obtained in the preceding section, while the last section is supplementary and contains some applications.

1. MONADS, T-ALGEBRAS AND THE MONAD L-FUZZ

1.0. There are several equivalent notions of a monad over a category $\boldsymbol{K}$ [5]. A monad (T, η, μ) in the monoid form consists of an endofunctor $T: \boldsymbol{K} \to \boldsymbol{K}$, and two natural transformations $\eta: \mathrm{Id} \to T$ and $\mu: T^2 \to T$, such that – composition left before right –

$$\eta_{TA}\mu_A = 1_{TA} = T\eta_A\mu_A\,, \quad T\mu_A\mu_A = \mu_{TA}\mu_A$$

for every object A. The Kleisli category $\boldsymbol{K}_T$ of (T, η, μ) has the same object class as $\boldsymbol{K}$ and the morphism classes $\boldsymbol{K}_T(A, B) = \boldsymbol{K}(A, TB)$ with the morphism composition

$$\alpha \circ \beta = \alpha T\beta\mu_C$$

where $\alpha \in K_T(A, B)$, $\beta \in K_T(B, C)$. There exists a pair of adjoint functors $(\Delta, \#)$ between K and K_T, given by

$$A^\Delta = A\,, \quad f^\Delta = f\eta_B\,,$$
$$A^\# = TA\,, \quad \alpha^\# = T\alpha\mu_B$$

if $f \in K(A, B)$ and $\alpha \in K_T(A, B)$. $(\Delta, \#)$ generates the given monad for $TA = A^{\Delta\#}$, $Tf = f^{\Delta\#}$, η being the unit of the adjunction and μ the natural transformation associated with the counit ε [6], [3]. The Kleisli construction gives rise to the definition of a monad $(T, \eta, \circ)$ in a clone form: here T is an object map of K, $\eta = (\eta_A)_{A \in |K|}$ a family of object maps $\eta_A\colon A \to TA$ and $\circ$ a family $(\circ_{ABC})_{A,B,C \in |K|}$ of mapings

$$\circ_{ABC}\colon K(A, TB) \times K(B, TC) \to K(A, TC)$$

such that (object indices in the composition sign will be omitted)

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)\,,$$
$$\alpha \circ \eta_B = \alpha\,,$$
$$(f^\Delta) \circ \beta = f\beta$$

for all composable morphisms α, β, γ, f. By the functor properties of Δ, $\#$ we have the identities

$$(fg)^\Delta = f^\Delta \circ g^\Delta\,, \quad 1_A^\Delta = \eta_A\,,$$
$$(\alpha \circ \beta)^\# = \alpha^\# \beta^\#\,, \quad \eta_A^\# = 1_{TA}\,.$$

Replacing $\circ$ in the clone form by a family $\# = (\#_{AB})_{A,B \in |K|}$ of mappings

$$\#_{AB}\colon K(A, TB) \to K(TA, TB)$$

satisfying (object indices omitted)

$$\eta_A \alpha^\# = \alpha\,,$$
$$\eta_A^\# = 1_{TA}\,,$$
$$(\alpha\beta^\#)^\# = \alpha^\# \beta^\#$$

one gets the notion of a monad in the extension form $(T, \eta, \#)$, connected with the monoid form and the clone form by the relations

(1) $$\alpha^\# = T\alpha\mu_B = 1_{TA} \circ \alpha\,,$$

(2) $$\mu_A = 1_{T^2A} \circ 1_{TA} = 1_{TA}^\#$$

for $\alpha \in K(A, TB)$.

Eilenberg and Moore [7], [3] generated the monad (T, η, μ) by a pair of adjoint functors (F^T, U^T) between K and the category K^T of T-algebras. Such a T-algebra (A, h) consists of an object A in K and a K-morphism $h\colon TA \to A$ satisfying

$$\eta_A h = 1_A\,, \quad Th\ h = \mu_A h\,.$$

$f \in \boldsymbol{K}(A, A')$ is a $\boldsymbol{K}^T$-morphism from (A, h) to (A', h') iff $Tf\, h = h'f$. $\boldsymbol{K}^T$-morphisms are combined by the composition law of $\boldsymbol{K}$. Evidently (TA, μ_A) is a T-algebra and the adjunction is given by

$$F^T A = (TA, \mu_A)\,, \quad F^T f = Tf\,,$$
$$G^T(A, h) = A\,, \qquad G^T g = g$$

where $f \in \boldsymbol{K}(A, A')$, $g \in \boldsymbol{K}^T((A, h), (A', h'))$.

1.1. In Zadeh's classical paper on fuzzy sets [8] characteristic functions are ranging over the interval [0, 1] of real numbers. With respect to inf, sup this is a special case of a brouwerian lattice [9] (or complete JID-lattice [4]). Precisely, a lattice L is called *brouwerian* iff it is complete and the intersection distributive over the suprema

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i)\,.$$

We establish a fuzzification making use of a fixed brouwerian lattice L. Let A be a set and

$$TA = L^A\,.$$

$p \in TA$ may be interpreted as a fuzzy set on A, $(a)p$ is the grade of membership of a in p. By $\alpha\colon A \to TB$ to every $a \in A$ we attribute a fuzzy set $(a)\,\alpha$ on B and adopting notation similar to that of conditional probability we set

$$\alpha(b/a) := (b)\,((a)\,\alpha)\,,$$

$a \in A$, $b \in B$. In particular, $\eta_A\colon A \to TA$ is defined by

$$\eta_A(a'/a) := \begin{cases} 1\,, & a' = a\,, \\ 0\,, & a' \neq a\,, \end{cases}$$

where 1 denotes the greatest and 0 the smallest element of the lattice L.

$\alpha\colon A \to TB$ and $\beta\colon B \to TC$ are composed to $\alpha \circ \beta\colon A \to TC$ by

$$(\alpha \circ \beta)\,(c/a) := \bigvee_{b \in B} (\alpha(b/a) \wedge \beta(c/b))\,,$$

$a \in A$, $c \in C$, with $\bigvee$ the supremum and $\wedge$ the intersection in L. One verifies without difficulty (see also [1]) that $(T, \eta, \circ)$ is a monad in the clone form over the category ***Set***, which will be denoted L-Fuzz in what follows. Consequently, T is an endofunctor ***Set*** $\to$ ***Set*** and η must be the natural transformation Id $\to T$, which can be tested also immediately. An easy computation using 1.0 (1), (2) gives the essential parts of L-Fuzz in the other monad forms

$$a^{\#}(b/p) = \bigvee_{a \in A} ((a)\, p \wedge \alpha(b/a))\,,$$

$b \in B$, $p \in TA$, $\alpha\colon A \to TB$, or

(1) $$\mu_A(a/\Phi) = \bigvee_{p \in TA} ((p)\, \Phi \wedge (a)\, p)\,,$$

$\Phi \in T^2 A$, $a \in A$.

The functor T transforms $f: A \to B$ into $Tf: TA \to TB$ so that

$$(2) \qquad Tf(b/p) = \bigvee\{(a)\, p/a \in A,\ (a) f = b\}\,,$$

$b \in B,\ p \in TA$.

2. CHARACTERIZATION OF T-ALGEBRAS OF L-FUZZ

2.0. In the sequel (A, h) always means a T-algebra of L-Fuzz with the additional assumption $A \neq \emptyset$. If no confusion arises η_A and μ_A often will be written without subscripts. It will be of advantage to distinguish typographically the elements of L - small greek letters with the exception of the bounds 0, 1 – from those of A – small Roman characters.

2.1. Together with $(L: \bigwedge, \bigvee, 0, 1)$ also $TA = L^A$ is a brouwerian lattice with respect to inf, sup, c_0, c_1 defined by

$$(x)(\inf p_i) := \bigwedge_{i \in I} (x)\, p_i\,,$$

$$(x)(\sup p_i) := \bigvee_{i \in I} (x)\, p_i\,,$$

$$(x)\, c_0 := 0\,, \quad (x)\, c_1 := 1$$

for every $x \in A$ and every family $(p_i/i \in I)$ of fuzzy sets $p_i \in TA$.

2.2. Definition. If (A, h) is a T-Algebra and $a, b \in A$, set

$$a \leqq b \quad \text{iff} \quad (\sup \{(a)\, \eta_A, (b)\, \eta_B\})\, h = b\,,$$

where sup denotes the operation from 2.1.

2.3. Lemma. *The relational system $(A, \leqq)$ has the following properties:*
(1) *$(A, \leqq)$ is a partial order;*
(2) *h is order-preserving;*
(3) $\forall x \in A\ \forall M \subseteq TA$;
if $M \subseteq h^{-1}(x)$ then $(\sup M)\, h = x$;
(4) $\forall X \subseteq A$:
$(\sup \{(x)\, \eta/x \in X\})\, h$ is the supremum of X in the partial order $(A, \leqq)$;
(5) *h is σ-preserving (that is, h preserves suprema).*

Proof. (1): The antisymmetry of $\leqq$ obviously holds. Reflexivity is an immediate consequence of the identity $\eta_A h = 1_A$ for T-algebras. Transitivity will be established by choosing suitable maps $\Phi \in T^2 A$ and applying the second identity for T-algebras $\mu h = Th\, h$:
Supposing $a \leqq b,\ b \leqq c$ let for $p \in TA$

$$(p)\, \Phi := \begin{cases} 1 & \text{if} \quad p = (c)\, \eta \quad \text{or} \quad p = \sup \{(b)\, \eta, (a)\, \eta\}\,, \\ 0 & \text{else}\,, \end{cases}$$

$$(p)\,\Phi' := \begin{cases} 1 & \text{if} \quad p = (a)\,\eta \quad \text{or} \quad p = \sup\{(b)\,\eta, (c)\,\eta\}\,, \\ 0 & \text{else}\,. \end{cases}$$

By 1.1 (1),

$$(\Phi)\,\mu = \sup\{(a)\,\eta, (b)\,\eta,\ (c)\,\eta\} = (\Phi')\,\mu\,,$$

while 1.1 (2) yields

$$Th(z/\Phi) = \begin{cases} 1 & \text{if} \quad z = (c)\,\eta h = c \quad \text{or} \quad z = (\sup\{(a)\,\eta, (b)\,\eta\})\,h = b\,, \\ 0 & \text{else}\,, \end{cases}$$

thus $(\Phi)\ Th\ h = (\sup\{(c)\,\eta, (b)\,\eta\})\,h = c$.

A similar computation with Φ' shows that

$$(\Phi')\ Th\ h = (\sup\{(a)\,\eta, (c)\,\eta\})\,h\,,$$

therefore $c = (\sup\{(a)\,\eta, (c)\,\eta\})\,h$, which was to be shown.

(2): The lattice ordering in TA is given by components: if $p, q \in TA$,

$$p \leqq q \quad \text{iff} \quad \forall a \in A\colon\ (a)\,p \leqq (a)\,q\,.$$

(We use $\leqq$ for the ordering of L, TA and A, the particular meaning being clear from the context).

If $p, q \in TA$, $p \leqq q$, set for an arbitrary $r \in TA$

$$(r)\,\Phi := \begin{cases} 1 & \text{if} \quad r = p \quad \text{or} \quad r = q\,, \\ 0 & \text{else}\,. \end{cases}$$

Then we obtain for every $z \in A$

$$\mu(z/\Phi) = (1 \wedge (z)\,p) \vee (1 \wedge (z)\,q) = (z)\,q\,,$$
$$Th(z/\Phi) = \eta(z/(p)\,h) \vee \eta(z/(q)\,h)\,,$$

therefore $(\Phi)\,\mu = q$, $(\Phi)\ Th = \sup\{(p)\,h\eta, (q)\,h\eta\}$ and finally

$$(q)\,h = (\Phi)\,\mu h = (\Phi)\ Th\ h = (\sup\{(p)\,h\eta, (q)\,h\eta\})\,h\,,$$

that is $(p)\,h \leqq (q)\,h$, in accordance with 2.2.

(3): Since h preserves the order,

$$x \leqq (p)\,h \leqq (\sup M)\,h$$

and because of $\sup M \leqq \sup h^{-1}(x)$ we only have to prove the equality $(\sup h^{-1}(x))\,h = x$. Taking

$$(p)\,\Phi := \begin{cases} 1 & \text{if} \quad p \in h^{-1}(x)\,, \\ 0 & \text{else} \end{cases}$$

we have $(\Phi)\,\mu = \sup h^{-1}(x)$ and $(\Phi)\ Th = (x)\,\eta$, and from the both T-algebra identities we get

$$x = (x)\,\eta h = (\Phi)\ Th\ h = (\Phi)\,\mu h = (\sup h^{-1}(x))\,h\,.$$

(4): If $X = \emptyset$ then

$$\sup \{(x)\,\eta / x \in X\} = c_0 \in L^A,$$

c_0 being the constant 0-map. Since $\eta_A h = 1_A$, h is surjective and therefore $(c_0)\,h$ must be the smallest element of A. Taking $x \in X \neq \emptyset$ and

$$a := (\sup \{(x)\,\eta / x \in X\})\,h$$

we have $x \leqq a$ by (2). If b is any upper bound of X in A, then for every $x \in X$

$$b = (\sup \{(x)\,\eta, (b)\,\eta\})\,h$$

and taking into account (3), 2.2 we find

$$b = (\sup \{\sup \{(x)\,\eta / x \in X\}, (b)\,\eta\})\,h\,.$$

Setting

$$(p)\,\Phi = \begin{cases} 1 & \text{if} \quad p = \sup \{(x)\,\eta / x \in X\} \quad \text{or} \quad p = (b)\,\eta\,, \\ 0 & \text{else} \end{cases}$$

one gets

$$(\Phi)\,\mu = \sup \{\sup \{(x)\,\eta / x \in X\}, (b)\,\eta\}\,,$$
$$(\Phi)\,Th = \sup \{(a)\,\eta, (b)\,\eta\}$$

and therefore $a \leqq b$.

(5): If $M \subseteq TA$ and $X := \{(p)\,h / p \in M\}$, then (4) implies

$$(\sup \{(p)\,h\eta / p \in M\})\,h = \operatorname{Sup} \{(p)\,h / p \in M\}\,,$$

Sup denoting the supremum operation in $(A, \leqq)$. Defining $(p)\,\Phi = 1$ if $p \in M$, 0 else, 1.1 (1), (2), immediately yield $(\Phi)\,\mu = \sup M$ and $Th\,(z/\Phi) = 1$ if $M \cap h^{-1}(z) \neq \emptyset$, 0 else. Therefore

$$(\Phi)\,Th = \sup \{(p)\,h\eta / p \in M\}\,,$$
$$\operatorname{Sup} \{(p)\,h / p \in M\} = (\Phi)\,Th\,h = (\Phi)\,\mu h = (\sup M)\,h.$$

2.4. We list the conclusions of 2.3, some of them involving merely the basic facts of the lattice or set theory:

(1) A is sup-complete.

(2) With respect to Sup and Inf defined by

$\operatorname{Inf} X := \operatorname{Sup} \{a / a \in A,\ \forall x \in X: a \leqq x\}$,

A is a complete lattice.

(3) ker h (the kernel of h) is an equivalence relation in TA, which is sup-compatible and separates the set $\{(a)\,\eta / a \in A\}$.

(4) Every equivalence class of ker h contains its supremum.

These suprema of equivalence classes are the object of the forthcoming considerations.

2.5. Definition. If (A, h) is a T-algebra, $a \in A$, then $g(a) := \sup \{p / p \in TA,\ (p)\,h \leqq a\}$.

Remark. g maps a to $\sup h^{-1}(a)$, because h is surjective and 2.3 (5) has been proved.

2.6. Lemma. Supposing 2.5, we have

$$\forall p \in TA \quad \forall a \in A: (p)\, h \leqq a \quad \text{iff} \quad p \leqq (a)\, p \, .$$

The assertion is a direct consequence of 2.5 and, as is well known, it is equivalent to (h, g) being a Galois connection between TA and the dual A^d of the lattice A. g is called *residuated to h*. (This differs from the terminology in [10], where h would be called residuated. Further properties of the Galois connections are listed in 2.7 below, see e.g. the article just quoted.)

2.7. For every $X \subseteq A,\ a \in A,\ p \in TA,$

(1) $(\operatorname{Inf} X)\, g = \inf\{(x)\, g / x \in X\}$,

(2) $1_{TA} \leqq hg$,

(3) $gh \leqq 1_A$, even $gh = 1_A$,

(4) $(a)\, g = \sup\{p / p \in TA,\ (p)\, h = a\}$,

(5) $(p)\, h = \operatorname{Inf}\{a / a \in A,\ p \leqq (a)\, g\}$.

2.8. As we have seen the underlying set A of a T-algebra (A, h) is a complete lattice. Simultaneously, the map h must have a residuated map g. Therefore it seems reasonable to ask for those maps g from a complete lattice A to the brouwerian lattice TA, for which the map h, being now defined by equation 2.7 (5), produces a T-algebra (A, h) of the monad L-Fuzz. In other words, our aim is the characterization of a T-algebra by residuated maps. To this end some further properties of g must be investigated. We start with an auxiliary notion which will be useful in the sequel.

2.9. Definition. Let $\Phi \in T^2A$.
Φ is concentrated iff $\exists p \in TA\ \exists \alpha \in L\ \forall q \in TA$:

$$(q)\, \Phi = \begin{cases} \alpha & \text{if} \quad q = p\, , \\ 0 & \text{else}\, . \end{cases}$$

2.10. Lemma. *Suppose h is a σ-preserving map $TA \to A$ and $\eta_A h = 1_A$. Then (A, h) is a T-algebra iff for every concentrated Φ,*

$$(\Phi)\, \mu_A h = (\Phi)\, Th\, h\, .$$

Proof. The implication from the left to the right is obvious. For the opposite direction take into account that together with TA also T^2A must be a brouwerian lattice. Denoting here for simplicity the supremum operation in T^2A, TA and A with the same symbol sup, an arbitrary $\Phi \in T^2A$ can be represented in the form

$$\Phi = \sup\{\Phi_p / p \in TA\}\, ,$$

$$(q)\, \Phi_p := \begin{cases} (p)\, \Phi & \text{if} \quad q = p\, , \\ 0 & \text{else} \end{cases}$$

by a set of concentrated Φ_p's.

Tf is always σ-preserving without any other supposition of f than that of f being a map. For, if $f\colon B \to C$, $c \in C$, $p_i \in TB$ for $i \in I$, then

$$\begin{aligned}
&(c)\,(\sup\{p_i/i \in I\}\ Tf) = Tf\,(c/\sup\{p_i/i \in I\}) \\
&= \bigvee\{(b)\,(\sup\{p_i/i \in I\})/b \in B,\ (b)\,f = c\} \\
&= \bigvee\{\bigvee_{i\in I}(b)\ p_i/b \in B,\ (b)\,f = c\} \\
&= \bigvee_{i\in I}\bigvee\{(b)\ p_i/b \in B,\ (b)\,f = c\} \\
&= \bigvee_{i\in I} Tf\,(c/p_i) = (c)\,(\sup\{(p_i)\ Tf/i \in I\})\,.
\end{aligned}$$

Also μ_A is σ-preserving since (TA, μ_A) is a T-algebra and 2.3 (5) holds.

h is supposed to be σ-preserving, therefore for every $\Phi \in T^2A$,

$$\begin{aligned}
(\Phi)\ \mu h &= (\sup_p \Phi_p)\ \mu h = \sup_p((\Phi_p)\ \mu h) \\
&= \sup_p((\Phi_p)\ Th\ h) = (\sup_p \Phi_p)\ Th\ h \\
&= (\Phi)\ Th\ h\,.
\end{aligned}$$

2.11. Let $\to$ denote the implication operation in the brouwerian lattice L, that is

$$\alpha \to \beta := \bigvee\{\gamma/\gamma \in L,\ \alpha \wedge \gamma \leqq \beta\}\,,$$

$a, \beta \in L$. By components it can be carried over to TA:

$$(a)\,(p \to q) := (a)\ p \to (a)\ q\,,$$

$a \in A$; $p, q \in TA$. For every $\alpha \in L$ let c_α be the constant map $A \to L$ with value α. If $g\colon A \to TA$, the Kleisli composition $g \circ g$ makes sense.

2.12. Lemma. *Suppose (A, h) is a T-algebra and g is defined by 2.5. Then*
(1) $\eta_A \leqq g$,
(2) $g \circ g = g$,
(3) $\forall a \in A\ \ \forall \alpha \in L$:
$\alpha \leqq g(\mathrm{Inf}\,\{x/x \in A,\ c_\alpha \to (a)\ g \leqq (x)\ g\}/a)$.

Proof. (1): By 2.7 (2) and $\eta_A h = 1_A$ we have for every $a \in A$:

$$(a)\ \eta \leqq (a)\ \eta\ hg = (a)\ g\,.$$

(2): Both μ_A and Tg are σ-preserving (see the proof in 2.10), therefore order preserving and (1) implies for every $a \in A$

$$\begin{aligned}
(a)\ g &= (a)\,(\eta \circ g) = (a)\ \eta\ Tg\ \mu \leqq (a)\ g\ Tg\ \mu \\
&= (a)\,(g \circ g)\,.
\end{aligned}$$

On the other hand,

$$(a)\,(g \circ g)\ h = (a)\ g\ Tg\ \mu h = (a)\ g\ Tg\ Th\ h\,,$$

and with regard to 2.7 (3) the last expression equals

$$(a)\, g\; T1_A\, h = (a)\, gh = a\,,$$

showing $(a)\,(g \circ g)\, h \leqq (a)\, g$ by definition of g in 2.5.

(3): For $p \in TA$, $\alpha \in L$,

$$\alpha p := \inf \{c_\alpha, p\}$$

denotes the "α-cut" of p. The concentrated $\Phi \in T^2 A$ defined by $(q)\,\Phi = \alpha$ if $q = p$, 0 else, fulfils

$$(\Phi)\, Th = \alpha((p)\, h)\, \eta\,, \quad (\Phi)\, \mu = \alpha p\,.$$

Since $(\alpha p)\, h = (\alpha((p)\, h)\, \eta)\, h$, for every $a \in A$ we have the implications: if $\alpha p \leqq (a)\, g$, then $\alpha((p)\, h)\, \eta \leqq (a)\, g$, or equivalently, $p \leqq c_\alpha \to (a)\, g$ implies $\alpha \leqq g((p)\, h/a)$. The assertion is now verified by 2.7 (5).

2.13. Now we are able to find the characteristic conditions on g to be a residuated map of a T-algebra map h. Their necessity is formulated in the following

Theorem. *If (A, h) is a T-algebra of the monad L-Fuzz, then A is a complete lattice with respect to the partial order defined in* 2.2. *The map g introduced by* 2.5 *is injective, δ-preserving and satisfies* 2.12 (2), (3).

Proof. 2.4 (2), 2.7 (3), 2.7 (1), 2.12 (2), (3).

The selected properties are also sufficient to get a T-algebra (A, h) defining its map by means of 2.7 (5). The exact formulation is given in the next point:

2.14. Theorem. *Suppose $(A, \leqq)$ is a complete lattice, $(T, \eta, \circ)$ the monad L-Fuzz and $g\colon A \to TA$ satisfies*

(1) *g injective,*

(2) *g δ-preserving,* (2.7 (1)),

(3) $g \circ g = g$,

(4) $\forall a \in A\ \forall \alpha \in L$:

$$\alpha \leqq g(\mathrm{Inf}\,\{x/x \in A,\ c_\alpha \to (a)\, g \leqq (x)\, g\}/a).$$

If $h\colon TA \to A$ is defined by

$$(p)\, h = \mathrm{Inf}\,\{x/x \in A,\ p \leqq (x)\, g\}\,,$$

then (A, h) is a T-algebra and for every $a \in A$

$$(a)\, g = \sup \{p/p \in TA,\ (p)\, h \leqq a\}\,.$$

Proof. First of all, (h, g) is shown to be a Galois connection between TA and A^d, the dual of A. Evidently h is monotone and also g is monotone by (2). If $a \in A$ then, by (2),

$$\begin{aligned}(a)\, ghg &= (\mathrm{Inf}\,\{x/(a)\, g \leqq (x)\, g\})\, g \\ &= \inf \{(x)\, g/(a)\, g \leqq (x)\, g\} = (a)\, g\,,\end{aligned}$$

and the injectivity of g yields $gh = 1_A$. For every $p \in TA$,

$$(p)\, hg = (\operatorname{Inf}\{x/p \leqq (x)\, g\})\, g = \inf\{(x)\, g/p \leqq (x)\, g\} \geqq p\,,$$

therefore $hg \geqq 1_{TA}$, leading to the desired result on (h, g). But then, as is well known from the theory of Galois connection, also the last statement of the theorem concerning g is true.

The next preparatory step is to demonstrate

(i) $(a)\, \eta \leqq (a)\, g$,
(ii) $a \leqq b$ iff $g(a/b) = 1$
for all $a, b \in A$.

The brouwerian implication has the property $c_1 \to (a)\, g = (a)\, g$, and (4), (2) imply

$$1 \leqq g\,(\operatorname{Inf}\{x/c_1 \to (a)\, g \leqq (x)\, g\}/a)$$
$$a = \operatorname{Inf}\{x/(a)\, g \leqq (x)\, g\}\,.$$

Therefore $1 \leqq g(a/a)$ and in virtue of the implication $\eta(a'/a) = 0$ if $a' \neq a$, (i) has been shown.

If $a \leqq b$, then (2) and, further, (i) imply

$$(a)\, g = \inf\{(a)\, g, (b)\, g\}\,,$$
$$1 = g(a/a) = g(a/a) \wedge g(a/b)\,,$$

therefore $g(a/b) = 1$.

Supposing $g(a/b) = 1$, (3) gives for every $c \in A$

$$g(c/b) = \bigvee_{x \in A} (g(x/b) \wedge g(c/x))$$
$$\geqq g(a/b) \wedge g(c/a) = g(c/a)\,,$$

consequently $(a)\, g \leqq (b)\, g$, and as previously shown $gh = 1_A$, therefore $a \leqq b$.

Now we can verify the T-algebra identities. Because of (i), monotony of h and $gh = 1_A$, for every $a \in A$ we have

$$(a)\, \eta h \leqq (a)\, gh = a\,.$$

From the properties of the Galois connection (h, g) we obtain $(a)\, \eta hg \geqq (a)\, \eta$, which implies $g(a/(a)\, \eta h) \geqq \eta(a/a) = 1$. By (ii) we conclude that

$$a \leqq (a)\, \eta h\,,$$

completing the proof of $\eta_A h = 1_A$.

The more complicated second identity

$$(\Phi)\; Th\; h = (\Phi)\; \mu_A h\,,$$

$\Phi \in T^2A$, will be verified only for concentrated Φ.

This will do, since h is σ-preserving because of the Galois properties, and 2.10 completes the proof.

Defining Φ by

$$(q)\,\Phi = \alpha \quad \text{if} \quad q = p\,, \quad 0 \text{ else}\,,$$

$\alpha \in L$, $p \in TA$ being arbitrary but fixed, we assert the following:

(iii) $\forall a \in A$: if $(\Phi)\,Th\,h \leqq a$ then $(\Phi)\,\mu h \leqq a$.

Setting $z := (p)\,h$ we get $(\Phi)\,Th = \alpha(z)\,\eta$ and $(\Phi)\,\mu = \alpha p$ (see 2.12 for notation). Supposing now the premise of (iii):

$$(\alpha(z)\,\eta)\,h = \operatorname{Inf}\{x/x \in A,\ \alpha(z)\,\eta \leqq (x)\,g\} \leqq a\,,$$

one concludes by (2)

$$\inf\{(x)\,g/x \in A,\ \alpha(z)\,\eta \leqq (x)\,g\} \leqq (a)\,g\,,$$

in particular $\alpha \leqq g(z/a)$.

Since

$$(\Phi)\,\mu h = (\alpha p)\,h \leqq a \quad \text{iff} \quad \alpha p \leqq (a)\,g\,,$$

it suffices to prove the validity of the relation on the right. This relation is a consequence of $p \leqq (z)\,g$ and

$$\alpha \wedge (y)\,p \leqq \alpha \wedge g(y/z) \leqq g(z/a) \wedge g(y/z)$$
$$\leqq \bigvee_{x \in A} (g(x/a) \wedge g(y/x)) = g(y/a)\,,$$

$y \in A$, again with help of (3) in the last step. Consequently, (iii) is valid.

Taking $a = (\Phi)\,Th\,h$ we get

$$(\Phi)\,\mu h \leqq (\Phi)\,Th\,h\,.$$

Supposing now the validity of the other relation

$$(\Phi)\,\mu h = (\alpha p)\,h \leqq a\,,$$

we immediately see that $\alpha p \leqq (a)\,g$ and since for every $x \in A$

$$\alpha p \leqq (x)\,g \quad \text{iff} \quad p \leqq c_\alpha \to (x)\,g\,,$$

we have

$$\{x/c_\alpha \to (a)\,g \leqq (x)\,g\} \subseteq \{x/p \leqq (x)\,g\}\,,$$
$$\operatorname{Inf}\{x/c_\alpha \to (a)\,g \leqq (x)\,g\} \geqq \operatorname{Inf}\{x/p \leqq (x)\,g\} = (p)\,h\,.$$

By (ii) and (3), if $a, x, y \in A$ and $x \leqq y$ then

$$g(x/a) = \bigvee_{z \in A} (g(z/a) \wedge g(x/z)) \geqq g(y/a) \wedge g(x/y)$$
$$= g(y/a) \wedge 1 = g(y/a)\,.$$

This together with the last estimate of Inf and (4) finally yields

$$g(\operatorname{Inf}\{x/c_\alpha \to (a)\,g \leqq (x)\,g\}/a) \leqq g((p)\,h/a)\,,$$
$$\alpha \leqq g((p)\,h/a) = g(z/a)\,,$$
$$\alpha(z)\,\eta \leqq (a)\,g\,,$$
$$(\Phi)\,Th\,h = (\alpha(z)\,\eta)\,h \leqq (a)\,gh = a\,.$$

Therefore $(\Phi)\,Th\,h \leqq (\Phi)\,\mu h$.

3. INDEPENDENCE OF THE CHARACTERIZING CONDITIONS

3.0. The characterizing conditions (1)–(4) of Theorem 2.14 will be shown to be independent. The corresponding counterexamples are constructed with $L = \{0, 1\}$, $A = L \times L$. The individual maps g will be given in the form of a matrix (g_{ij}), $1 \leqq i$, $j \leqq 4$, where $g_{ij} = g(j/i)$ and the elements of A (0, 0), (0, 1), (1, 0), (1, 1) are assigned to the rows and columns 1, 2, 3, 4, respectively. The Kleisli composition is performed by the max-min-product of matrices (that is $\vee$, $\wedge$ instead of $+$, $\cdot$).

3.1. $g_{ij} = 1$, $1 \leqq i, j \leqq 4$.
Obviously g is δ-preserving and idempotent with respect to $\circ$. As for condition (4), it is sufficient to take $\alpha = 1$:

$$\begin{aligned} 1 &\leqq g(\operatorname{Inf}\{x/c_1 \to (y)\, g \leqq (x)\, g\}/y) \\ &= g(\operatorname{Inf}\{x/(y)\, g \leqq (x)\, g\}/y) = g(y/y)\,. \end{aligned}$$

But evidently, g is not injective.

3.2. $g_{ij} = \delta_{ij}$ (Kronecker symbol).
g is injective and idempotent. (4) can be shown as in 3.1.
g is not δ-preserving.

3.3.

$$g = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

g is injective and δ-preserving. Computation of $g \circ g$ results in the matrix from 3.1, hence (2) is violated. (4) is valid as in the preceding examples.

3.4.

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

g is injective, δ-preserving and idempotent.
Since $g(1/1) \neq 1$, (4) does not hold.

4. SPECIAL LATTICES AS UNDERLYING OBJECTS OF T-ALGEBRAS

4.1. Example. The brouwerian lattice L is itself the underlying object of a T-algebra (L, h) with the morphism

$$(p)h := \bigvee_{\alpha \in L} (\alpha \wedge (\alpha)p)\,.$$

$p \in TL$. This can be verified without difficulty by directly testing the monad identities.

The residuated map g is proved just to be the implication of L interpreting $g\colon L \to TL$ as a binary operation $L \times L \to L$: for every $\alpha, \beta \in L$,

$$\begin{aligned} g(\beta/\alpha) &= (\beta)\,(\sup\{p/\bigvee\{\gamma/\gamma \in L,\ \gamma \wedge (\gamma)\,p \leqq \alpha\}) \\ &= (\beta)\,(\sup\{p/\forall\gamma \in L\colon (\gamma)\,p \leqq \gamma \to \alpha\}) \\ &= (\beta)\,(\sup\{p/p \leqq 1_L \to c_\alpha\}) = (\beta)\,(1_L \to c_\alpha) = \\ &= \beta \to \alpha\,. \end{aligned}$$

The validity of conditions (1)–(4) from 2.14 can be restated using the well known identities concerning implication (see [11]).

$1 \to \alpha = \alpha$, therefore $(\alpha)\,g \neq (\beta)\,g$ if $\alpha \neq \beta$, and g must be injective. δ-preservation is expressed by

$$\beta \to \bigwedge_{i\in I} \alpha_i = \bigwedge_{i\in I} (\beta \to \alpha_i)\,.$$

The identity $\alpha \to \alpha = 1$ together with $\eta_L \leqq g$ gives $g \leqq g \circ g$. On the other hand,

$$(\gamma \to \alpha) \wedge (\beta \to \gamma) \leqq \beta \to \alpha\,,$$

therefore

$$(g \circ g)\,(\beta/\alpha) = \bigvee_{\gamma\in L} ((\gamma \to \alpha) \wedge (\beta \to \gamma)) \leqq g(\beta/\alpha)\,,$$

so that $\circ$ is idempotent. Condition (4) amounts to

$$\alpha \leqq \bigwedge\{\xi/c_\alpha \to (\beta)\,g \leqq (\xi)\,g\} \to \beta\,. \tag{4'}$$

By the series of equivalences

$$\begin{aligned} &c_\alpha \to (\beta)\,g \leqq (\xi)\,g\,, \\ &\forall\gamma \in L\colon \alpha \to (\gamma \to \beta) \leqq (\gamma \to \xi)\,, \\ &\forall\gamma \in L\colon \gamma \to (\alpha \to \beta) \leqq (\gamma \to \xi)\,, \\ &\forall\gamma \in L\colon \gamma \wedge (\alpha \to \beta) \leqq \xi\,, \\ &\bigvee_{\gamma\in L} (\gamma \wedge (\alpha \to \beta)) = (\alpha \to \beta) \wedge 1 \leqq \xi\,, \end{aligned}$$

(4′) is reduced to $\alpha \leqq (\alpha \to \beta) \to \beta$ or equivalently, $\alpha \wedge (\alpha \to \beta) = \alpha \wedge \beta \leqq \beta$.

4.2. Application of 2.14

Theorem. *Let* $(A, \leqq)$ *be a complete lattice and* δ *a meet-irreducible element of* L, $\delta \neq 1$.

If mappings $g\colon A \to TA$, $h\colon TA \to A$ *are defined by*

$$\begin{aligned} &g(b/a) = 1 \quad \textit{if} \quad b \leqq a\,, \quad \textit{else}\ \delta\,; \quad a, b \in A\,, \\ &(p)\,h = \operatorname{Inf}\{a/a \in A,\ p \leqq (a)\,g\}\,; \quad p \in TA\,, \end{aligned}$$

then (A, h) *is a T-algebra.*

Proof. By 2.14 it is sufficient to prove conditions (1)–(4). If $a, c \in A$, $a \nleqq c$, then $g(a/c) = \delta \neq 1 = g(a/a)$, which yields injectivity of g.

For every $X \subseteq A$, $b \in A$

$$g(b/\operatorname{Inf} X) = 1 \quad \text{iff} \quad b \leqq \operatorname{Inf} X \quad \text{iff}$$
$$\forall x \in X: b \leqq x \quad \text{iff} \quad \bigwedge_{x \in X} g(b/x) = 1 \, .$$

g being two-valued, (2) follows. Having again the codomain of g in mind, (3) follows from

$$(g \circ g)(b/a) = \bigvee_{x \in A} (g(x/a) \wedge g(b/x)) = 1$$

iff $\exists x \in A: x \leqq a$, $b \leqq x$ iff $g(b/a) = 1$.

Evidently, (4) holds for $\alpha \leqq \delta$. If $\alpha \nleqq \delta$ then $\alpha \to \delta \leqq \delta$, because

$$\alpha \wedge (\alpha \to \delta) \leqq \delta \, , \quad (\alpha \vee \delta) \wedge ((\alpha \to \delta) \vee \delta) = \delta$$

and δ is supposed to be meet-irreducible.

Now, for every $a, y \in A$

$$\alpha \to g(y/a) \begin{cases} = 1 & \text{if} \quad y \leqq a \, , \\ \leqq \delta & \text{else} \, . \end{cases}$$

Consequently $\operatorname{Inf} \{x/c_\alpha \to (a)\, g \leqq (x)\, g\} = a$ and $\alpha \leqq 1 = g(a/a)$.

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