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ENDOMORPHISMS AND CONNECTED COMPONENTS OF PARTIAL MONOUNARY ALGEBRAS

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Homomorphisms and endomorphisms of (complete) monounary algebras were investigated in [1], [3]–[6]; for the case of partial monounary algebras cf. [2].

Let $A \neq \emptyset$ be a set, card $A \geqq 2$. We denote by F the set of all partial mappings of the set A into A. If $f \in F$, then (A, f) is said to be a *partial monounary algebra*. To each $f \in F$ there corresponds a partition P_f of the set A consisting of all connected components of (A, f) (cf. § 1 below). Let End (A, f) be the set of all endomorphisms of (A, f).

Let F_0 be the set of all mappings of the set A into A. In [1] it was shown that for each $f \in F_0$ the relation

(1a) $$(g \in F_0 \mathbin{\&} \operatorname{End}(A, f) = \operatorname{End}(A, g)) \Rightarrow P_f = P_g$$

is valid.

In this paper it will be proved that for $f \in F$ the analogous relation

(1) $$(g \in F \mathbin{\&} \operatorname{End}(A, f) = \operatorname{End}(A, g)) \Rightarrow P_f = P_g$$

need not hold, but that the set of nonisomorphic types of partial monounary algebras (A, f) such that (1) fails to hold for f is small (independently of the cardinality of the set A). Namely, we shall describe partial mappings $f_1, f_2 \in F$ such that, whenever $f \in F$ and f does not satisfy (1), then (A, f) is isomorphic either to (A, f_1) or to (A, f_2). (Cf. Thm. 4.6, type τ and π.)

Further, we shall establish some results on the relation between (A', f') and (A', g'), where $f, g \in F$, $A' \in P_f$ and $f' = f \mid A'$, $g' = g \mid A'$ (under the assumption that f fulfils (1)).

1. PRELIMINARIES

Let (A, f) be a partial monounary algebra. We shall denote by D_f the set of all $x \in A$ such that $f(x)$ does not exist. A mapping $H: A \to A$ is called an *endomorphism* of the partial monounary algebra (A, f) (cf. [2]), if the following relation is valid:

$$(\forall x \in A - D_f)\,(H(x) \in A - D_f \mathbin{\&} H(f(x)) = f(H(x)))\,.$$

The system of all endomorphisms of (A, f) will be denoted by the symbol End(A, f).

Let N be the set of all positive integers. For each $x \in A$ we put $f^0(x) = x$. Let $n \in N$. If $f^m(x)$ is defined for each $m \in N \cup \{0\}$, $m < n$, and if $f^{n-1}(x) \notin D_f$, then we put $f^n(x) = f(f^{n-1}(x))$. Further, denote $f^{-n}(x) = \{y \in A : f^n(y) = x\}$ for each $n \in N$. For $x, y \in A$ we shall write $x \equiv_f y$, if there exist $m, n \in N \cup \{0\}$ such that $f^n(x) = f^m(y)$. The relation $\equiv_f$ is an equivalence relation on A and the elements of the set $A/\equiv_f$ are called *connected components of the algebra* (A, f). If $A/\equiv_f$ has one element, then we shall say that (A, f) is connected. The connected component containing the element $x \in A$ will be denoted by $K_f(x)$.

Let Ord be the class of all ordinals; denote $\mathrm{Ord}_0 = \mathrm{Ord} \cup \{\infty_1, \infty_2\}$. We put $\alpha < \infty_1 < \infty_2$ for each $\alpha \in \mathrm{Ord}$. For each $f \in F$, a mapping s_f of the set A into the class Ord_0 was defined in [2].

The following propositions (T), (T0)–(T3), which will be often used in the sequel, are immedaite consequences of 3.3 and 4.8 [2].

(T) If (A, h) is a partial monounary algebra, $H \in \mathrm{End}\,(A, h)$, $x \in A$, then $s_h(x) \leqq s_h(H(x))$.

(T0) If (A, h) is a partial monounary algebra, $x \in A$, $n \in N \cup \{0\}$, $h^{-n}(x) \neq \emptyset$, $h^{-n-1}(x) = \emptyset$, then $s_h(x) = n$.

(T1) Let (A, h) be a partial monounary algebra, $x \in D_h$, $y \in A - K_h(x)$. Then $s_h(x) \leqq s_h(y)$ if and only if there exists $H \in \mathrm{End}\,(A, h)$ such that the following conditions are fulfilled:

(i) $H(x) = y$,

(ii) $H(h^{-n}(x)) \subseteq h^{-n}(y)$ for each $n \in N$,

(iii) $H(t) = t$ for each $t \in A - \bigcup_{n \in N \cup \{0\}} h^{-n}(x)$.

(T2) Let (A, h) be a partial monounary algebra, $x, y \in A$, $x \neq h(x) = h(y)$. Then $s_h(x) \leqq s_h(y)$ if and only if there exists $H \in \mathrm{End}\,(A, h)$ such that (i), (ii), (iii) from (T1) are valid.

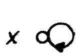
(T3) Let (A, h) be a partial monounary algebra, $x, y \in A$, $s_h(y) \neq \infty_i$ $(i = 1, 2)$, $y \in f^{-k}(x)$, $k \in N$. Then $s_f(x) > s_f(y)$.

In what follows, the notation $s_f(x) = \infty$ means that either $s_f(x) = \infty_1$ or $s_f(x) = \infty_2$. (Thus $s_f(x) \neq \infty$ means that neither $s_f(x) = \infty_1$ nor $s_f(x) = \infty_2$ holds.)

Let (A, f) be a partial monounary algebra. By A_2^f we shall denote the set of all $x \in A$ such that there exists $y \in D_f$ with $y \equiv_f x$. Further, let $A_1^f = A - A_2^f$.

Remark. In the figures we use the following notation:

– a pair of elements $x, y \in A$ with $f(x) = y$;



– an element $x \in A$ with $f(x) = x$;

– an element $z \in D_f$.

1.1. Definition. We say that a partial monounary algebra (B, h) is of *type* τ, π, δ or γ, if it fulfils the following condition (τ), (π), (δ) or (γ), respectively (cf. Fig. 1):

(τ) card $A > 1$ and there is $x \in B$ such that $h(t) = x$ for each $t \in B$;
(π) there is $x \in B$ such that $h(x) = x$ and $\emptyset \neq D_h = B - \{x\}$;
(δ) $h(t) = t$ for each $t \in B$;
(γ) $B = D_h$.

If (B, h) is of type τ or of type π, $x \in B$ and $h(x) = x$, then we also say that (B, h) is of type τ or of type π with the end x.

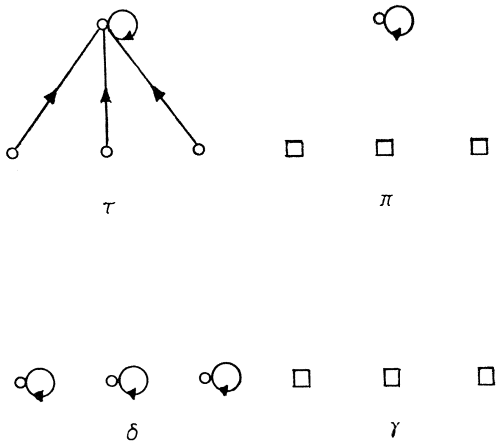


Fig. 1.

1.2. Definition. Let (B, h_1) and (B, h_2) be partial monounary algebras such that

(i) $B = \{x_i : i \in N\} \cup \bigcup_{i \in N, i > 1} B_i$, where x_i, $i \in N$, are distinct elements, B_i, $i \in N$, $i > 1$, are disjoint sets and $x_i \notin B_j$ for $i, j \in N$, $j > 1$;
(ii) $h_1(b_{i+1}) = x_{i+2}$ for each $b_i \in B_i \cup \{x_i\}$, $i \in N$, and $h_1(x_1) = x_2$;
(iii) $h_2(b_{i+1}) = x_i$ for each $b_{i+1} \in B_{i+1} \cup \{x_{i+1}\}$, $i \in N$, and $x_1 \in D_{h_2}$.

The algebra (B, h_1) or (B, h_2) is said to be of *type* σ or ϱ, respectively (cf. Fig. 2). If (B, h_1) is of type σ or ϱ and there are x_i (for each $i \in N$) and B_i (for each $i \in N$, $i > 1$) fulfilling (i), (ii) or (i), (iii), respectively, then we write

$$(B, h_1) \in \sigma(x_1, x_2, \ldots, B_2, B_3, \ldots),$$
$$(B, h_2) \in \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots).$$

1.3. Definition. Let (B, h_1) and (B, h_2) be of *type* τ and π, respectively, with the end x. We shall write

$$(B, h_1) = (B, h_2)^\tau, \quad (B, h_2) = (B, h_1)^\pi.$$

1.4. Definition. Let (B, h_1) and (B, h_2) be of *type* δ and γ, respectively. We shall write

$$(B, h_1) = (B, h_2)^{\delta}, \quad (B, h_2) = (B, h_1)^{\gamma}.$$

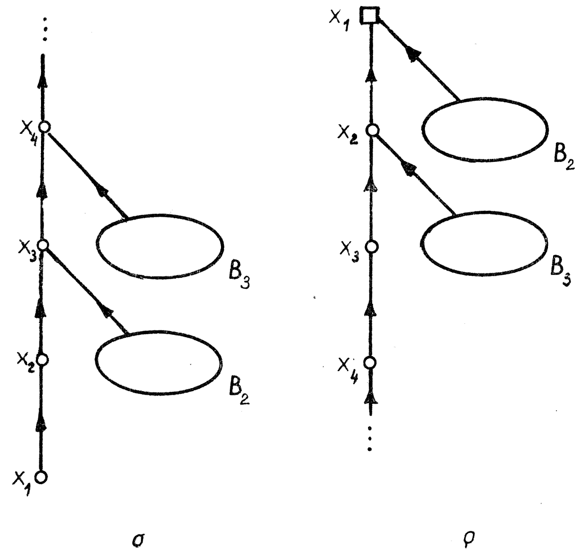


Fig. 2.

1.5. Definition. Let $(B, h_1) \in \sigma(x_1, x_2, \ldots, B_2, B_3, \ldots)$, $(B, h_2) \in \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots)$. We shall write

$$(B, h_1) = (B, h_2)^{\sigma}, \quad (B, h_2) = (B, h_1)^{\varrho}.$$

2. AUXILIARY RESULTS

Now let (A, f) be a partial monounary algebra. Further assume that (A, g) is a partial monounary algebra with the property End $(A, f) =$ End (A, g).

2.0. Lemma. *Let* $x \in A, f(x) = x$. If $A \neq D_g$, *then* $x \notin D_g$ *and* $g(x) = x$.

Proof. Let $H(t) = x$ for each $t \in A$. It is obvious that $H \in$ End (A, f), hence $H \in$ End (A, g). This implies that either $x \in D_g$ or

$$x = H(g(x)) = g(H(x)) = g(x).$$

Let $y \in A - D_g$ and suppose that $x \in D_g$. From the definition of an endomorphism it follows that for no $G \in$ End (A, g) the relation $G(y) = x$ is valid, which is a contradiction with the fact that $H(y) = x$ and $H \in$ End (A, g).

2.1. Lemma. *Let* $x \in A_1^f - D_g$. *Then* $g(x) \in K_f(x)$.

Proof. We can use the proof of Lemma 4 from [1].

2.2. Corollary. *Let* $x \in A_1^f$. *Then* $g(K_f(x)) \subseteq K_f(x)$.

2.3. Lemma. *Let* $z \in D_g \cap A_1^f$, $x \in f^{-1}(z)$. *Then* $x \in D_g$.

Proof. Define a mapping $H: A \to A$ as follows: $H \mid K_f(x) = f \mid K_f(x)$ and $H \mid (A - K_f(x)) = \mathrm{id} \mid (A - K_f(x))$. Since $H \in \mathrm{End}\,(A, f)$, then $H \in \mathrm{End}\,(A, g)$. We have $H(x) = z \in D_g$, thus $x \in D_g$.

2.4. Lemma. *In* $A_1^f \cap D_g$ *there exist no distinct elements* x_1, x_2, x_3 *such that* $f(x_1) = x_2$, $f(x_2) = x_3$.

Proof. Suppose that such x_1, x_2, x_3 exist. Let $s_g(x_1) \leqq s_g(x_2)$. From (T1) (for g, x_1, x_2 instead of h, x, y) it follows that there is $H \in \mathrm{End}\,(A, g)$ with $H(x_1) = H(x_2) = x_2$, $H(x_3) = x_3$. Then $H \in \mathrm{End}\,(A, f)$ and we obtain

$$x_2 = H(x_2) = H(f(x_1)) = f(H(x_1)) = f(x_2) = x_3\,,$$

a contradiction. Now let $s_g(x_1) \geqq s_g(x_2)$. According to (T1) there exists $H_1 \in \mathrm{End}\,(A, g)$ such that $H_1(x_1) = H_1(x_2) = x_1$, $H_1(x_3) = x_3$. Thus $H_1 \in \mathrm{End}\,(A, f)$ and

$$x_1 = H_1(x_2) = H_1(f(x_1)) = f(H_1(x_1)) = f(x_1) = x_2\,,$$

which is a contradiction.

2.5. Lemma. *Let* $z \in D_g \cap A_1^f$. *Then* $f^{-1}(z) - \{z\} = \emptyset$.

Proof. Assume that $x \in f^{-1}(z) - \{z\}$. From 2.3 we infer that $x \in D_g$. First let $s_g(x) \leqq s_g(z)$. According to (T1) there is $H \in \mathrm{End}\,(A, g)$ such that $H(x) = H(z) = z$ and then

$$z = H(z) = H(f(x)) = f(H(x)) = f(z)\,. \tag{1}$$

Since $z \in D_g$, 2.0 and (1) imply that $A = D_g$. Thus $0 = s_g(z) \leqq s_g(x) = 0$. Therefore $s_g(z) \leqq s_g(x)$ and (T1) implies that there is $G \in \mathrm{End}\,(A, g)$ such that $G(z) = G(x) = x$. Hence $G \in \mathrm{End}\,(A, f)$ and we obtain

$$G(f(x)) = G(z) = x \neq z = f(x) = f(G(x))\,, \tag{2}$$

a contradiction.

2.6. Lemma. *Let* $z_1, z_2 \in A_1^f \cap D_g$, $z_1 \in K_f(z_2)$. *Then there is* $n \in N$ *such that* $f^n(z_1) = f^n(z_2)$.

Proof. Since $z_1 \in K_f(z_2)$, there exist $m, n \in N$ with $f^n(z_1) = f^m(z_2)$. We can assume that $s_g(z_1) \leqq s_g(z_2)$. From (T1) it follows that there is $H \in \mathrm{End}\,(A, g)$ such that $H(z_1) = H(z_2) = z_2$. Then $H \in \mathrm{End}\,(A, f)$ and we obtain

$$f^n(z_2) = f^n(H(z_1)) = H(f^n(z_1)) = H(f^m(z_2)) =$$
$$= f^m(H(z_2)) = f^m(z_2) = f^n(z_1)\,.$$

2.7. Lemma. *Let $z \in D_g \cap A_1^f$, $x \in A$, $n \in N$ and suppose that $f^{n'}(x) \neq f^n(z) = f^n(x) \neq f^{n'}(z)$ for each $n' \in N \cup \{0\}$, $n' < n$. Then $x \in D_g$.*

Proof. Suppose that $x \notin D_g$. From 2.5 it follows that $f^{-1}(z) = \emptyset$. From the assumption of the lemma we obtain that there exists the least positive integer m such that there is $x' \in A - D_g$ with the property

$$(1) \qquad f^{m'}(x') \neq f^m(z) = f^m(x') \neq f^{m'}(z) \quad \text{for each} \quad m' \in N \cup \{0\}\,, \quad m' < m\,.$$

Put

$$H(t) = \begin{cases} f^{k_2-k_1}(x')\,, & \text{if there are} \quad 0 \leqq k_1 \leqq k_2 < m \quad \text{with} \quad f^{k_1}(t) = f^{k_2}(z)\,, \\ t & \text{otherwise}\,. \end{cases}$$

Then $H \in \text{End}\,(A, f)$, thus $H \in \text{End}\,(A, g)$. Denote $U = \{t \in A: H(t) \neq t\}$. Since $H(x') = x'$, the relation $x' \in K_g(z)$ implies $H(z) = z$, a contradiction; therefore $x' \notin K_g(z)$. Analogously, if $H(t) = t$ for some $t \in A$, then $t \notin K_g(z)$. Thus we have

$$(2) \qquad K_g(z) \subseteq U\,.$$

The relation $s_g(z) \leqq s_g(H(z)) = s_g(x')$ is valid and (T1) implies that there exists $G \in \text{End}\,(A, g)$ such that $G(z) = x'$, $G(t) = t$ for each $t \in A - K_g(z)$, $G(g^{-k}(z)) \subseteq g^{-k}(x')$ for each $k \in N$. Let $t \in U$, i.e. $f^{k_1}(t) = f^{k_2}(z)$, where $0 \leqq k_1 \leqq k_2 < m$. We get

$$f^{k_2}(x') = f^{k_2}(G(z)) = G(f^{k_2}(z)) = G(f^{k_1}(t)) = f^{k_1}(G(t))\,,$$

hence $G(t) \in f^{-k_1}(f^{k_2}(x'))$. Since $t \in f^{-k_1}(f^{k_2}(z))$ and $f^{-k_1}(f^{k_2}(z)) \cap f^{-k_1}(f^{k_2}(z)) = \emptyset$, the relation $G(t) \neq t$ is valid. Therefore $t \in K_g(z)$, hence

$$(3) \qquad U \subseteq K_g(z)\,.$$

From (2) and (3) we obtain

$$(4) \qquad K_g(z) = U = \{t \in A: H(t) \neq t\}\,.$$

Denote $y' = g(x')$. Since $H \in \text{End}\,(A, g)$, there exists $G_1 \in \text{End}\,(A, g)$ such that

$$G_1(t) = \begin{cases} g(H(t))\,, & \text{if} \quad t \in K_g(z)\,, \\ t & \text{otherwise}\,. \end{cases}$$

(The mapping is correctly defined, since $g(H(z)) = g(x')$ exists.) Then $G_1 \in \text{End}\,(A, f)$. Put $f^m(z) = u$. We have $H(u) = u$, thus (4) implies that $u \notin K_g(z)$ and then $G_1(u) = u$. We obtain

$$(5) \qquad u = f^m(z) = G_1(f^m(z)) = f^m(G_1(z)) = f^m(g(x')) = f^m(y')\,.$$

First assume that $m > 1$. Put

$$G_2(t) = \begin{cases} f(t) & \text{for} \quad t \in K_f(x')\,, \\ t & \text{otherwise}\,. \end{cases}$$

Then $G_2 \in \text{End}(A, f)$. We have $G_2(z) = f(z)$ und by virtue of the definition of H and (4) we obtain that $G_2(z) \in K_g(z)$. Denote $v = f^{m-1}(z)$. Then $v \in K_g(z)$, but $G_2(v) = f(v) = u \notin K_g(z)$, which is a contradiction with the relation $G_2 \in \text{End}(A, g)$. Hence we obtain that $m = 1$. From (5) it follows that

$$f(y') = u = f(x') = f(z). \tag{6}$$

Suppose that $s_f(y') \leqq s_f(x')$ and that $f(y') \neq y'$. Then (T2) (for f, y', x' instead of h, x, y) implies that there is $H_1 \in \text{End}(A, f)$ with $H_1(y') = H_1(x') = x'$. Thus $H_1 \in \text{End}(A, g)$ and

$$g(x') = g(H_1(x')) = H_1(g(x')) = H_1(y') = x'.$$

In view of 2.0 we infer that $f(x') = x'$, which is a contradiction, since then $f^m(z) = f^0(x')$, $0 < m$. Hence either $s_f(y') \leqq s_f(x')$ and $f(y') = y'$, or $s_f(y') > s_f(x')$. In the first case $s_f(y') = s_f(x') = \infty$, $f(y') = y'$, hence consider the possibility $s_f(y') \geqq s_f(x')$. Since $f(x') \neq x'$, we can use (T2) which implies that there is $H_2 \in \text{End}(A, f)$ such that $H_2(x') = H_2(y') = y'$. We have $H_2 \in \text{End}(A, g)$ and

$$g(y') = g(H_2(x')) = H_2(g(x')) = H_2(y') = y'.$$

In view of 2.0, $f(y') = y'$. Further, from (4), (6) and from the definition of H we obtain that $K_g(z) = \{z\}$.

First assume that $f^{-1}(x') \neq \emptyset$, $s \in f^{-1}(x')$. Put $H_3(z) = s$, $H_3(t) = t$ for each $t \in A - \{z\}$. Since $K_g(z) = \{z\}$, the relation $H_3 \in \text{End}(A, g)$ is valid, thus $H_3 \in \text{End}(A, f)$. Because of $K_g(z) = \{z\}$ and $g^{-1}(y') \supseteq \{x'\}$ we have $y' \neq z$ and then we get

$$g(x') = y' = H_3(y') = H_3(f(y')) = H_3(f(z)) = f(H_3(z)) = f(s) = x',$$

which yields a contradiction as above. Hence $f^{-1}(x') = \emptyset$. Therefore $0 = s_f(x') \leqq s_f(z) = 0$ and (T2) implies that there is $H_4 \in \text{End}(A, f)$ such that $H_4(x') = H_4(z) = z$. Then $H_4 \in \text{End}(A, g)$, a contradiction, since $x' \notin D_g$ and $z \in D_g$.

2.8. Lemma. *Let $x \in A_1^f$ and let x belong to a cycle consisting of k elements, $k > 1$. Then $K_f(x) \cap D_g = \emptyset$.*

Proof. Assume that $K_f(x) \cap D_g \neq \emptyset$, $z \in K_f(x) \cap D_g$. We denote by C the cycle of the component $K_f(x)$. There is the least $i \in N \cup \{0\}$ such that $f^i(z) \in C$. Put $f^i(z) = u$. From 2.5 it follows that $z \notin C$, hence $i \geqq 1$. Let $k_1 \in N$ be such that $kk_1 - i \geqq 0$ and put $x' = f^{kk_1 - i}(u)$. Obviously, $x' \in C$ and $f^{-1}(x') - \{x'\} \neq \emptyset$, thus 2.5 implies that $x' \notin D_g$. Analogously, if $y' \in C$, we obtain that $y' \notin D_g$. Thus $C \cap D_g = \emptyset$. Denote $g(x') = w$. In view of 2.1 we have $w \in K_f(x')$. There exists $G \in \text{End}(A, f)$ such that $G(t) \in C$ for each $t \in K_f(x')$ and $G(t) = t$ for each $t \in (A - K_f(x')) \cup C$. Then

$$w = g(x') = g(G(x')) = G(g(x')) = G(w),$$

which implies that $w \in C$. Since $C \cap D_g = \emptyset$, $w = g(x')$ exists. By induction, $g^m(x')$ exists and $g^m(x') \in C$ for each $m \in N$. This yields that $K_g(x') \cap D_g = \emptyset$ and hence $z \notin K_g(x')$. Put

$$G_1(t) = \begin{cases} g(t), & \text{if} \quad t \in K_g(x'), \\ t & \text{otherwise}. \end{cases}$$

Evidently $G_1 \in \text{End}(A, g)$, thus $G_1 \in \text{End}(A, f)$ and we obtain

$$g(x') = G_1(x') = G_1(f^{kk_1 - i}(u)) = G_1(f^{kk_1 - i}(f^i(z)) =$$
$$= f^{kk_1}(G_1(z)) = f^{kk_1}(z) = f^{kk_1 - i}(f^i(z)) = f^{kk_1 - i}(u) = x'.$$

According to 2.0 this implies that $f(x') = x'$, which is a contradiction.

2.9. Corollary. *Let* $x \in A_1^f$, $s_f(x) = \infty$, $f(x) \neq x$. *Then* $K_f(x) \cap D_g = \emptyset$.

Proof. Suppose that there is $z \in K_f(x) \cap D_g$. From 2.8 we obtain that there is no cycle C with card $C > 1$ in $K_f(x)$. There exists the least nonnegative integer n such that $s_f(f^n(z)) = \infty$. Put $f^n(z) = u$. If $f(u) = u = z$, then 2.5 implies that $f^{-1}(z) = \{z\}$ and that $K_f(z) = \{z\}$, which is a contradiction with the assumption. Hence $f(u) = u \neq z$ or $f(u) \neq u$. In both cases there exists $x' \neq u$ such that $f^{n'}(x') \neq f^n(z) = f^n(x') \neq f^{n'}(z)$ for each $n' \in N \cup \{0\}$, $n' < n$, and $s_f(x') = \infty$. According to 2.7 we obtain that $x' \in D_g$ and then according to 2.5 the relation $f^{-1}(x') - \{x'\} = \emptyset$ is valid. This is a contradiction, since $s_f(x') = \infty$ and $f(x') \neq x'$.

3. ALGEBRAS OF SOME SPECIAL TYPES

As above, let (A, f) and (A, g) be partial monounary algebras with $\text{End}(A, f) = \text{End}(A, g)$. In this section we shall study algebras of some special types, e.g. $\tau, \pi, \delta, \gamma$, and we shall also prove some results concerning components of partial monounary algebras.

3.1. Lemma. *Let* $x \in A$, $f(x) = x$ *and* $s_f(t) \neq \infty$ *for each* $t \in K_f(x) - \{x\}$. *Further let* $K_f(x) \cap D_g \neq \emptyset$. *Then*

(i) (A, f) *and* (A, g) *is of type* τ *and* π, *respectively, with the end* x,

or

(ii) (A, f) *and* (A, g) *is of type* δ *and* γ, *respectively.*

Proof. Choose an arbitrary element $z \in K_f(x) \cap D_g$. First assume that $x \notin D_g$. Then 2.0 implies that $g(x) = x$. Since $K_f(x) \subseteq A_1^f$, we obtain that $f(z)$ exists; denote $f(z) = y$.

Suppose that $y \neq x$. Put

$$H(t) = \begin{cases} t, & \text{if} \quad t \in K_g(z), \\ x & \text{otherwise}. \end{cases}$$

Then $H \in \text{End}(A, g)$, thus $H \in \text{End}(A, f)$. If $y \notin K_g(z)$, then

$$x = H(y) = H(f(z)) = f(H(z)) = f(z) = y \neq x,$$

which is a contradiction. Hence $y \in K_g(z)$ and there is $i \in N$ such that $g^i(y) = z$. Since $z \neq x$, there exists the least positive integer n such that $f^n(z) = x$. Put

$$H_1(t) = \begin{cases} f^{n-1}(t), & \text{if } t \in K_f(x), \\ t & \text{otherwise}. \end{cases}$$

We have $n \geqq 1$, thus the mapping H_1 is correctly defined and $H_1 \in \text{End}(A, f)$. Then $H_1 \in \text{End}(A, g)$ and we get

$$x \neq f^{n-1}(z) = H_1(z) = H_1(g^i(y)) = g^i(H_1(y)) = g^i(f^{n-1}(y)) =$$
$$= g^i(f^n(z)) = g^i(x) = x,$$

a contradiction. Therefore $f(z) = y = x$. From 2.7 it follows that if $z' \in K_f(x) - \{x\}$ and $f(z') = f(z) = x$, then $z' \in D_g$. Further, 2.5 implies that $f^{-1}(z') = \emptyset$. Hence

(1) $\quad K_f(x) - \{x\} \subseteq D_g \quad$ and $\quad f(t) = x = g(x) \quad$ for each $\quad t \in K_f(x)$.

Now let $z' \in K_f(x) - \{x\}$ and assume that $g^{-1}(z') \neq \emptyset$, $u \in g^{-1}(z')$. Then $u \notin K_f(x)$ in virtue of (1). Put

$$H_2(t) = \begin{cases} x, & \text{if } t \in K_f(x), \\ t & \text{otherwise}. \end{cases}$$

Then $H_2 \in \text{End}(A, f)$, thus $H_2 \in \text{End}(A, g)$ and we get

$$x = H_2(z') = H_2(g(u)) = g(H_2(u)) = g(u) = z',$$

which is a contradiction. Hence $K_g(z') = \{z'\}$ for each $z' \in K_f(x) - \{x\}$. Further let $v \in A - K_f(x)$. Then $0 = s_g(z) \leqq s_g(v)$ and (T1) implies that there is $H_3 \in \text{End}(A, g)$ such that $H_3(z) = H_3(v) = v$, $H_3(x) = x$, which is a contradiction, since

$$x = H_3(x) = H_3(f(z)) = f(H_3(z)) = f(v) \notin K_f(x).$$

Hence $A - K_f(x) = \emptyset$ and by (1) we obtain that (i) is valid.

Now assume that $x \in D_g$. Then 2.0 implies that $A = D_g$. If $A = \{x\}$, then (ii) holds. Let $w \in A - \{x\}$. Put

$$G_1(t) = \begin{cases} w, & \text{if } t = x, \\ t & \text{otherwise}. \end{cases}$$

Evidently $G_1 \in \text{End}(A, g)$, hence $G_1 \in \text{End}(A, f)$,

$$w = G_1(x) = G_1(f(x)) = f(G_1(x)) = f(w)$$

and the condition (ii) is valid.

3.2. Lemma. (a) *If (A, f) is of type τ with the end x and $g \neq f$, then $(A, g) = (A, f)^\pi$.*

(b) *If (A, f) is of type π with the end x and $g \neq f$, then $(A, g) = (A, f)^\tau$.*

(c) *If (A, f) is of type δ and $g \neq f$, then $(A, g) = (A, f)^\gamma$.*

(d) *If (A, f) is of type γ and $g \neq f$, then $(A, g) = (A, f)^\delta$.*

Proof. (a) If (A, f) is of type τ with the end x and $D_g = K_f(x) \cap D_g \neq \emptyset$, then in view of 3.1, (A, g) is o type π with the end x. Now let $D_g = \emptyset$. Then 2.0 implies that $g(x) = x$. Let $z \in A - \{x\}$ and put

$$H(t) = \begin{cases} x, & \text{if} \quad t = z, \\ t & \text{otherwise}. \end{cases}$$

The mapping H belongs to $\operatorname{End}(A, f)$ and $H^{-1}(x) = \{x, z\}$. Thus $H \in \operatorname{End}(A, g)$ and we obtain

$$H(g(z)) = g(H(z)) = g(x) = x,$$

which implies $g(z) \in \{x, z\}$. From 2.0 it follows that $g(z) \neq z$, thus $g(z) = x = f(z)$, $g = f$ – a contradiction.

(b) Let (A, f) be of type π with the end x. If $A = D_g$, then there is $H \in \operatorname{End}(A, g)$ with $H(x) \neq x$, which is a contradiction with the fact that $H \in \operatorname{End}(A, f)$. Therefore $A \neq D_g$ and thus 2.0 implies that $g(x) = x$. Then $D_g \subseteq D_f$ and $D_g \neq D_f$ since $g \neq f$. Now let $y \in D_f - D_g$ and put

$$H_1(t) = \begin{cases} x, & \text{if} \quad t = y, \\ t & \text{otherwise}. \end{cases}$$

We have $H_1 \in \operatorname{End}(A, f)$, hence $H_1 \in \operatorname{End}(A, g)$ and

$$x = g(x) = g(H_1(y)) = H_1(g(y)),$$

which yields that $g(y) \in \{x, y\}$. If $g(y) = y$, then 2.0 implies that $f(y) = y$, a contradiction. Thus $g(y) = x$. Assume that $D_g \neq \emptyset$, $z \in D_g$. Then there is $G \in \operatorname{End}(A, f)$ such that $G(y) = z$, and this is a contradiction, since $G \in \operatorname{End}(A, g)$. Therefore $(A, g) = (A, f)^\tau$.

(c) Let (A, f) be of type δ and let (A, g) be not of type γ, i.e. $A \neq D_g$. Since $f(y) = y$ for each $y \in A$, 2.0 implies that $g(y) = y$ for each $y \in A$, thus $g = f$.

(d) Suppose that (A, f) is of type γ and (A, g) is not of type δ, $f \neq g$. Then there is $x \in A - D_g$ with $g(x) = y \neq x$. Put

$$H(t) = \begin{cases} y, & \text{if} \quad t = x, \\ t & \text{otherwise}. \end{cases}$$

Obviously $H \in \operatorname{End}(A, f)$, hence $H \in \operatorname{End}(A, g)$ and we have

$$y = H(y) = H(g(x)) = g(H(x)) = g(y).$$

Using 2.0 we obtain that $f(y) = y$, a contradiction with the relation $y \in D_f$.

3.3. Lemma. *Suppose that (A, f) fails to be of type π. Let $y \in A_1^f$, $x \in g^{-1}(y) \cap$ $\cap (A - A_1^f)$. Then $f(y) = g(y) = y$ and $x \in D_f$.*

Proof. Put $H_1(t) = f(t)$ for each $t \in K_f(y)$, $H_1(t) = t$ for each $t \in A - K_f(y)$. Then $H_1 \in \text{End}(A, f)$, thus $H_1 \in \text{End}(A, g)$, which implies that

$$f(y) = H_1(y) = H_1(g(x)) = g(H_1(x)) = g(x) = y .$$

According to 2.0 the relation $g(y) = y$ is valid. Since $x \notin A_1^f$, there exists $z \in K_f(x) \cap$ $\cap D_f$. Let $z = f^i(x)$ for some $i \in N \cup \{0\}$. First assume that $i > 0$. We have $g(x) = y$, $g(y) = y$ and $z \in D_g$, thus $z \notin K_g(x)$. Put $H_2(t) = y$ for each $t \in K_g(y)$, $H_2(t) = t$ for each $t \in A - K_g(y)$. Then $H_2 \in \text{End}(A, g)$, $H_2 \in \text{End}(A, f)$ and

$$z = H_2(z) = H_2(f^i(x)) = f^i(H_2(x)) = f^i(y) = y ,$$

which is a contradiction. Hence $i = 0$ and $x \in D_f$.

3.4. Lemma. *Suppose that (A, f) fails to be of type π and let $y \in A_1^f$, $x \in g^{-1}(y) \cap$ $\cap (A - A_1^f)$. Denote $C_0 = \{y\}$, $C_1 = f^{-1}(y) - C_0$ and $C_n = f^{-1}(C_{n-1})$ for each $n \in N$, $n > 1$. If $u \in C_n$, $n \in N \cup \{0\}$, then $g(u) = f(u)$.*

Proof. We shall proceed by induction. Lemma 3.3 implies that $f(y) = g(y) = y$, $x \in D_f$. Now let $n \in N$, $u \in C_n$. Since $f(u) \neq u$, we obtain from 3.3 (for u instead of y) that $g^{-1}(u) \subseteq A_1^f$. According to 2.1 (for the elements belonging to $g^{-1}(u)$ instead of the element x) the relation $g^{-1}(u) \subseteq K_f(u)$ is valid, hence $g^{-1}(u) \subseteq K_f(y)$. Assume that $u \in D_g$. From 2.9 it follows that $s_f(t) \neq \infty$ for each $t \in K_f(y) - \{y\}$. Then 3.1 implies that (A, g) is of type π with the end y, but this is a contradiction with $x \neq y$, $g(x) = y$. Therefore $u \notin D_g$, $C_n \cap D_g = \emptyset$. Denote $u_1 = g(u)$ and put

$$H(t) = \begin{cases} f(t), & \text{if } t \in \bigcup_{k \in N \cup \{0\}} f^{-k}(f^{n-1}(u)) , \\ t & \text{otherwise} . \end{cases}$$

Then $H \in \text{End}(A, f)$, hence $H \in \text{End}(A, g)$ and we obtain

$$H(u_1) = H(g(u)) = g(H(u)) = g(f(u)) .$$

Since $f(u) \in C_{n-1}$, we have $g(f(u)) = f(f(u))$, hence

(1) $H(u_1) = f^2(u)$, $u_1 \in H^{-1}(f^2(u))$.

This implies that

(2) if $n = 1$, then $u_1 \in \{y, u\}$,

(3) if $n = 2$, then $u_1 \in \{y\} \cup C_1$,

(4) if $n > 2$, then $u_1 \in C_{n-1}$.

First let $n = 1$. If $u_1 = u$, i.e. $g(u) = u$, then 2.0 implies that $f(u) = u$, a contradiction. Therefore $g(u) = u_1 = b = f(u)$.

Now let $n = 2$ and $u_1 = y$. Denote $v = f(u)$. Then $f(v) = g(v)$, hence $v \neq g(v) = g(u) \neq u$. Since there exists no $H_1 \in \text{End}(A, f)$ with $H_1(v) = H_1(u) = u$, we infer from (T2) that $s_g(u) < s_g(v)$ and there is $H_2 \in \text{End}(A, g)$ with $H_2(u) = H_2(v) = v$. Then

$$y \neq v = H_2(v) = H_2(f(u)) = f(H_2(u)) = f(v) = y,$$

which is a contradiction.

Further let $n \geqq 2$ and $u_1 \in C_{n-1}$. Suppose that $g(u) \neq f(u)$ and denote $v = f(u)$. From (1) and from the definition of H it follows that

(5) $u_1 \neq f(u_1) = H(u_1) = f^2(u) = f(v) \neq v$.

Since $u_1, v \in C_{n-1}$, the induction hypothesis implies

(6) $f(u_1) = g(u_1)$, $f(v) = g(v)$.

If $s_f(u_1) \leqq s_f(v)$, then by (5) and (T2) there is $H_3 \in \text{End}(A, f)$ with $H_3(u_1) = H_3(v) = v$, $H_3(u) = u$. Then $H_3 \in \text{End}(A, g)$ and

$$u_1 = g(u) = g(H_3(u)) = H_3(g(u)) = H_3(u_1) = v,$$

which is a contradiction. Hence $s_f(u_1) > s_f(v)$ and there is no $H_4 \in \text{End}(A, f)$ such that $H_4(u_1) = H_4(v) = v$. Thus (T2) implies that $s_g(u_1) > s_g(v)$ and that there exists $H_5 \in \text{End}(A, g)$ with $H_5(u_1) = H_5(v) = u_1$, $H_5(u) = u$. Since $H_5 \in \text{End}(A, f)$, we obtain

$$u_1 = H_5(v) = H_5(f(u)) = f(H_5(u)) = f(u) = v,$$

a contradiction.

3.5. Lemma. *Suppose that (A, f) fails to be of type π and let $y \in A_1^f$, $x \in g^{-1}(y) \cap (A - A_1^f)$. Further let $y \neq u \in f^{-1}(y)$. Then $g(v) = f(v)$ for each $v \in K_f(x) - \{x\}$.*

Proof. From 3.3 we have $f(y) = g(y) = y$, $x \in D_f$ and in view of 3.4, $g(u') = f(u')$ for each $u' \in K_f(y)$. Further, $x \neq g(x) = y = g(u) \neq u$. If $s_g(x) \geqq s_g(u)$, then (T2) implies that there is $H \in \text{End}(A, g)$ with $H(x) = H(u) = x$, which is a contradiction with the fact that $H \in \text{End}(A, f)$, since $x \in D_f$, $u \notin D_f$. Hence

(0) $s_g(x) < s_g(u)$

and there exists $G_1 \in \text{End}(A, g)$ such that

(1) $G_1(x) = G_1(u) = u$, $G_1(g^{-n}(x)) \subseteq g^{-n}(u)$ for each $n \in N$, $G_1(t) = t$ for each $t \in A - K_g(x)$.

Let $v \in f^{-j}(x)$, $j \in N$. Since $G_1 \in \text{End}(A, f)$, we get

$$u = G_1(x) = G_1(f^j(v)) = f^j(G_1(v))$$

and thus there exists $v' \in f^{-j}(u)$ with $G_1(v) = v'$. Then $v' \in g^{-j}(u)$ (according to 3.4), $G_1(v) = v' \neq v$ and (1) implies that

(2) $v \in g^{-j}(x)$.

Hence $v \notin D_g$. Further we have

(3) $G_1(g(v)) = g(G_1(v)) = g(v') = f(v')$.

We shall prove the relation $g(v) = f(v)$ by induction with respect to j. If $j = 1$, then (2) implies $g(v) = x = f(v)$. Now let $j > 1$ and suppose that $g(v_1) = f(v_1)$ for each $v_1 \in f^{-j+1}(x)$. Denote $v_1 = f(v)$, $v_1' = G_1(v_1)$. In virtue of (3) we obtain

$$v_1' = G_1(v_1) = G_1(f(v)) = f(G_1(v)) = f(v') = G_1(g(v)),$$

hence $g(v) \in G_1^{-1}(v_1') = \{v_1'\} \cup g^{-j+1}(x)$. The induction hypothesis yields

$$g(v) \in \{v_1'\} \cup f^{-j+1}(x).$$

If we assume that $g(v) = v_1'$, then we get a contradiction with 3.3, since $v_1' \in A_1^f$, $v \in g^{-1}(v_1') \cap (A - A_1^f)$ and $f(v_1') \neq v_1'$. Hence

(4) $g(v) \in f^{-j+1}(x)$.

Put $g(v) = v_2$ and let $v_1 \neq v_2$. Further let i be the least positive integer such that $f^i(v_1) = f^i(v_2)$ (since $v_1, v_2 \in f^{-j+1}(x)$, such i does exist and $i < j$). Denote $a = f^{i-1}(v_1)$, $b = f^{i-1}(v_2)$. Then $\{a, b\} \subseteq f^{-j+i}(x)$ and using the induction hypothesis we get

(5) $g(a) = f(a)$, $g(b) = f(b)$, $g(v_1) = f(v_1)$, $g(v_2) = f(v_2)$.

This implies

(6) $f(a) = f(b)$, $g(a) = g(b)$.

If $s_f(b) \leqq s_f(a)$, then (T2) implies that there exists $G_2 \in \text{End}(A, f)$ such that $G_2(a) = G_2(b) = a$, $G_2(v) = v$, $G_2(v_2) \neq v_2$, hence $G_2 \in \text{End}(A, g)$ and

$$v_2 \neq G_2(v_2) = G_2(g(v)) = g(G_2(v)) = g(v) = v_2,$$

which is a contradiction. Thus $s_f(b) > s_f(a)$ and there is no $H_1 \in \text{End}(A, f)$ with $H_1(a) = H_1(b) = a$. From this and from (T2) we obtain that $s_g(b) > s_g(a)$ and that there exists $G_3 \in \text{End}(A, g)$ such that $G_3(a) = G_3(b) = b$, $G_3(v) = v$, $G_3(v_1) \neq v_1$. Hence $G_3 \in \text{End}(A, f)$ and

$$v_1 \neq G_3(v_1) = G_3(f(v)) = f(G_3(v)) = f(v) = v_1,$$

a contradiction.

3.6. Lemma. *Suppose that (A, f) fails to be of type π and let $y \in A_1^f$, $x \in g^{-1}(y) \cap (A - A_1^f)$. Then $K_f(y) = \{y\}$.*

Proof. According to 3.3 we have $f(y) = g(y) = y$, $x \in D_f$. Assume that there is $u \in f^{-1}(y) - \{y\}$. The assumption of 3.5 is satisfied, hence $g(v) = f(v)$ for each $v \in K_f(x) - \{x\}$. Further, 3.4 implies that $g(t) = f(t)$ for each $t \in K_f(y)$. Thus we obtain

(1) $s_g(x) \geqq s_f(x)$.

Since $f(u) \neq u$, 3.3 implies that $g^{-1}(u) \subseteq A_1^f$ and from 2.1 we get that $g^{-1}(u) \subseteq \subseteq K_f(u)$. Then $f(g^{-1}(u)) = g(g^{-1}(u)) = u$, therefore $g^{-1}(u) \subseteq f^{-1}(u)$. Moreover, from 3.4 we obtain $f^{-1}(u) \subseteq g^{-1}(u)$, hence

(2) $f^{-1}(u) = g^{-1}(u)$.

It can be shown by induction that $f^{-n}(u) = g^{-n}(u)$ for each $n \in N$ and thus

(3) $s_f(u) = s_g(u)$.

Similarly as in the proof of 3.5 we have

(0) $s_g(x) < s_g(u)$.

From (1), (0) and (3) it follows that

$$s_f(x) \leqq s_g(x) < s_g(u) = s_f(u)$$

is valid and hence there is $u' \in f^{-1}(u)$ with the property

(4) $s_f(x) \leqq s_f(u')$.

Then (T1) and (4) imply that there exists $H \in \text{End}\,(A, f)$ such that $H(x) = H(u') = = u'$, $H(y) = y$. Hence $H \in \text{End}\,(A, g)$ and

$$y = H(y) = H(g(x)) = g(H(x)) = g(u') = f(u') = u \neq y\,,$$

which is a contradiction.

3.7. Lemma. *Suppose that (A, f) fails to be of type π and let $y \in A_1^f$, $x \in g^{-1}(y) \cap \cap (A - A_1^f)$. Then $K_f(x) = \{x\}$.*

Proof. The assertions of 3.3, 3.4 and 3.6 are valid. Assume that $w \in f^{-1}(x)$. Put

$$G(t) = \begin{cases} g(t)\,, & \text{if} \quad t \in K_g(y)\,, \\ t & \text{otherwise.} \end{cases}$$

Obviously $G \in \text{End}\,(A, g)$, hence $G \in \text{End}\,(A, f)$ and

$$y = G(x) = G(f(w)) = f(G(w))\,,$$

thus $G(w) \in f^{-1}(y) = \{y\}$, i.e. $G(w) = y$. Since $w \neq y$ and $g^2(w) = y$, $w \in K_g(y)$, we get $G(w) = g(w)$. Therefore

$$w \in g^{-1}(y) \cap (A - A_1^f)\,, \quad w \notin D_f$$

and this is a contradiction with 3.3.

3.8. Lemma. *Suppose that (A, f) fails to be of type π. Then $g^{-1}(A_1^f) \subseteq A_1^f$.*

Proof. Assume that there are elements $y \in A_1^f$, $x \in g^{-1}(y) \cap (A - A_1^f)$. Hence the assertions of 3.3, 3.6 and 3.7 hold. Since (A, f) is not of type π, we obtain that $A \neq \{x, y\}$ and there is $s \in A - (\{x, y\} \cup D_f)$. Then there exists $H \in \text{End}\,(A, f)$ such that $H(x) = s$, $H(y) = y$. Thus $H \in \text{End}\,(A, g)$ and

$$y = H(y) = H(g(x)) = g(H(x)) = g(s)\,.$$

We have

(1) $s \in g^{-1}(y) \cap (A - A_1^f)$,
since for $s \in A_1^f$, 2.1 implies that $s \in K_f(y)$, a contradiction. Then (1) and 3.3 imply that $s \in D_f$ and this is a contradiction as well.

3.9. Lemma. *Let $x_0 \in A_1^f \cap D_g$ and $s_f(t) \neq \infty$ for each $t \in K_f(x_0)$. Then*

(i) *there is $x \in K_f(x_0) - D_g$;*

(ii) *there is the least integer $n \in N \cup \{0\}$ with $f^n(x) = f^m(g(x))$ for some $m \in \in N \cup \{0\}$;*

(iii) $n < m$.

Proof. Since $s_f(t) \neq \infty$ for each $t \in K_f(x_0)$, there exists no cycle in $K_f(x_0)$ and hence 2.4 implies that $K_f(x_0) - D_g \neq \emptyset$. Let $x \in K_f(x_0) - D_g$. According to 2.1 we get $g(x) \in K_f(x)$ and then (ii) is valid. Let us suppose that $n \geqq m$. Then $n > 0$, since in the opposite case $m = n = 0$, $g(x) = x$ and hence $f(x) = x$ with respect to 2.0, $s_f(x) = \infty$, which is a contradiction. Thus $f^{-1}(f^n(x)) \neq \emptyset$ and 2.5 implies that

(1) $f^\alpha(x) \notin D_g$ for each $\alpha \geqq n$.

Put

$$H(t) = \begin{cases} f(t), & \text{if } t \in K_f(x), \\ t & \text{otherwise}. \end{cases}$$

Then $H \in \operatorname{End}(A, f)$, thus $H \in \operatorname{End}(A, g)$ and we obtain

$$(2) \qquad g(f^n(x)) = g(H^n(x)) = H^n(g(x)) = f^n(g(x)) = f^{n-m}(f^m(g(x))) = \\ = f^{n-m}(f^n(x)),$$

$$g(f^{n+1}(x)) = g(H^{n+1}(x)) = H^{n+1}(g(x)) = f^{n+1}(g(x)) = \\ = f^{n-m}(f^{n+1}(x)).$$

Denote $i_1 = n - m$, $z = f^n(x)$, $z' = f^{n+1}(x)$. Thus we have $g(z) = f^{i_1}(z)$, $g(z') = = f^{i_1}(z')$. By (1) and (2) the relations $g(z) \notin D_g$, $g(z') \notin D_g$ are valid and the induction with respect to k yields that

$$(3) \qquad g^k(z) = f^{ki_1}(z), \quad g^k(z') = f^{ki_1}(z') \quad \text{for each} \quad k \in N \cup \{0\}$$

holds. Then $K_g(z) \cap D_g = \emptyset$ and $K_g(z') \cap D_g = \emptyset$. Since $z \in A_1^g - D_g$, $z' = f(z)$, Lemma 2.1 (for z, z', g instead of $x, g(x), f$) implies that $z' \in K_g(z)$. Thus there are $k_1, k_2 \in N \cup \{0\}$ such that $g^{k_1}(z) = g^{k_2}(z')$ and then

$$f^{k_1 i_1 + n}(x) = f^{k_1 i_1}(f^n(x)) = f^{k_1 i_1}(z) = g^{k_1}(z) = g^{k_2}(z') = \\ = f^{k_2 i_1}(z') = f^{k_2 i_1}(f^{n+1}(x)) = f^{k_2 i_1 + n + 1}(x).$$

There is no cycle in $K_f(x)$, therefore

$$k_1 i_1 + n = k_2 i_1 + n + 1,$$

$$(k_1 - k_2)\, i_1 = 1,$$

which implies that $i_1 = 1$. In virtue of (3) we get that $z \in A_1^g$, $g^k(z) = f^k(z)$ for each $k \in N \cup \{0\}$. Since $x_0 \in K_f(x) \cap D_g$, we obtain that $K_g(x_0) \cap K_g(z) = \emptyset$. From the fact that $\{f^k(z): k \in N \cup \{0\}\} \subseteq K_g(z)$ we infer that in the system of components $\{K_g(t): t \in K_f(x)\}$ there is a component $K_g(z')$, $z' \in K_f(x)$ such that

(4) $K_g(z) \cap K_g(z') = \emptyset$ and $f(z') \in K_g(z)$.

Because (A, f) is not of type τ, hence (according to 3.2) (A, g) is not of type π and 3.8 implies

(5) $f^{-1}(A_1^g) \subseteq A_1^g$.

Then (4) and (5) imply that $z' \in f^{-1}(K_g(z)) \subseteq A_1^g$, which is a contradiction with 2.1, since $z' \in A_1^g - D_f$, $f(z') \in K_g(z) \neq K_g(z')$.

3.10. Lemma. *Suppose that the assumption of* 3.9 *is satisfied and let* x, n, m *be as in* 3.9. *Then*

(i) $m - n = 1$,

(ii) *there are* $a_1, a_2, \ldots, a_k$ *such that if* $j \in N$, $j < k$, *then* $g(a_j) = a_{j+1} \in f^{-1}(a_j)$, $a_k \in D_g$.

Proof. Denote $i = m - n$. Put $a = f^n(g(x))$, $b = f^n(x)$. If $n = 0$, then $x = f^m(g(x))$, $m > 0$, hence for each $n \in N \cup \{0\}$ we have $f^{-1}(b) \neq \emptyset$ and 2.5 implies that $b \notin D_g$. Put

$$H(t) = \begin{cases} f(t), & \text{if } t \in K_f(x), \\ t & \text{otherwise}. \end{cases}$$

Then $H \in \operatorname{End}(A, f)$, $H \in \operatorname{End}(A, g)$ and

$$f^i(a) = f^{m-n}(f^n(g(x))) = f^m(g(x)) = f^n(x) = b,$$

$$g(b) = g(f^n(x)) = g(H^n(x)) = H^n(g(x)) = f^n(g(x)) = a.$$

Denote $a_1 = g(b)$. Thus $a_1 \in f^{-i}(b)$. If $a_1 \notin D_g$, then put $a_2 = g(a_1)$. By induction, if a_j is defined and $a_j \notin D_g$, $j \in N$, then put $a_{j+1} = g(a_j)$. The induction yields that

(1) if $j \in N$, $a_j \notin D_g$, then $a_{j+1} \in f^{-1}(a_j)$.

Therefore

$$\infty \neq s_f(a_1) > s_f(a_2) > \ldots,$$

which is a decreasing sequence of ordinals. Thus there is $k \in N$ such that $a_k \in D_g$. According to 2.5 we have $f^{-1}(a_k) = \emptyset$ and (1) implies that $a_k \in f^{-ki}(b)$.

Now pur $b' = f^{i-1}(a)$. Then either $i = 1$ or $i > 1$ and $f^{-1}(b') \neq \emptyset$. Consider the case $i > 1$. It follows from 2.5 that $b' \notin D_g$ and we obtain

$$g(b') = g(f^{i-1}(a)) = g(H^{i-1}(a)) = H^{i-1}(g(a)) = $$
$$= f^{i-1}(g(a_1)) = f^{i-1}(a_2).$$

Put $g(b') = a'_1$. Hence

$$a'_1 = f^{i-1}(a_2) \in f^{i-1}(f^{-i}(a_1)) \subseteq f^{-1}(a_1) = f^{-1}(a) \subseteq$$
$$\subseteq f^{-1}(f^{-(i-1)}(b')) \subseteq f^{-i}(b') .$$

By induction, if a'_j is defined, $a'_j \notin D_g$, $j \in N$, then we denote $a'_{j+1} = g(a'_j)$. The induction with respect to $j \in N$ yields that

(2) if $j \in N$, $a'_j \notin D_g$, then $a'_{j+1} \in f^{-i}(a_j)$.

Thus

$$\infty \neq s_f(a'_1) > s_f(a'_2) > \dots$$

and analogously as above there is $k' \in N$ such that $a'_{k'} \in D_g$. According to 2.5 we get $f^{-1}(a'_{k'}) = \emptyset$ and in view of (2) we have $a'_{k'} \in f^{-k'i}(b')$. Hence

$$f^{ik}(a_k) = b = f(b') = f(f^{ik'}(a'_{k'})) = f^{ik'+1}(a'_{k'}) ,$$

thus 2.6 yields that $ik = ik' + 1$, $i(k - k') = 1$, a contradiction. Therefore $i = 1$. Then (1) can be expressed as follows:

$$\text{if} \quad j \in N, \ a_j \notin D_g, \quad \text{then} \quad a_{j+1} \in f^{-1}(a_j) ,$$

and the proof is complete.

3.11. Lemma. *Suppose that the assumption of* 3.9 *is satisfied and let* $a_1, a_2, \dots, a_k$ *be as in* 3.10. *Denote* $x_1 = a_k$ *and* $x_{j+1} = f(x_j)$ *for each* $j \in N$. *Then*

(i) $g(x_j) = x_{j-1}$ *for each* $j \in N$, $j > 1$, *and* $x_1 \in D_g$.

Proof. Since there is no cycle in $K_f(x)$, the elefents x_j are mutually distinct. From 3.10 (ii) we obtain that $x_1 \in D_g$, $x_1 = a_k = g(a_{k-1})$, $x_2 = f(a_k) = a_{k-1}, \dots$ $\dots, x_k = f(a_2) = a_1$, which implies

(1) $x_1 \in D_g$, $g(x_2) = x_1, \dots, g(x_k) = x_{k-1}$.

Further we have

(2) $g(x_{k+1}) = g(f(x_k)) = g(f(a_1)) = a_1 = x_k$.

Since $f^{-1}(x_j) \neq \emptyset$ for each $j \in N$, $j > 1$, Lemma 2.5 implies that $x_j \notin D_g$. Hence $\{x_j : j \in N, j > 1\} \cap D_g = \emptyset$. By induction for $j \in N$, $j \geqq k$, we obtain

(3) $g(x_{j+1}) = x_j$.

From (1)–(3) we get that (i) is valid.

3.12. Lemma. *Suppose that the assumption of* 3.9 *is satisfied and let* $\{x_j : j \in N\}$ *be as in* 3.11. *Denote* $B_j = f^{-1}(x_{j+1}) - \{x_j\}$ *for each* $j \in N$, $j > 1$. *Then*

(i) $K_f(x) = K_g(x)$,

(ii) $(K_f(x), f) \in \sigma(x_1, x_2, \dots, B_2, B_3, \dots)$,

(iii) $(K_g(x), g) \in \varrho(x_1, x_2, \dots, B_2, B_3, \dots)$.

Proof. Let $p, q \in N$, $p \geqq q > 1$ and let $y \in f^{-p+q}(x_p)$. Assume $y \in D_g$. Then 2.7 implies that there is $\alpha \in N$ such that $f^\alpha(x_1) = f^\alpha(y)$. Hence

$$x_{\alpha+1} = f^\alpha(x_1) = f^\alpha(y) \in f^{\alpha-p+q}(x_p),$$

therefore $x_{\alpha+1} = f^{\alpha-p+q}(x_p) = x_{\alpha-p+q+p} = x_{\alpha+q}$. This yields that $q = 1$, a contradiction, thus

(1) $y \notin D_g$.

Further, assume that $s_f(y) \geqq q$, i.e. that $f^{-q}(y) \neq \emptyset$. Then there are $y_1, y_2 \in A$ such that $y_1 \in f^{-q}(y)$, $y_2 = f(y_1)$. We obtain

$$f^{p-1}(x_1) = x_p = f^{p-q}(y) = f^{p-q}(f^q(y_1)) = f^p(y_1) = f^{p-1}(y_2),$$

and from 2.7 (because of $x_1 \in D_g$) we infer that $y_2 \in D_g$. Then $f^{-1}(y_2) = \emptyset$ by 2.5, which is a contradiction. Thus

(2) $s_f(y) < q$.

Further, since $x_1 \in f^{-(q-1)}(x_q)$, we have

(3) $s_f(x_q) \geqq q - 1$.

Let $s \in N$, $s > 1$, $c \in f^{-1}(x_{s+1}) - \{x_s\}$. By (1) and (2) (for $s + 1, s, c$ instead of p, q, y) we have $c \notin D_g$ and $s_f(c) \leqq s - 1$. Put

$$H(t) = \begin{cases} f(t), & \text{if } t \in K_f(x), \\ t & \text{otherwise}. \end{cases}$$

Obviously $H \in \operatorname{End}(A, f)$, thus $H \in \operatorname{End}(A, g)$. Then

$$f(g(c)) = H(g(c)) = g(H(c)) = g(f(c)) = g(x_{s+1}) = x_s,$$

$g(c) \in f^{-1}(x_s)$. Suppose that $d = g(c) \neq x_{s-1}$. According to (2) we have $s_f(d) \leqq s - 2$, and by (3) the relation $s_f(x_{s-1}) \geqq s - 2$ is valid, hence $s_f(d) \leqq s_f(x_{s-1})$. Moreover, $f(d) = f(x_{s-1})$, thus (T2) implies that there exists $H_1 \in \operatorname{End}(A, f)$ such that $H_1(d) = H_1(x_{s-1}) = x_{s-1}$, $H_1(c) = c$. Then $H_1 \in \operatorname{End}(A, g)$, which implies

$$x_{s-1} \neq d = g(c) = g(H_1(c)) = H_1(g(c)) = H_1(d) = x_{s-1},$$

a contradiction. Therefore

(4) $g(c) = x_{s-1}$.

Now let $s > 2$ and suppose that $f^{-1}(c) \neq \emptyset$, $t \in f^{-1}(c)$. From (1) it follows that $t \notin D_g$ and then

$$f(g(t)) = H(g(t)) = g(H(t)) = g(f(t)) = g(c) = x_{s-1},$$

$g(t) \in f^{-1}(x_{s-1})$. If we assume that $u = g(t) \neq x_{s-2}$, we get analogously as above that $s_f(u) \leqq s - 3 \leqq s_f(x_{s-2})$ and $f(u) = x_{s-1} = f(x_{s-2})$, thus there is $H_2 \in \operatorname{End}(A, f)$ such that $H_2(u) = H_2(x_{s-2}) = x_{s-2}$, $H_2(t) = t$, which is a contradiction, since

$$x_{s-2} \neq u = g(t) = g(H_2(t)) = H_2(g(t)) = H_2(u) = x_{s-2}.$$

Therefore $g(t) = x_{s-2}$. Since $s_f(c) \leqq s - 1 \leqq s_f(x_s)$ and $f(c) = x_{s+1} = f(x_s)$, according to (T2) there exists $H_3 \in \operatorname{End}(A, f)$ such that $H_3(c) = H_3(x_s) = x_s$. Then

$$s_g(c) \leqq s_g(H_3(c)) = s_g(x_s)$$

and since $g(c) = x_{s-1} = g(s)$, we infer from (T2) that there is $H_4 \in \operatorname{End}(A, g)$ such that $H_4(c) = H_4(x_s) = x_s$, $H_4(t) = t$. Hence $H_4 \in \operatorname{End}(A, f)$ and we obtain

$$x_s = H_4(c) = H_4(f(t)) = f(H_4(t)) = f(t) = c\,,$$

a contradiction. Thus $f^{-1}(c) = \emptyset$.

Further let $v \in f^{-1}(x_2) - \{x_2\}$. From 2.7 it follows that $v \in D_g$. If $s_g(x_1) \leqq s_g(v)$, then (T1) yields that there exists $H_5 \in \operatorname{End}(A, g)$ such that $H_5(x_1) = H_5(v) = v$, $H_5(x_2) \neq x_2$, which implies

$$x_2 \neq H_5(x_2) = H_5(f(x_1)) = f(H_5(x_1)) = f(v) = x_2\,.$$

Hence $s_g(v) < s_g(x_1)$. Put $H_6(v) = x_1$, $H_6(w) = w$ for each $w \in A - \{v\}$. Then $H_6 \in \operatorname{End}(A, f)$, thus $H_6 \in \operatorname{End}(A, g)$. Therefore $K_g(v) = \{v\}$ and there is $H_7 \in \operatorname{End}(A, g)$ such that $H_7(v) = H_7(x_2) = x_2$, which is a contradiction, since $H_7 \notin \operatorname{End}(A, f)$. Thus

$$f^{-1}(x_2) = \{x_1\}\,.$$

In particular, we have proved that $K_f(x) \subseteq K_g(x)$. Then 3.8 implies that $g^{-1}(K_f(x)) \subseteq A_1^f$ and thus $g^{-1}(K_f(x) \subseteq K_f(x)$ according to 2.1. Hence (i) holds. The definition of the types σ, ϱ immediately implies that (ii) and (iii) are valid.

4. CONNECTED COMPONENTS OF (A, f)

In this section we shall first prove some results of auxiliary character and the theorems concerning connected components of (A, f) and (A, g), where $\operatorname{End}(A, f) = \operatorname{End}(A, g)$.

4.1. Lemma. *Let $v, v' \in D_f$, $v \neq v'$. If $z \in K_f(v)$ and $g(z) = z'$, then $z' \notin K_f(v')$.*

Proof. Suppose that $z' \in K_f(v')$. If $s_f(v) \geqq s_f(v')$, then (T1) implies that there is $H \in \operatorname{End}(A, f)$ such that $H(z) = z$, $H(z') \in K_f(v)$. Thus $H \in \operatorname{End}(A, g)$ and

$$z' \neq H(z') = H(g(z)) = g(H(z)) = g(z) = z'\,,$$

which is a contradiction.

Let $s_f(v) < s_f(v')$. There exist $i, j \in N \cup \{0\}$ such that $f^i(z) = v$, $f^j(z') = v'$. According to (T1) there are $H_1 \in \operatorname{End}(A, f)$ and $y \in K_f(v')$ such that $f^i(y) = v'$ and $H_1(z) = y$, $H_1(z') = z'$. Then we have

$$z' = H_1(z') = H_1(g(z)) = g(H_1(z)) = g(y)\,.$$

Since $s_f(v) < s_f(v')$, there is $v'' \in f^{-1}(v')$ such that $s_f(v) \leqq s_f(v'')$. According to (T1) there exist $H_2 \in \mathrm{End}\,(A, f)$ and $u \in f^{-i}(v'')$ such that $H_2(z) = u$, $H_2(z') = z'$. Then

$$z' = H_2(z') = H_2(g(z)) = g(H_2(z)) = g(u) .$$

Further, $g(u) \neq u$ and $g(y) \neq y$, since in the opposite case we get from 2.0 that $f(u) = u$ or $f(y) = y$, which is a contradiction. Thus

(1) $u \neq g(u) = z' = g(y) \neq y$.

Hence (T2) implies that there is $H_3 \in \mathrm{End}\,(A, g)$ such that $H_3(z') = z'$ and either $H_3(u) = H_3(y) = u$ or $H_3(u) = H_3(y) = y$. Then

$$v' = f^j(z') = f^j(H_3(z')) = H_3(f^j(z')) = H_3(v') .$$

In the former case we have

$$v' = H_3(v') = H_3(f^i(y)) = f^i(H_3(y)) = f^i(u) = v'' \neq v'$$

and in the latter we get

$$v' = H_3(v') = H_3(f^{i+1}(u)) = f^{i+1}(H_3(u)) = f^{i+1}(y) ,$$

which is a contradiction, too, since $f^i(y) = v' \in D_f$.

4.2. Lemma. *Suppose that (A, f) is not of type π and let $A' \subseteq A_2^f$ be a connected component of (A, f). Then*

(i) $g(A') \subseteq A'$,

(ii) $g^{-1}(A') \subseteq A'$.

Proof. First assume that $g(A') \nsubseteq A'$, i.e. there are $v \in A'$ and $y \notin A'$ such that $g(v) = y$. According to 4.1 the case $y \in A_2^f$ yields a contradiction (in 4.1 the possibility $z = v$ was included). Hence $y \in A_1^f$. Then 3.8 implies that $g^{-1}(K_f(y)) \subseteq A_1^f$, which is a contradiction with $v \in g^{-1}(y) \cap A_2^f$. Hence (i) holds.

Now suppose that there are $u \in A'$ and $x \in A'$ such that $g(x) = u$. According to 2.1 we infer that $x \notin A_1^f$ and according to 3.8 we get that $x \notin A_2^f$, which is a contradiction.

4.3. Lemma. *Let the assumption of* 4.2 *hold. Then A' is a connected component of (A, g).*

Proof. It follows from 4.2 that A' is closed with respect to the partial operation g. Suppose that A' is not a connected component of (A, g). Then A' is a union of mutually disjoint sets of the form $K_g(t)$. Thus in the system $\{K_g(t)\colon t \in A'\}$ there are two sets $K_g(r)$ and $K_g(w)$ such that $K_g(r) \neq K_g(w)$ and $f(r_1) = w_1$ for some $r_1 \in K_g(r)$, $w_1 \in K_g(w)$. Denote $A'' = K_g(r)$. Assume that $A'' \subseteq A_1^g$. From 2.2 (interchanging f and g) we obtain $f(A'') \subseteq A''$, a contradiction with the fact that $w_1 \notin A''$. Hence $A'' \subseteq A_2^g$. If we suppose that (A, g) is of type π, then 3.2 implies that (A, f) is of type τ and there is no component in (A, f) which is a subset of A_2^f. Therefore (A, g) is not of type π, $A'' \subseteq A_2^f$ is a connected component of (A, g) and then 4.2 implies that $f(A'') \subseteq A''$, which is a contradiction.

4.4. Lemma. *Let (A, f) be neither of type τ nor of type π and let $A' \subseteq A_1^f$ be a connected component of (A, f). Then A' is a connected component of (A, g).*

Proof. From 2.2 it follows that $g(A') \subseteq A'$ and from 3.8 we obtain that $g^{-1}(A') \subseteq \subseteq A_1^f$. Let $y \in g^{-1}(A')$, $g(y) = x \in A'$. Since $y \in A_1^f$, we get from 2.1 that $g(y) \in K_f(y)$, hence $x \in K_f(y)$, $y \in A'$. Thus

(1) $g(A') \subseteq A'$, $g^{-1}(A') \subseteq A'$.

Suppose that B' is a connected component of (A, g) such that $B' \subseteq A'$. If $B' \subseteq A_2^g$, then 4.3 (f and g interchanged) implies that B' is a connected component of (A, f), thus $B' = A'$. If $B' \subseteq A_1^g$, then from (1) (again with f and g interchanged and for B' instead of A') we get

$$f(B') \subseteq B', \quad f^{-1}(B') \subseteq B',$$

which yields that $B' = A'$.

4.5. Lemma. (a) *Let A' be a connected component of (A, f) and let $(A', f \mid A')$ be of type σ. If $g' = g \mid A' \neq f \mid A' = f'$, then $(A', g') = (A', f')^\varrho$.*

(b) *Let A' be a connected component of (A, f) and let $(A', f \mid A')$ be of type ϱ. If $g' = g \mid A' \neq f \mid A' = f'$, then $(A', g') = (A', f')^\sigma$.*

Proof. (a) Let $(A', f \mid A')$ be of type σ. Then 4.4 implies that A' is a connected component of (A, g). Theorem 3 [1] yields that if $A' \cap D_g = \emptyset$, then $g \mid A' = f \mid A'$. Suppose that $g \mid A' \neq f \mid A'$. Hence $A' \cap D_g \neq \emptyset$ and in view of 3.12, $(A', g') = = (A', f')^\varrho$.

(b) Let $(A', f \mid A')$ be of type ϱ. From 4.3 we infer that A' is a connected component of (A, g).

First assume that $A' \subseteq A_1^g$. Hence $A' \cap D_f \neq \emptyset$ and 2.9 implies (f and g interchanged) that either

(i) $s_g(t) \neq \infty$ for each $t \in A'$

or

(ii) there is $x \in A'$ such that $g(x) = x$ and $s_g(t) \neq \infty$ for each $t \in A' - \{x\}$.

In the case (ii) we get from 2.0 (since $A \neq D_f$) that $f(x) = x$, which is a contradiction with the fact that (A', f') is of type ϱ. Thus (i) is valid and the assumptions of 3.12 are satisfied (for arbitrary $x \in A'$ and with f and g interchaged). This implies that (A', g') is of type σ, (A', f') is of type ϱ and $(A', g') = (A', f')^\sigma$.

Now assume that $A' \subseteq A_2^g$. Since (A', f') is of type ϱ, we have $(A', f') \in \in \varrho(x_1, x_2, \ldots, B_2, B_3, \ldots)$. Suppose that $x_1 \notin D_g$ and $y = g(x_1)$. Since for each $i \in N$ there is $H \in \operatorname{End}(A, f)$ such that $H(x_1) = x_i$, we get that $x_i \notin D_g$ for each $i > 1$. Further, if $B_i \cap D_g \neq \emptyset$, then $x_i \notin D_g$ (since A' is a connected component of (A, g)), but there is $H_1 \in \operatorname{End}(A, f)$ such that $H_1(B_i) = x_i$, which is a contradiction with the relation $H_1 \in \operatorname{End}(A, g)$. Therefore $x_1 \in D_g$. Further, there exists $G \in \operatorname{End}(A, f)$ such that $G(x_1) = x_2$, $G(x_2) = x_3, \ldots$. Then according to (T)

$$s_g(x_1) \leqq s_g(G(x_1)) = s_g(x_2) \leqq s_g(G(x_2)) = s_g(x_3) \leqq \ldots,$$

thus

(1) $s_g(x_1) \leqq s_g(x_i)$ for each $i \in N$.

Moreover, since for $i \in N$, $i > 1$, the relations $x_i \in K_g(x_1)$, $x_1 \in D_g$ are valid, (T3) implies

(2) $s_g(x_i) < s_g(x_1)$ or $s_g(x_i) = s_g(x_1) = \infty$.

From (1) and (2) we obtain that $s_g(x_i) = \infty$ for each $i \in N$. Further, $s_g(b_i) \neq \infty$ for each $b_i \in B_i$, $i \in N$, $i > 1$, since in the opposite case there exists $G_1 \in \operatorname{End}(A, g)$ with $G_1(x_1) = b_i$, a contradiction with $G_1 \in \operatorname{End}(A, f)$. Therefore

(3) there exist mutually distinct elements y_i for $i \in N$ such that $\{y_i\colon i \in N\} \subseteq \subseteq \{x_i\colon i \in N\}$ and $x_1 = y_1$, $g(y_{i+1}) = y_i$ for each $i \in N$.

Suppose that $\{y_i\colon i \in N\} \neq \{x_i\colon i \in N\}$, i.e. there exists $j \in N$, $j > 1$, such that $x_j \notin \{y_i\colon i \in N\}$. Since A' is a connected component of (A, g), we have $x_j \in g^{-k}(y_1)$ for some $k \in N$. Put

$$G_2(t) = \begin{cases} y_m, & \text{if } \ t \in g^{-m}(y_1), \quad m \in N \cup \{0\}, \\ t & \text{otherwise}. \end{cases}$$

Then $G_2 \in \operatorname{End}(A, g)$, hence $G_2 \in \operatorname{End}(A, f)$ and because $G_2(x_j) = y_k \neq x_j$, we get

$$f^{j-1}(x_j) = x_1 = G_2(x_1) = G_2(f^{j-1}(x_j)) = f^{j-1}(G_2(x_j)) = f^{j-1}(y_k).$$

Since $y_k \in \{x_i\colon i \in N\}$, this implies that $y_k = x_j$, which is a contradiction. Hence

(4) $\{y_i\colon i \in N\} = \{x_i\colon i \in N\}$.

Further, for $i \in N$, $i > 1$, the following assertion holds:

(5) $(\forall G_3 \in \operatorname{End}(A, f))\,[(G_3(x_1) = x_i \,\&\, \{G_3^j(x_1)\colon j \in N \cup \{0\}\} = \{x_j\colon j \in N\}) \Rightarrow \Rightarrow i = 2]$.

Namely, if $k \in N$ and $G_3(x_1) = x_{1+k}$, then $G_3(x_{1+k}) = x_{1+2k}$, $G_3(x_{1+2k}) = x_{1+3k}, \ldots$, and from

(6) $x_2 \in \{x_1, x_{1+k}, x_{1+2k}, \ldots\}$

we get that $k = 1$. By an analogous method to that applied when proving (5) we obtain (if we interchange f and g, x_i and y_i) that the following relation is valid:

(5′) $(\forall H_3 \in \operatorname{End}(A, g))\,[(H_3(y_1) = y_i \,\&\, \{H_3^j(y_1)\colon j \in N \cup \{0\}\} = \{y_j\colon j \in N\}) \Rightarrow \Rightarrow i = 2]$.

Therefore (4) and the relations $y_1 = x_1$ and $\operatorname{End}(A, f) = \operatorname{End}(A, g)$ imply that

(7) $y_2 = x_2$

holds. Further we have

(8) $(\forall G_4 \in \operatorname{End}(A, f)\,(G_4(x_1) = x_2 \Rightarrow G_4(x_2) = x_3)$,

(8′) $(\forall H_4 \in \operatorname{End}(A, g))\,(H_4(y_1) = y_2 \Rightarrow H_4(y_2) = y_3)$,

which implies

(9) $y_3 = x_3$.

The induction on $i \in N$ gives

(10) $y_i = x_i$ for each $i \in N$.

Now let $i \in N$, $i > 1$ and $b_i \in B_i$. Put

$$G_5(t) = \begin{cases} x_i, & \text{if } t = b_i, \\ t & \text{otherwise}. \end{cases}$$

Obviously $G_5 \in \text{End}(A, f)$, thus $G_5 \in \text{End}(A, g)$, and (10) implies

$$G_5(g(b_i)) = g(G_5(b_i)) = g(x_i) = x_{i-1}.$$

Since $g(b_i) \neq b_i$, we get $G_5(g(b_i)) = g(b_i)$. Hence we have proved that $g \mid A' = f \mid A'$, which is a contradiction with the assumption.

The above results will be summarized in the following Theorems 4.6 and 4.7. (We shall repeat all the assumptions therein.)

4.6. Theorem. *Let (A, f) be a partial monounary algebra which is neither of type τ nor of type π. If (A, g) is a partial monounary algebra with* End $(A, f) =$ End (A, g), *then $P_f = P_g$.*

Proof. The assertion follows from 4.3 and 4.4.

4.7. Theorem. *Let (A, f) be a partial monounary algebra which is of none of the type $\tau, \pi, \delta, \gamma$. If (A, g) is a partial monounary algebra such that* End $(A, f) =$ End (A, g), *A' is a connected component of (A, f) and $g' = g \mid A' \neq f \mid A' = f'$, then the following assertions hold:*

(a) *If $A' \subseteq A_1^f$ and (A', f') is not of type σ, then $A' \subseteq A_1^g$.*

(b) *If $A' \subseteq A_1^f$ and (A', f') is of type σ, then $A' \subseteq A_2^g$ and $(A', g') = (A', f')^\varrho$.*

(c) *If $A' \subseteq A_2^f$ and (A', f') is not of type ϱ, then $A' \subseteq A_2^g$.*

(d) *If $A' \subseteq A_2^f$ and (A', f') is of type ϱ, then $A' \subseteq A_1^g$ and $(A', g') = (A', f')^\sigma$.*

Proof. 1) Let $A' \subseteq A_1^f$. If $A' \cap D_g = \emptyset$, then $A' \subseteq A_1^g$. If $A' \cap D_g \neq \emptyset$, then 2.9 implies that either

(i) $s_f(t) \neq \infty$ for each $t \in A'$,

or

(ii) there is $x \in A'$ such that $f(x) = x$ and $s_f(t) \neq \infty$ for each $t \in A' - \{x\}$.

In the case (ii) the assumptions of 3.1 are satisfied and hence (A, f) is either of type τ or of type δ, which is a contradiction. Therefore (i) holds and the assumptions of 3.12 are fulfilled, which implies that (A', f') is of type σ.

(a) If $A' \subseteq A_1^f$ and (A', f') fails to be of type σ, then we have already proved that $A' \cap D_g = \emptyset$ and hence $A' \subseteq A_1^g$.

(b) Let $A' \subseteq A_1^f$ and let (A', f') be of type σ. Then 4.5 implies that $(A', g') = (A', f')^\varrho$, $A' \subseteq A_2^g$.

2) Let $A' \subseteq A_2^f$. If $A' \cap D_g \neq \emptyset$, then $A' \subseteq A_2^g$. If $A' \cap D_g = \emptyset$, then 2.9 (with f and g interchanged) implies that either

(iii) $s_g(t) \neq \infty$ for each $t \in A'$,

or

(iv) there is $x \in A'$ such that $g(x) = x$ and $s_g(t) \neq \infty$ for each $t \in A' - \{x\}$.

If (iv) holds, then the assumptions of 3.1 are satisfied (with f and g interchanged) and then (A, g) is of type τ or δ which yields that (A, f) is of type π or γ according to 3.2. This is a contradiction, hence (iii) is valid and the assumptions of 3.12 are fulfilled (if we interchange f and g), thus (A', g') is of type σ and then 4.5 (a) implies that (A', f') is of type ϱ.

(c) If $A' \subseteq A_2^f$ and (A', f') fails to be of type ϱ, then $A' \cap D_g \neq \emptyset$ and $A' \subseteq A_2^g$.

(d) If $A' \subseteq A_2^f$ and (A', f') is of type ϱ, then 4.5 (b) implies that $(A', g') = (A', f')^\sigma$, $A' \subseteq A_1^g$.

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