Adolf Karger

# Affine Darboux motions

*Czechoslovak Mathematical Journal*, Vol. 35 (1985), No. 3, 355–372

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

# AFFINE DARBOUX MOTIONS

ADOLF KARGER, Praha

(Received October 18, 1982)

## I. INTRODUCTION

The Euclidean Darboux motion in $E_3$ has the following properties:

a) All trajectories are plane curves.
b) All trajectories are ellipses or straight line segments.
c) All trajectories are affinely equivalent.
d) It is cylindrical (it splits into a product of a plane motion with a translation).
e) It has infinitely many straight trajectories.

It is easy to show that the condition a) implies all the others, but only b) is equivalent to it. Therefore if we want to generalize the concept of Darboux motion to the $n$-dimensional space and to more general groups, it is not apparent which of the conditions a), ..., e) should be preserved, because the condition a) does not imply the others in a more general situation. In the present paper we define the affine Darboux motions as those having the properties a) and c) because in this case we may describe Darboux motions by a rather simple analytic condition. There are examples of affine Darboux motions with no conical section as a trajectory, which are not cylindrical.

There are also affine Darboux motions with only plane trajectories such that the trajectories are not affinely equivalent. Concerning e) we shall show later that affine Darboux motions have as many straight trajectories as possible, but this property is not characteristic. This will be seen from examples given in the second part of the paper.

## II. GENERAL PROPERTIES OF AFFINE DARBOUX MOTIONS

Let $G$ be a Lie subgroup of the general affine group $GA_n$ in an $n$-dimensional affine space, let $g(t)$ be a one-parametric motion from $G$ of the moving affine space $\bar{A}_n$ in the fixed affine space $A_n$, $t \in I$. Let us fix a base $\bar{R}_0 = \{\bar{A}_0, \bar{f}_1, \ldots, \bar{f}_n\}$ or $R_0 = \{A_0, f_1, \ldots, f_n\}$ in $\bar{A}_n$ or $A_n$, respectively. By a *frame* in $\bar{A}_n$ or $A_n$ we mean any base of the form $\bar{R} = \bar{R}_0 \,.\, g$ or $R = R_0 \,.\, g$ in $\bar{A}_n$ or $A_n$, respectively, for any $g \in G$.

A *moving frame* of the motion $g(t)$ is any pair of frames $(\bar{R}, R)$ such that $g(t)\,[\bar{R}(t)] = R(t)$ with the action $g[\bar{R}_0 g_0] = R_0 g g_0$, where $g, g_0 \in G$. (In what follows, all functions are supposed to be sufficiently differentiable.)

If $(\bar{R}, R)$ is a moving frame of the motion $g(t)$, denote $R' = R\varphi$, $\bar{R}' = \bar{R}\psi$, $\varphi - \psi = \omega$, $\varphi + \psi = \eta$. Then $\varphi, \psi \in \boldsymbol{G}$, where $\boldsymbol{G}$ is the Lie algebra of $G$. Further denote by $\Omega_k$ the operator of the $k$-th derivative of the trajectory of a point $\bar{X} \in \bar{A}_n$ at $X = g(t)\,\bar{X}$, so $X^{(k)} = \Omega_k X$. Then (see [3])

$$(1) \qquad \Omega_1 = \omega\,, \quad \Omega_{k+1} = \varphi\Omega_k - \Omega_k\psi + \Omega_k'\,.$$

**Definition 1.** A motion $g(t)$ from $G$ is called a $D_r$ *motion* (*a motion having the Darboux property of degree* $r$) if there exist unique functions $\alpha_1(t), \ldots, \alpha_r(t)$ such that

$$(2) \qquad \Omega_{r+1} = \sum \alpha_i \Omega_i\,.$$

Remark. The number $r$ is the least number with the property (2) because $\Omega_{k+1} = \sum \beta_j \Omega_j$ with $k < r$ contradicts the unicity of $\alpha_i$.

Remark. The $D_r$ property is a geometrical property of the motion as it does not depend on the choice of the parameter $t$ or of the moving frame. To see this consider a parameter change $t = t(\tau)$ with $\mathrm{d}t/\mathrm{d}\tau \neq 0$. Let us denote by tilda operators obtained with respect to $\tau$ and by a prime the derivative with respect to $\tau$. Then we get $\tilde{\Omega}_1 = \Omega_1 \,.\, t'$. Further, let $\tilde{\Omega}_k = \Omega_k (t')^k + \sum_{j=1}^{k-1} \Omega_j \gamma_j$. Then $\tilde{\Omega}_{k+1} = \Omega_{k+1}(t')^{k+1} + \sum_{j=1}^{k-1} (\Omega_{j+1} t' \gamma_j + \Omega_j \gamma_j') + \Omega_k\, k(t')^{k-1}\, t''$. So $\tilde{\Omega}_{r+1} = \sum_{i=1}^{r} \alpha_i \Omega_i$ implies $(t')^{r+1}\, \Omega_{r+1} = \sum_{i=1}^{r} \beta_i \Omega_i$.

Similarly, if $h = h(t)$ is a change of the moving frame, $(\tilde{\bar{R}}, \tilde{R}) = (\bar{R}h, Rh)$, we get $\tilde{\varphi} = h^{-1}\varphi h + h^{-1}h'$, $\tilde{\psi} = h^{-1}\psi h + h^{-1}h'$ and $\tilde{\Omega}_1 = h^{-1}\Omega_1 h$. Further, let $\tilde{\Omega}_k = h^{-1}\Omega_k h$. Then $\tilde{\Omega}_{k+1} = \tilde{\varphi}\tilde{\Omega}_k - \tilde{\Omega}_k\tilde{\psi} + \tilde{\Omega}_k' = (h^{-1}\varphi h + h^{-1}h')(h^{-1}\Omega_k h) - (h^{-1}\Omega_k h)(h^{-1}h + h^{-1}h') + (h^{-1})'\,\Omega_k h + h^{-1}\Omega_k' h + h^{-1}\Omega_k h' = h^{-1}\Omega_{k+1} h$, where we have used $(h^{-1})' = -h^{-1}h'h^{-1}$.

Let us denote by $M(\boldsymbol{G})$ the associative algebra generated by $\boldsymbol{G}$ in the associative algebra $M_{n+1}$ of matrices of degree $n + 1$.

**Theorem 1.** *Every motion in* $G$ *is a* $D_r$ *motion for some* $r$, *where* $r \leqq \dim M(\boldsymbol{G}) \leqq n^2 + n$.

Remark. The statement of the theorem is to be understood locally in the following sense: Let $g(t)$ be defined on an open interval $I$. Then there is a number $r$ and an open interval $J \subset I$ such that the statement holds on $J$.

**Lemma 1.** *Let* $x_1(t), \ldots, x_m(t), \ldots$ *be vector functions in* $R^n$ *defined on an open interval* $I$. *Then there exist* $m \in N \cup \{0\}$, $m \leqq n$ *and an open interval* $J \subset I$ *such that* $x_{m+1} = \sum_{i=1}^{m} \alpha_i x_i$, *where* $\alpha_i$ *are uniquely defined (differentiable) functions.*

Proof. Denote by $m$ the maximal natural number or zero such that all subdeterminants of order $m+1$ of vectors $x_1, \ldots, x_{m+1}$ equal zero on $I$ and at least one subdeterminant of order $m$ of vectors $x_1, \ldots, x_m$ is different from zero at some $t \in I$. Then $m \leqq n$. Let us further suppose that the nonzero subdeterminant is the determinant consisting of the first $m$ coordinates, det $|x_i^j(t_0)| \neq 0$, $i, j = 1, \ldots, m$. Then this determinant is different from zero on an open interval $J \subset I$.

Let us write

$$x_i = \begin{pmatrix} y_i^j \\ z_i^s \end{pmatrix}, \quad i, j = 1, \ldots, m \; ; \quad s = m+1, \ldots, n \, .$$

To prove our lemma we have to solve the system of linear equations

$$\begin{pmatrix} y_i \\ z_i \end{pmatrix} (\alpha_i) = \begin{pmatrix} y_{m+1} \\ z_{m+1} \end{pmatrix}$$

for the unknown column $(\alpha_i)$. The Equations $(y_i)(\alpha_i) = y_{m+1}$ have a unique (differentiable) solution $(\alpha_i)$. Further,

$$0 = \begin{vmatrix} y_1, \ldots, y_m, y_{m+1} \\ z_1^k, \ldots, y_m^k, z_{m+1}^k \end{vmatrix} = \begin{vmatrix} y_1, \ldots, y_m, \sum \alpha_i y_i \\ z_1^k, \ldots, z_m^k, z_{m+1}^k \end{vmatrix} = \begin{vmatrix} y_1, \ldots, y_m, 0 \\ z_1^k, \ldots, z_m^k, z_{m+1}^k - \sum_{i=1}^m \alpha_i z_i^s \end{vmatrix}$$

for every $k$, $m+1 \leqq k \leqq n$ and so $z_{m+1}^k = \sum_{i=1}^m \alpha_i z_i^k$ and $x_{m+1} = \sum \alpha_i x_i$ with $\alpha_i$ unique.

Proof of Theorem 1. We have $\Omega_1 \in M(G)$, because $\Omega_1 = \omega \in \boldsymbol{G}$. Further, if $\Omega_k \in M(G)$, then $\Omega_{k+1} = \varphi\Omega_k - \Omega_k\psi + \Omega_k' \in M(G)$, as $\varphi, \psi \in \boldsymbol{G}$. So we apply Lemma 1. Finally, $M(G) \subset M(GA_n)$ and dim $M(GA_n) = n^2 + n$.

Remark. For some subgroups of $GA_n$ we really get dim $M(G) < n^2 + n$. For instance, if $G$ is the group of Euclidean motions in $E_2$, we have dim $M(G) = 4$ and $n^2 + n = 6$.

The geometric characterization of the $D_r$ property is given in the following

**Theorem 2.** *A motion in $G$ has the $D_r$ property iff there exists a regular curve in $A_r$ such that the trajectory of any point is an affine image of this curve and $r$ is the least number with this property. If a motion has the $D_r$ property, then the trajectory of any point lies in a subspace of $A_n$ of dimension at most $r$.*

Remark. The last statement of Theorem 2 is void for $r \geqq n$. The detailed formulation of the statement from Theorem 2 which is equivalent to the $D_r$ property is as follows: There exists a curve $V(t)$ in $A_r$ such that for each point $\bar{X} \in \bar{A}_n$ there exists an affine mapping $f: A_r \to A_n$ such that $f(V(t)) = g(t)\,\bar{X}$ for all $t \in I$. Here $f$ is not supposed to be regular as the trajectories may lie in subspaces of different dimensions. By a *regular curve* we mean a curve $X(t)$ such that the $r$-th osculating space has dimension $r$ at each $t(X', \ldots, X^{(r)}$ are linearly independent for each $t)$.

Remark. Theorem 2 shows that the $D_r$ property is an affine property of a motion in $G$. This means that the $D_r$ property is preserved if the group $G$ is imbedded in $GA_n$. It follows that if we find all $D_r$ motions in $GA_n$, we also know all $D_r$ motions for all subgroups of $GA_n$.

Proof of Theorem 2. We have $X^{(r+1)} - \sum_{i=1}^{r} \alpha_i X^{(i)} = 0$ and so the trajectory of $\bar{X} \in \bar{A}_n$ can be expressed as $X(t) = A_0 + \sum_{i=1}^{r} m_i(t) f_i$, where $A_0$ is a point and $f_i$ are constant vectors (not necessarily independent) and functions 1, $m_1(t), \ldots, m_r(t)$ form a fundamental system of solutions of the equation $y^{(r+1)} - \sum_{i=1}^{r} \alpha_i y^{(i)} = 0$.

Hence we see that if the motion has the $D_r$ property, all trajectories lie in subspaces of dimension at most $r$. Each trajectory is an affine image of the curve $V(t) = B_0 + \sum_{i=1}^{r} m_i(t)\, e_i$, where $\{B_0, e_1, \ldots, e_r\}$ is a base in $A_r$. The curve $V(t)$ is regular in our sense, as the Wronski determinant of functions $m'_1(t), \ldots, m'_r(t)$ is different from zero, because they come from the fundamental system of solutions of a differential equation.

Conversely, let $V(t) = B_0 + \sum_{i=1}^{s} m_i(t)\, e_i$ be a regular curve in $A_s$ such that each trajectory is its affine image. Consider the system of linear equations $m^j_{(s+1)} = \sum_{i=1}^{s} \alpha_i m_j^{(i)}$, $j = 1, \ldots, s$, for unknowns $\alpha_i$. As $\det m_j^{(i)} \neq 0$, this system has a unique solution and so the functions $1, m_1(t), \ldots, m_s(t)$ form a fundamental system of solutions of $y^{(s+1)} - \sum_{i=1}^{s} \alpha_i y^{(i)} = 0$ and each trajectory satisfies this equation. Finally, $s < r$ contradicts the unicity of the functions $\alpha_i$.

Remark. Theorem 2 is a generalization of the known fact that cycloids (as trajectories of a cycloidal motion in plane) are affine images (projections) of a helix. In another words, a cycloidal plane motion is a $D_3$ motion. It is not the only Euclidean $D_3$ plane motion; the $D_3$ plane motions are characterized as motions with one straight trajectory (see [6]).

**Theorem 3.** *A motion $g(t) \in G$ has the $D_r$ property iff $g(t)$ can be expressed as $g(t) = \sum_{i=0}^{r} m_i(t)\, M_i$, where $M_i$ are constant matrices, $m_0 = 1$, $\det |m_i^{(j)}| \neq 0$ and $g(t) \in G$.*

Proof. If $g(t)$ is a motion then $X(t) = g(t)\, \bar{X}$ is the trajectory of $\bar{X}$. The operator of the $k$-th derivative of the trajectory of $\bar{X}$ is therefore expressed by $\Omega_k = g^{(k)}$ in a fixed frame in $A_n$. So the $D_r$ property also means that $g(t)$ satisfies the differential equation

$$g^{(r+1)} - \sum_{i=1}^{r} \alpha_i g^{(i)} = 0$$

and the statement follows.

**Definition 2.** A motion $g(t) \subset G$ is called an $F_s$ *motion*, if the trajectory of any point lies in a subspace of dimension $s$ and $s$ is the least number with this property.

Remark. Each $F_s$ motion is also (locally) a $D_r$ motion for some $r$, where of course $s \leqq r$. A natural question to ask at this moment is how large $r$ may be for a given $s$ in a given group $G$. For instance, we shall show later that for $G = GA_n$, $s = 1$ the answer is $1 \leqq r \leqq n + 1$. If the group $G$ is a proper subgroup of $GA_n$ we may get stronger results. For instance, for Euclidean motions we get for $n = 3$ and $s = 2$ that either $r = 2$ or the motion is a plane motion (and $r \leqq 4$). The problem of generalization of this particular result to any $n$ is still open (see [3]).

**Definition 3.** We say that the motion $g(t)$ in $G$ *splits*, if there exist nontrivial subgroups $G_1$ and $G_2$ of $G$ such that $G_1 \cap G_2 = \{e\}$, $g_1 g_2 = g_2 g_1$ for all $g_1 \in G_1$, $g_2 \in G_2$ and $g(t) = g_1(t)\, g_2(t)$ with $g_1(t) \in G_1$, $g_2(t) \in G_2$ for all $t \in I$.

Remark. The basic problem concerning Darboux motions is to describe how $D_r$ motions split into $D_r$ factors and to find all these factors in a given group for small $r$. (For large $r$ we are near to the general motion and the problems become complicated.)

For instance, it is known that all $F_2$ similarity motions in $E_3$ split into the product of a plane motion with a motion in a straight line (see [4]). Only partial results are known about splitting of $F_2$ or $D_2$ affine motions (see [5]). All non-splitting Euclidean $D_2$ motions in $E_n$ are found in [3]. Some examples are also given below in this paper.

We shall now investigate some other geometric properties of Darboux motions.

**Definition 4.** Let $v_1, \ldots, v_r$ be vectors in $A_n$, let $X_1, \ldots, X_r$ be columns of their coordinates in some base $R_0 = \{A_0, e_i\}$, so $v_i = R_0 X_i$ (the first coordinate of a vector is zero and is omitted). Let $|X_1, \ldots, X_r|^j$, $j = 1, \ldots, \binom{n}{p}$ denote all subdeterminants of order $r$ of columns $X_1, \ldots, X_r$.

**Lemma 2.** *Let $A$ be a matrix of order $n$. Then there exist numbers $\alpha_{jk}$ such that*

$$|X_1, \ldots, AX_i, \ldots, X_r|^j = \sum_{k=1}^{\binom{n}{r}} \alpha_{jk} |X_1, \ldots, X_r|^k$$

*for all columns $X_i$.*

Proof. $\sum_{i=1}^{r} |X_1, \ldots, AX_i, \ldots, X_r|^j$ is a skew-symmetric $r$-linear function of $X_i$. Let $f$ be any skew-symmetric $r$-linear function of vectors, let $e_1, \ldots, e_n$ be a base. Then $X_i = a_i^\alpha e_\alpha$, $\alpha = 1, \ldots, n$; $i = 1, \ldots, r$ with the summation omitted. Further,

$$\begin{aligned} f(X_1, \ldots, X_r) &= f(a_1^{\alpha_1} e_{\alpha_1}, \ldots, a_r^{\alpha_r} e_{\alpha_r}) = a_1^{\alpha_1} \ldots a_r^{\alpha_r} f(e_{\alpha_1}, \ldots, e_{\alpha_r}) = \\ &= \sum_{\alpha_1 < \ldots < \alpha_r} f(e_{\alpha_1}, \ldots, e_{\alpha_r}) \sum_{\Pi} \operatorname{sgn} \Pi a_1^{\alpha_1} \ldots a_r^{\alpha_r} = \\ &= \sum_{\alpha_1 < \ldots < \alpha_r}^{n} f(e_{\alpha_1}, \ldots, e_{\alpha_r}) \, |X_1, \ldots, X_r|^{\alpha_1 < \ldots < \alpha_r} . \end{aligned}$$

Let $g(t)$ be a motion in $G$, let $\bar{X} \in \bar{A}$ be a fixed point. For the coordinates $X$ of $\bar{X}$ in the frame $\bar{R}$ we have

(3) $$X' = -\psi X .$$

(We have $\bar{X} = \bar{R}X$ and so $\bar{X}' = 0 = \bar{R}'X + \bar{R}X' = \bar{R}(\psi X + X')$.)

If $F(x_i, t)$, $i = 1, \ldots, n$ is a function of $n + 1$ variables, denote

$$\frac{\mathrm{d}F}{\mathrm{d}\psi} = \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} \psi X , \quad \text{where} \quad \frac{\partial F}{\partial x} = \left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right).$$

**Lemma 3.**

$$\frac{\mathrm{d}}{\mathrm{d}\psi} |X', \ldots, X^{(r)}|^j = \sum_{i=1}^{\binom{n}{r}} \alpha_{ij}(t) \, |X', \ldots, X^{(r)}|^i + |X', \ldots, X^{(r-1)}, X^{(r+1)}|^j ,$$

*where $X^{(k)}$ denotes the k-th derivative of the trajectory of $\bar{X}$ at $X$.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}\psi} |X', \ldots, X^{(r)}|^j = \frac{\mathrm{d}}{\mathrm{d}\psi} |\Omega_1 X, \ldots, \Omega_r X|^j =$$

$$= \sum_{i=1}^{r} |\Omega_1 X, \ldots, \Omega_i' X - \Omega_i \psi X, \ldots, \Omega_r X|^j =$$

$$= \sum_{i=1}^{r} |\Omega_1 X, \ldots, \Omega_{i+1} X - \varphi \Omega_i X, \ldots, \Omega_r X|^j =$$

$$= |\Omega_1 X, \ldots, \Omega_{r-1} X, \Omega_{r+1} X|^j - \sum_{i=1}^{r} |\Omega_1 X, \ldots, \varphi \Omega_i X, \ldots, \Omega_r X|^j ,$$

because $\Omega_i' - \Omega_i \psi = \Omega_{i+1} - \varphi \Omega_i$. The statement follows now from Lemma 2.

**Lemma 4.** *Let a system of linear differential equations* (3) *be given. Let $F_i(X, t)$ be functions of $n + 1$ variables and let $\gamma_{ij}(t)$ $i, j = 1, \ldots, k$ be functions such that*

(4) $$\frac{\mathrm{d}}{\mathrm{d}\psi} F_i = \sum_{i,j=1}^{k} \gamma_{ij} F_j$$

*for all $X$. Then, if a solution of* (3) *satisfies $F_i(X(t_0), t_0) = 0$ at some $t_0$, the identity $F_i(X(t), t) = 0$ holds for all $t$.*

Proof. Throughout the proof we shall use the matrix notation, so $(\mathrm{d}/\mathrm{d}\psi) F = \gamma F$. Consider the system of ordinary differential equations $Y' = \gamma Y$. Denote by $f$ the matrix whose columns form a fundamental system of solutions of this equations. Then $f$ is a regular matrix and

$$\frac{\mathrm{d}}{\mathrm{d}\psi} (f^{-1} F) = \frac{\mathrm{d}}{\mathrm{d}t} (f^{-1}) \, F + f^{-1} \frac{\mathrm{d}}{\mathrm{d}\psi} F = \frac{\mathrm{d}}{\mathrm{d}t} f^{-1} . F + f^{-1} \gamma F = 0$$

because $(d/d\psi) F = \gamma F$ and $f' = \gamma f$ implies $(f^{-1})' = -f^{-1}\gamma$.

This shows that the functions $f^{-1}F$ are the first integrals of (3). So if $X(t)$ is a solution of (3), we have $f^{-1}(t) F(X(t), t) = C$, where $C$ are constants. If now $F(X(t_0), t_0) = 0$, we have $C = 0$ and $F(X(t), t) = 0$ for all $t$.

**Theorem 4.** *Let $g(t)$ be a motion in G with the property that $|X', \ldots, X^{(r-1)}, X^{(r+1)}|$ is a consequence of $|X', \ldots, X^{(r)}|$ ($|X', \ldots, X^{(r-1)}, X^{(r+1)}|^i = \sum \beta_{ij}(t) |X', \ldots, X^{(r)}|^j$). Let the trajectory of any point $\bar{X}$ at some $t_0$ satisfy $|X', \ldots, X^{(r)}|^i = 0$ for all i and $|X', \ldots, X^{(r-1)}|^j \neq 0$ for some j. Then the trajectory of X lies in a subspace of $A_n$ of dimension $r - 1$.*

Proof. According to Lemma 4 we have $|X', \ldots, X^{(r)}|^i = 0$ for the whole trajectory of $X$. From the assumptions we get $X^{(r)} = \sum_{i=1}^{r-1} \alpha_i X^{(i)}$ in an interval around $t_0$ and the trajectory is an $r - 1$ dimensional curve.

**Corollary.** *Let $g(t)$ be a $D_r$ motion. Then the trajectory of any point satisfying $|X', \ldots, X^{(r)}|^i = 0$ for all i at some $t_0$ and $|X', \ldots, X^{(r-1)}|^j \neq 0$ for some j and all $t \in I$, lies in a subspace of dimension $r - 1$ and in no subspace of smaller dimension on I.*

Proof. We have $X^{(r)} = \sum_{i=1}^{r-1} \alpha_i X^i$ on $I$ with $X', \ldots, X^{(r-1)}$ linearly independent.

A similar corollary of Theorem 4 may be expressed as follows: *If $g(t)$ is a $D_r$ motion and the r-th osculating space of the trajectory of a point $\bar{X}$ has dimension less then r at $t_0$ and the $(r - 1)$-st osculating space at $t_0$ has dimension $r - 1$, then at least a piece of the trajectory of $\bar{X}$ around $t_0$ is an $r - 1$ dimensional curve.*

Remark. From Corollary of Theorem 4 we get for instance that if a $D_1$ motion has an instantaneous pole, then this pole remains fixed during the motion. Similarly, $D_2$ motions have the property that any point of the set of inflextion points, which is not a pole, has a straight trajectory.

Remark. The necessary condition from Theorem 4 is not sufficient to characterize the $D_r$ motions in general, because there exist motions with the property that all points of $|X', \ldots, X^{(r)}| = 0$ have trajectories in subspaces of dimension less than $r$, which are not $D_r$ motions. An example will be given later.

Similarly, the condition that $|X', \ldots, X^{(r)}| = 0$ implies $|X', \ldots, X^{(r-1)}, X^{(r+1)}| = 0$ does not yield that $(d/d\psi) |X', \ldots, X^{(r)}|$ is a consequence of $|X', \ldots, X^{(r)}|$ as the set $|X', \ldots, X^{(r)}| = 0$ may be empty.

The condition that $|X', \ldots, X^{(r)}| = 0$ implies $|X', \ldots, X^{(r-1)}, X^{(r+1)}| = 0$ is a necessary condition for all points of this set to have its trajectory in a subspace of dimension less then $r$. In many cases this condition is also sufficient. Let for instance $|X', \ldots, X^{(r)}|^i = 0$ be given by a single equation and let the solution set have sufficiently many points to determine its equation (it is an algebraic equation). This means that if $F(X) = 0$ for all points of $|X', \ldots, X^{(r)}| = 0$ with $F(X)$ algebraic, then $F(X) = |X', \ldots, X^{(r)}| . G(X, t)$, where $G(X, t)$ is a polynomial in $x_i$'s with its coefficients being functions of $t$. If this is the case and $|X', \ldots, X^{(r)}| = 0$ implies $|X', \ldots, X^{(r-1)}, X^{(r+1)}| = 0$, we get $\mathrm{d}|\mathrm{d}\psi\, |X', \ldots, X^{(r)}| = \alpha(t)\, |X', \ldots, X^{(r)}|$, as $(\mathrm{d}/\mathrm{d}\psi)\, |X', \ldots, X^{(r)}|$ is a polynomial of degree not higher than the degree of $|X', \ldots, X^{(r)}|$ and the necessary condition becomes also sufficient.

Such situation occurs for instance for $r = n$, $n$ odd and $|X', \ldots, X^{(r)}| = 0$ irreducible.

Similarly, let $M$ be a set given as a solution of some algebraic equations $G_i(X, t) = 0$ such that all points of $M$ satisfy $|X', \ldots, X^{(r)}| = 0$. Then the necessary condition for all points of $M$ to have trajectories in subspaces of dimension less than $r$ is $(\mathrm{d}/\mathrm{d}\psi)\, G_i(X) = 0$ on $M$. This condition is also sufficient, if it implies $(\mathrm{d}/\mathrm{d}\psi)\, G_i = \sum \alpha_{ij} G_j$ for all $X$. Such situation occurs in the case of $|X', \ldots, X^{(r)}| = 0$ reducible.

**Theorem 5.** *Let $g(t)$ be a $D_r$ motion in $G$. Further, let $X(t)$ be an isolated solution of $|X', X^{(k)}| = 0$, $k = 2, \ldots, r$ for each $t$, with $X' \neq 0$. Then if $X$ is not a singular point of all conditions $|X', X^{(k)}|^j = 0$, it has a straight trajectory.*

Proof. Let us denote by $F_i(X) = 0$, $i = 1, \ldots, s$ all the equations $|X', X^{(k)}| = 0$. Then we have for the rank

$$r\left(\frac{\partial F}{\partial x_j}(X)\right) = n\,,$$

because if

$$0 < r\left(\frac{\partial F_i}{\partial x_j}(X)\right) < n\,,$$

it is possible to express some of the variables as functions of the others and $X$ is not an isolated solution. If

$$r\left(\frac{\partial F_i}{\partial x_j}(X)\right) = 0\,,$$

$X$ is a singular point for all equations $F_i(X) = 0$. Further, we have $F_i(X(t), t) = 0$ and so

$$\sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j} x'_j + \frac{\partial F_i}{\partial t} = 0\,.$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}\psi} |X', X^{(k)}| = \frac{\mathrm{d}}{\mathrm{d}\psi} |\Omega_1 X, \Omega_k X| = |\Omega'_1 X - \Omega_1 \psi X, \Omega_k X| + |\Omega_1 X, \Omega'_k X - \Omega_k \psi X| =$$

$$= |\Omega_2 X - \varphi\Omega_1 X, \Omega_k X| + |\Omega_1 X, \Omega_{k+1} X - \varphi\Omega_k X| =$$

$$= |\Omega_2 X, \Omega_k X| + |\Omega_1 X, \Omega_{k+1} X| - |\varphi\Omega_1 X, \Omega_k X| - |\Omega_1 X, \varphi\Omega_k X|\,.$$

As $X'' = \alpha X'$ at $X$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\psi} |X', X^{(k)}| = 0$$

at $X$ which means

$$\frac{\partial F_i}{\partial t} - \frac{\partial F_i}{\partial x} \psi X = 0 .$$

Both the conditions together give

$$\frac{\partial F_i}{\partial x} (X' - \psi X) = 0$$

and so $X' = -\psi X$.

**Lemma 5.** *Let $g(t)$ be a $D_r$ motion in $G$. Then $\Omega_m = \sum_{i=1}^{r} \alpha_{mi}\Omega_i$ for all $m > r$.*

Proof. The lemma is easily proved by induction.

**Theorem 6.** *Let $g(t)$ be a real analytic $D_r$ motion in $G$. Let for a point $\bar{X} \in \bar{A}_n$ and $1 \leqq k \leqq r$*

$$|X', \ldots, X^{(k)}| = |X', \ldots, X^{(k-1)}, X^{(k+1)}| \ldots = |X', \ldots, X^{(k-1)}, X^{(r)}| = 0$$

*at $t = t_0$ with $X^{(i)} = (g(t)\bar{X})^i$. Then the trajectory of $X$ lies in a subspace of dimension less then $k$.*

Proof. According to Lemma 5 we have $X^{(m)} = \sum_{i=1}^{r} a_i^m X^{(i)}$ for $m > r$. Let $s \geqq 0$ be the maximal number such that $|X', \ldots, X^{(s)}| \neq 0$ at $t_0$. Then $s < k$. The trajectory of the point $X$ in the base $\{X, X', \ldots, X^{(s)}, Y_{s+1}, \ldots, Y_n\}$ ($Y_k$ arbitrary) will have the following Taylor expansion (put $t_0 = 0$):

$$X(t) = X(0) + \sum_{m=1}^{\infty} \frac{1}{m!} X^{(m)}(0)\, t^m = \sum_{m=1}^{\infty} \frac{t^m}{m!} \left(\sum_{i=1}^{r} a_i^m X^{(i)}\right) = \sum_{i=1}^{r} \left(\sum_{m=1}^{\infty} a_i^m \frac{t^m}{m!}\right) X^{(i)} ,$$

where $a_i^m = \delta_i^m$ for $1 \leqq m \leqq r$ and $X^{(i)}$ may be expressed by $X', \ldots, X^{(s)}$. This proves the statement.

Remark. In some groups all $D_r$ motions are real analytic for some $r$. This is always the case when there are no arbitrary functions in the expression of a general $D_r$ motion. Such motions are solutions of an autonomous system of differential equations with algebraic (and therefore analytic) right hand sides and hence they are given by analytic functions. For instance, all $D_2$ motions in $E_n$ are analytic. In the general case, if the above mentioned arbitrary functions are analytic, the $D_r$ motion is analytic as well. A $D_r$ motion is analytic also in the case when the functions $\alpha_i$ in $\Omega_{r+1} = \sum_{i=1}^{r} \alpha_i \Omega_i$ are analytic (see Theorem 3).

The author does not know whether Theorem 6 remains valid with the analyticity condition removed. In the differentiable case we are able to prove only a wekaer theorem, which is presented below.

**Lemma 6.**

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\,|X^{(i_1)},\ldots,X^{(i_s)}|^i = \sum_{\alpha=1}^{s}|X^{(i_1)},\ldots,X^{(i_\alpha+1)},\ldots,X^{(i_s)}|^i + \sum_{j=1}^{\binom{n}{s}}\alpha_{ij}|X^{(i_1)},\ldots,X^{(i_s)}|^j\,.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\,|X^{(i_1)},\ldots,X^{(i_s)}|^i = \sum_{\alpha=1}^{s}|X^{(i_1)},\ldots,(\Omega_{i_\alpha+1}-\varphi\Omega_{i_\alpha})\,X,\ldots,X^{(i_s)}|^i =$$

$$= \sum_{\alpha=1}^{s}|X^{(i_1)},\ldots,X^{(i_\alpha+1)},\ldots,X^{(i_s)}|^i - \sum_{\alpha=1}^{s}|X^{(i_1)},\ldots,\varphi X^{(i_\alpha)},\ldots,X^{(i_s)}|^i =$$

$$= \sum_{\alpha=1}^{s}|X^{(i_1)},\ldots,X^{(i_\alpha+1)},\ldots,X^{(i_s)}|^i - \sum_{j=1}^{\binom{n}{s}}\alpha_{ij}|X^{(i_1)},\ldots,X^{(i_s)}|^j$$

according to Lemma 1.

**Theorem 7.** *Let $g(t)$ be a $D_3$ motion in $G$. Then the points satisfying $|X', X''| = 0$, $|X', X'''| = 0$, $X' \neq 0$ have straight trajectories.*

Proof. Denote $|X', X''|^j = F_j$, $|X', X'''|^j = G_j$, $|X'', X'''|^j = H_j$. Then

$$\frac{\partial}{\partial\psi}F_j = G_j + \sum\alpha_{jk}F_k\,,$$

$$\frac{\partial}{\partial\psi}G_j = (1+\alpha_3)\,H_j + \alpha_2F_j + \sum\beta_{jk}G_k\,,$$

$$\frac{\partial}{\partial\psi}H_j = -\alpha_1F_j + \alpha_3H_j + \sum\gamma_{jk}H_k\,,$$

where $\Omega_4 = \sum_{i=1}^{3}\alpha_i\Omega_i$ and $\alpha_{jk}, \beta_{jk}, \gamma_{jk}$ are functions. If now $F_j = 0$ and $G_j = 0$ at some $X$ and $X' \neq 0$, then $X''$ and $X'''$ are linearly dependent and so $H_j = 0$. From Lemma 4 we get that $|X', X''| = 0$ around $t_0$ and the trajectory is on a straight line.

## III. EXAMPLES

**Example 1.** Let us consider an affine motion $g(t)$ in the affine plane, which is not centroaffine. Let us suppose that the condition $|X', X''| = F_2(X)$ for $g(t)$ is nontrivial and that all points satisfying $F_2(X) = 0$ have straight trajectories. We shall investigate conditions under which $g(t)$ is a $D_2$ motion.

A necessary condition for all points of $F_2(X) = 0$ to have straight trajectories is that $F_2(X) = 0$ implies $F_3(X) = |X', X'''| = 0$.

If $F_2(X) = 0$ has infinitely many points and is not a twice counted straight line, this condition implies that $F_3(X) = \alpha(t)\,F_2(X)$ and this latter condition is also sufficient. The remaining cases where $F_2(X) = 0$ is empty, a one point set or a twice counted straight line, must be treated separately.

Let us also remark that $F_2(X) = 0$ contains all instantaneous poles of $g(t)$. So if $F_2(X) = 0$ has only one point and the motion has a pole, the motion is a centroaffine motion.

Case I).

$$\omega = \begin{pmatrix} 0, & 0 \\ 0, & \omega_1 \end{pmatrix},$$

where $\omega_1$ is a regular $2 \times 2$ matrix and $F_2(X) = 0$ has at least two points and is not a straight line. ($F_2(X) = 0$ is nonempty as the motion has a pole.) Denote

$$\Omega_2 = \begin{pmatrix} 0, & 0 \\ \vartheta_2, & \Theta_2 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0, & 0 \\ \vartheta_3, & \Theta_3 \end{pmatrix}.$$

Then $F_2(X) = |\omega_1 x, \vartheta_2 + \Theta_2 x|$, $F_3(X) = |\omega_1 x, \vartheta_3 + \Theta_3 x|$, where $x = (x_1, x_2)^{\mathrm{T}}$. Let $\omega_1 x = y$, $y = (y_1, y_2)^{\mathrm{T}}$. Then $F_2(X) = |y, \vartheta_2 + \Theta_2 \omega_1^{-1} y|$ and $F_2(X) = 0$ is $(\vartheta_2)_1\, y_1 - (\vartheta_2)_1\, y_2 + c_{21} y_1^2 + (c_{22} - c_{11})\, y_1 y_2 - c_{12} y_2^2 = 0$ where $\Theta_2 \omega_1^{-1} = (c_{ij})$; $i, j = 1, 2$, and similarly for $F_3(X)$. This gives $\vartheta_3 = \alpha\vartheta_2$, $\Theta_3 \omega_1^{-1} = \alpha\Theta_2 \omega_1^{-1} + \beta E$ for some functions $\alpha(t)$, $\beta(t)$ and the motion is a $D_2$ motion.

If $F_2(X) = 0$ is a twice counted straight line, then this line passes through the origin and so $\vartheta_2 = 0$. From (1) we get $\vartheta_2 = \omega_1 \psi_0$ for

$$\psi = \begin{pmatrix} 0, & 0 \\ \psi_0, & \psi_1 \end{pmatrix}$$

and so $\psi_0 = 0$. $\varphi - \psi = \omega$ gives $\varphi_0 = 0$ and the motion is a centroaffine motion.

If $F_2(X) = 0$ is only a straight line (quadratic terms vanish), we have $\Theta_2 \omega_1^{-1} = \mu E$ and so $\Theta_2 = \mu\omega_1$. Then $\Theta_3 = (\mu^2 + \mu')\,\omega_1$ and so $F_3(X) = 0$ is also only a straight line (quadratic terms vanish as well). Then we get $\vartheta_3 = \alpha\vartheta_2$, as $\vartheta_2 \neq 0$ and they must be linearly dependent. So we may write $\Omega_3 = \alpha\Omega_2 + (\mu^2 + \mu' - \alpha\mu)\,\Omega_1$ and the motion is a $D_2$ motion, as $F_2$ is nontrivial ($\Omega_2$ and $\Omega_1$ are linearly independent, as $\vartheta_2 \neq 0$ and $\omega_0 = 0$).

Case II). $\omega_1$ is singular. Here we have two different possibilities for the Jordan normal form of $\omega_1$:

$$\text{a)}\quad \omega_1 = \begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix}, \qquad \text{b)}\quad \omega_1 = \begin{pmatrix} 0, & \lambda \\ 0, & 0 \end{pmatrix}, \quad \lambda \neq 0.$$

In these two cases we have to carry out all necessary computations. The details are left out as uninteresting. As the result we get that if $F_2(X) = 0$ is not empty, the motion has the $D_2$ property. So we have

**Theorem 8.** *Let $g(t)$ be an affine plane motion which is not centroaffine and* *such that $F_2(X)$ is not trivial and $F_2(X) = 0$ is not empty. Then if each point of $F_2(X)$ has a straight trajectory, $g(t)$ has the $D_2$ property. (If $\omega_1$ is regular, then $F_2(X) = 0$ is nonempty.)*

Remark. If the assumptions of Theorem 8 are satisfied, then each point of $F_2(X)$ has a straight trajectory iff $F_2(X) = 0$ implies $F_3(X) = 0$. We may also say that if an affine but not centroaffine motion has infinitely many inflexion points of order two at each instant, it has infinitely many straight trajectories.

**Example 2.** We shall describe all affine motions in $A_n$ which have only straight trajectories ($F_1$ motions).

Let $g(t)$ be a motion in $A_n$. Let us denote

$$\omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0, & 0 \\ \vartheta, & \Omega \end{pmatrix}.$$

Then $g(t)$ has only straight trajectories if the equation $|\omega X, \Omega_2 X| = 0$ is satisfied for all $X \in \bar{A}_n$. If we write $X = (1, x)^{\mathrm{T}}$, we get $|\omega_0 + \omega_1 x, \vartheta + \Theta x| = 0$ and therefore $|\omega_1 x, \Theta x| = 0$ must be satisfied for all $x$.

a) Classification of vector parts

We shall find $\omega_1$, $\Theta$ for all $F_1$ motions with $\omega_1 \neq 0$. Let us write

$$\omega_1 = \begin{pmatrix} \omega_2, & 0 \\ 0, & J \end{pmatrix},$$

where $\omega_1$ is in the normal (real) Jordan form, $\omega_2$ is regular and $J$ corresponds to the eigenvalue 0. ($\omega_1$ can be given the normal Jordan form in a suitable moving frame.)

Let us denote

$$x = \begin{pmatrix} \omega_2^{-1}, & 0 \\ 0, & E \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \Theta \begin{pmatrix} \omega_2^{-1}, & 0 \\ 0, & E \end{pmatrix} = \begin{pmatrix} A, & B \\ C, & D \end{pmatrix}.$$

We have to solve the equations

$$\begin{vmatrix} y, & Ay + Bz \\ Jz, & Cy + Dz \end{vmatrix} = 0 \quad \text{for all } y, z\,.$$

Let us use the indices $i, j$ for $y$ and $\alpha$, $\beta$ for $z$.

First, let rank $r(\omega_1) > 1$ (this condition implies that if deg $\omega_2 = 0$, then deg $J > 2$ and if deg $\omega_2 = 1$, deg $J > 1$). The case $r(\omega_1) = 1$ will be discussed separately.

i) Deg $\omega_2 = 1$. Consider the subdeterminant

$$\begin{vmatrix} y_1, & A_{11} y_1 + B_{1\beta} z_\beta \\ \varepsilon_\alpha z_{\alpha+1}, & C_{\alpha 1} y_1 + D_{\alpha\beta} z_\beta \end{vmatrix} = 0 \quad \text{with} \quad \varepsilon_\alpha = 0 \quad \text{or} \quad 1\,.$$

$$C_{\alpha 1} y_1^2 - \varepsilon_\alpha B_{1\beta} z_{\alpha+1} z_\beta + D_{\alpha\beta} y_1 z_\beta - A_{11} y_1 \varepsilon_{\alpha+1} = 0\,.$$

We may suppose $\varepsilon_1 = 1$, because $r(J) \geqq 1$ and so $C_{\alpha 1} = B_{1\alpha} = 0$.

ii) Deg $\omega_2 \geqq 2$. From

$$\begin{vmatrix} y_1, & A_{1i}y_i + B_{1\beta}z_\beta \\ y_\kappa, & A_{kj}y_j + B_{k\gamma}z_\gamma \end{vmatrix} = 0 \quad \text{we get} \quad y_1 z_\gamma B_{k\gamma} - B_{1\beta}z_\beta y_k = 0$$

and so $B_{i\alpha} = 0$.

From $$\begin{vmatrix} y_1, & A_{1i}y_i \\ \varepsilon_\alpha z_{\alpha+1}, & C_{\alpha j}y_j + D_{\alpha\beta}z_\beta \end{vmatrix} = 0 \quad \text{we get} \quad C_{\alpha i}y_1 y_i = 0 \quad \text{and} \quad C_{\alpha i} = 0\,.$$

So we always have $B = 0$, $C = 0$.

**Lemma 7.** *Let $A$ be regular, $|y, Ay| = 0$ for all $y$. Then $A = \lambda E$.*

Proof. Consider
$$\begin{vmatrix} y_i, & A_{ik}y_k \\ y_j, & A_{jl}y_l \end{vmatrix} = 0$$
for $i \neq j$.

It remains to prove that $D = \lambda J$. We have to consider separate cases similarly as above.

i) deg $\omega_2 = 0$. Then deg $J > 2$, $r(J) \geqq 2$ and hence $J$ has at least two nonzero elements; so, let $J_{12} = 1$ and $J_{\alpha,\alpha+1} = 1$ for some $\alpha > 1$. Then

$$\begin{vmatrix} z_2, & D_{1\beta}z_\beta \\ z_{\alpha+1}, & D_{\alpha\gamma}z_\gamma \end{vmatrix} = 0\,,$$

which is $D_{\alpha\gamma}z_2 z_\gamma - D_{1\beta}z_{\alpha+1}z_\beta = 0$. Because $z_2 z_\gamma = z_{\alpha+1}z_\beta$ only if $\beta = 2$ and $\gamma = \alpha + 1$, we get

$D_{\alpha\gamma} = 0$ for $\gamma \neq \alpha + 1$, $D_{1\beta} = 0$ for $\beta \neq 2$ and $D_{\alpha,\alpha+1} = D_{12}$. If $J$ has only zero's in the $\alpha$-th row, we get

$$\begin{vmatrix} z_2, & D_{12}z_2 \\ 0, & D_{\alpha\beta}z_\beta \end{vmatrix} = D_{\alpha\beta}z_2 z_\beta = 0 \quad \text{and} \quad D_{\alpha\beta} = 0 \quad \text{for all } \beta\,.$$

ii) deg $\omega_2 \geqq 1$. Then

$$\begin{vmatrix} y_1, & \lambda y_1 \\ \varepsilon_\alpha z_{\alpha+1}, & D_{\alpha\varrho}z_\beta \end{vmatrix} = y_1(D_{\alpha\beta}z_\beta - \lambda\varepsilon_\alpha z_{\alpha+1}) = 0$$

and so $D_{\alpha\varrho} = \lambda\varepsilon_\alpha\delta^\beta_{\alpha+1}$ and $D = \lambda J$.

This shows that the vector part of $\Omega_2$ satisfies the $D_1$ condition.

b) Solution for the translation part

Let us write $\omega_0 = (a, b)^\mathrm{T}$ similarly as in a). We may suppose that $a = 0$ in a suitable moving frame. So we have to discuss two cases:

i) $\omega_0 \neq 0$. Then $\vartheta = \mu\omega_0$ for some $\mu$, as $|\omega_0, \vartheta| = 0$.

1) deg $\omega_2 \geqq 1$. Then

$$\begin{vmatrix} y_1, & \lambda y_1 \\ \varepsilon_\alpha z_{\alpha+1}b_\alpha, & \varepsilon_\alpha\lambda z_{\alpha+1} + \mu b_\alpha \end{vmatrix} = b_\alpha y_1(\lambda - \mu) = 0$$

and $b_\alpha \neq 0$ for some $\alpha$.

2) $\deg \omega_2 = 0$. Then

$$\begin{vmatrix} z_2 + b_1, & \lambda z_2 + \mu b_1 \\ \varepsilon_\alpha z_{\alpha+1} + b_\alpha, & \varepsilon_\alpha \lambda z_{\alpha+1} + \mu b_\alpha \end{vmatrix} = 0\,, \quad \alpha > 1\,,$$

gives $b_\alpha(\lambda - \mu) = 0$ and $b_1 \varepsilon_\alpha(\lambda - \mu) = 0$.

As $\deg J > 2$, $r(J) \geqq 2$, we have either $b_\alpha \neq 0$ for some $\alpha > 1$ or $b_1 \neq 0$ and $\varepsilon_\alpha = 1$ for suitable $\alpha$. So $\lambda = \mu$.

ii) $\omega_0 = 0$. In a similar way as in i) we get $\vartheta = 0$.

**Theorem 9.** *Let $g(t)$ be an affine motion with only straight trajectories, such that the* rank $r(\omega_1) > 1$. *Then it is a $D_1$ motion.*

c) Cases when $g(t)$ is not a $D_1$ motion

i) $\deg \omega_2 = 1$. Then

$$\omega = \begin{pmatrix} 0, & 0, & 0 \\ 0, & 1, & 0 \\ \omega_\alpha, & 0, & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0, & 0, & 0 \\ a, & A, & B_\alpha \\ b_\alpha, & C_\alpha, & D_{\alpha\beta} \end{pmatrix},$$

$$\begin{vmatrix} y, & Ay + B_\alpha z_\alpha + a \\ \omega_\alpha, & C_\alpha y + D_{\alpha\varrho} z_\beta + b_\alpha \end{vmatrix} = C_\alpha y^2 + D_{\alpha\beta} y z_\beta + y(b_\alpha - A\omega_\alpha) - B_\alpha \omega_\alpha z_\alpha - a\omega_\alpha = 0$$

and so $C = 0$, $D = 0$. If $\omega_\alpha \neq 0$ for some $\alpha$, we get $B = 0$, $a = 0$, $b = A\omega_\alpha$ and the motion has the $D_1$ property. So $\omega_\alpha = 0$ and $b_\alpha = 0$.

In a suitable moving frame we have

$$\eta = \begin{pmatrix} 0, & 0, & 0 \\ b_1, & 0, & b_{12} \\ 0, & b_{21}, & 0 \end{pmatrix} \quad \text{and} \quad \Omega_2 = 1/2 \begin{pmatrix} 0, & 0, & 0 \\ -b_1, & 2, & -b_{12} \\ 0, & b_{21}, & 0 \end{pmatrix},$$

and therefore $b_{21} = 0$. The matrix of the motion can be written in the form

(5) $$g(t) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ f_1, & t, & f_i, & 0 \\ 0, & 0, & E, & 0 \\ 0, & 0, & 0, & E \end{pmatrix}, \quad i = 2, \dots, k\,,$$ where $f_j$, $j = 1, \dots, k$ are arbitrary functions. If $W(1, t, f_i) \neq 0$, $g(t)$ has the property $D_{k+1}$, where $k \leqq n$.

ii) $\deg \omega_2 = 0$. Then $\deg J \geqq 2$. Computations similar as in i) show that the only possibility is

$$\omega = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ b_2, & 0, & 0, & b_{23} \\ 0, & 0, & 0, & 0 \end{pmatrix},$$

where $b_{23}$ is a row of $n - 2$ elements;

(6) $$g(t) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ f_1, & 1, & t, & f_i \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & E \end{pmatrix}, \quad i = 2, \dots, k - 1 \quad \text{and } f_j, \quad j = 1, \dots, k - 1$$

are arbitrary functions. If $W(1, t, f_j) \neq 0$, $g(t)$ has the property $D_{k+1}$, where $k \leqq \leqq n - 1$.

**Theorem 10.** *Let $g(t)$ be an affine motion with only straight trajectories and* rank $r(\omega_1) = 1$. *Then $g(t)$ is a $D_k$ motion with $k \leqq n + 1$. All such motions for $k > 1$ are given by (5) and (6).*

d) Classification of $D_1$ motions

Now we shall describe all affine $D_1$ motions. They are motions for which $\Omega_2 = \alpha\Omega_1$ for some function $\alpha(t)$. This means that for each trajectory we have $X'' = \alpha(t) X'$. If we change the parameter $t$, $t = t(s)$, we get

$$\frac{dX}{ds} = X' \frac{dt}{ds}, \quad \frac{d^2X}{ds^2} = X'' \left(\frac{dt}{ds}\right)^2 + X' \frac{d^2t}{ds^2} = \left(\alpha \left(\frac{dt}{ds}\right)^2 + \frac{d^2t}{ds^2}\right) X' .$$

So we may always choose the parameter in such a way that $\Omega_2 = 0$. Such a parameter is determined up to a linear transformation, $t = as + b$.

Further, let $g(t)$ be a $D_1$ affine motion, $g(t)(\bar{R}_0) = R_0\, g(t) = R(t)$. If we take a special moving frame $(\bar{R}_0, R(t))$ of $g(t)$, we get $\Omega_1 = g^{-1}g'$, $\Omega_2 = g^{-1}g''$ and the equation $\Omega_2 = 0$ means that the $D_1$ motions satisfy the equation $g'' = 0$. So each $D_1$ affine motion $g(t)$ can be expressed as follows:

$$g(t) = \begin{pmatrix} 1, & 0 \\ A_0 t + B_0, & At + B \end{pmatrix}$$

where $A, B$ are constant $n \times n$ matrices, $A_0, B_0$ are constant columns. We may further suppose that $g(0) = e$ and so $B_0 = 0$ and $B = E$. Motions equivalent to $g(t)$ are now $g(t) = \gamma^{-1}\, g(t)\, \gamma$, $\gamma \in GA_n$. Let

$$g(t) = \begin{pmatrix} 1, & 0 \\ \tilde{A}_0 t, & \tilde{A} t + E \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 1, & 0 \\ \gamma_1, & \gamma_2 \end{pmatrix}.$$

Then

(7) $$\tilde{A}_0 = \gamma_2^{-1}(A_0 + A\gamma_1), \quad \tilde{A} = \gamma_2^{-1} A \gamma_2 .$$

This shows that we can take $A$ in the real Jordan form and $A_0 = 0$ for $A$ regular. This proves the following theorem:

**Theorem 11.** *Each affine $D_1$ motion can be written as a product $g(t) =$*

$= g_1(a_1 t + b_1) \ldots g_s(a_s t + b_s)$, *where* $a_i \neq 0$ *and* $g_i(\tau)$ *is one of the following*:

$$\text{a)}\quad g_i(\tau) = \begin{pmatrix} 1, & 0 \\ 0, & (\tau + 1)\, E + \tau J \end{pmatrix},$$

$$\text{b)}\quad g_i(\tau) = \begin{pmatrix} 1, & 0 \\ 0, & (\alpha\tau + 1)\, E + \tau F + \tau G \end{pmatrix},$$

$$\text{c)}\quad g_i(\tau) = \begin{pmatrix} 1, & 0 \\ \alpha\tau T, & E + \tau J \end{pmatrix},$$

*where* $F$ *and* $G$ *are of even degree*, $\alpha \in R$ *and* $J = (\delta_{\alpha+1,\beta})$,

$$F = \operatorname{Diag}\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}, \quad G = (\delta_{\alpha+2,\beta}), \quad T = (\delta_i, 1)$$

*and we have to use a suitable embedding of* $GA_m$ *into* $GA_n$:

$$\begin{pmatrix} 1, & 0 \\ t, & g \end{pmatrix} \to \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & E, & 0, & 0 \\ t, & 0, & g, & 0 \\ 0, & 0, & 0, & E \end{pmatrix}.$$

Proof. Theorem 11 follows from (7) and the real Jordan forms of $n \times n$ matrices.

**Example 3.** In this example we would like to show how the first part of the paper may be used in a more specific situation. We shall discuss the problem of straight trajectories in similarity plane kinematics in the light of theorems of Section 1.

Let us use the complex coordinate $z = x + iy$ in $E_2$. The similarity group $G$ is then given by matrices

$$g = \begin{pmatrix} 1, & 0 \\ z_1, & z_2 \end{pmatrix},$$

where $z_1, z_2 \in C$, $z_2 \neq 0$.

The matrix $\omega$ can be given the form

$$\omega = \begin{pmatrix} 0, & 0 \\ 0, & \alpha \end{pmatrix},$$

where we suppose $\alpha \neq 0$. Then $\alpha$ represents a regular matrix and from Example 2 we know that there is only one $F_1$ motion (which is $D_1$ at the same time). It preserves a point and may be expressed by the matrix

$$g(t) = \begin{pmatrix} 1, & 0 \\ 0, & 1 + \lambda t \end{pmatrix}$$

with $\lambda \in C$, $\lambda \neq 0$.

All the other motions with a fixed point are $D_2$ motions (they depend on two functions only) and they have no straight trajectory. So we may further suppose, that the motion has no fixed point. Let us call such motion simply a proper similarity motion. Then we have the following situation:

a) There are not proper $F_1$ or $D_1$ similarity motions.

b) A proper similarity motion is a $D_2$ motion iff it has infinitely many straight trajectories. (It is known that if a similarity motion has two straight trajectories, it has infinitely many.) So if a similarity motion has two straight trajectories, then all trajectories are affinely equivalent.

c) A proper similarity motion is a $D_3$ motion iff it has one straight trajectory.

To prove this statements, we have to introduce some more facts from plane similarity kinematics.

We may choose the parameter $t$ and the moving frame in such a way that

$$\omega = \begin{pmatrix} 1, & 0 \\ 0, & e^{i\beta} \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0, & 0 \\ 1, & \varkappa \end{pmatrix}, \quad \psi = \begin{pmatrix} 0, & 0 \\ 1, & \tilde{\varkappa} \end{pmatrix},$$

where $\varkappa - \tilde{\varkappa} = e^{i\beta}$, and $\beta, \varkappa = \varkappa_1 + i\varkappa_2$ are invariants of the motion. Using (1) we compute

$$\Omega_1 = \omega, \quad \Omega_2 = \begin{pmatrix} 0, & 0 \\ -e^{i\beta}, & e^{2i\beta} + i\beta' e^{i\beta} \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} 0, & 0 \\ -(\varkappa + 2i\beta')\, e^{i\beta} - e^{2i\beta}, & e^{3i} + 3i\beta' e^{2i\beta} + i\beta'' - (\beta')^2\, e^{i\beta} \end{pmatrix}.$$

From the formula $F_2 = |X', X''| = \operatorname{Im}(\bar{X}' \,.\, X'')$ and similarly for $F_3$ we get

$$F_2 = (\beta' + \sin\beta)(x^2 + y^2) + y,$$

$$F_3 = (\sin 2\beta + 3\beta' \cos\beta + \beta'')(x^2 + y^2) - x(\varkappa_2 + \sin\beta + 2\beta') + y(\cos\beta + \varkappa_1).$$

Using now the results from Example 1, we see that if $F_2 = 0$ implies $F_3 = 0$, then the motion is a $D_2$ motion, as $F_2$ is always nontrivial. As the instantaneous pole cannot have straight trajectory, the existence of two points with straight trajectories implies that $F_2 = 0$ and $F_3 = 0$ coincide.

Let now $F_2 = 0$ and $F_3 = 0$ be different. Then they have two common points. (We may suppose that $\varkappa_2 + \sin\beta + 2\beta' \neq 0$, because $\varkappa_2 + \sin\beta + 2\beta' = 0$ for $D_3$ motions leads to a $D_2$ motion.) The common point different from the origin can have a straight trajectory only if $F_2(X) = 0$ and $F_3(X) = 0$ implies $F_4(X) = |X', X^{(4)}| = 0$. As all the three of them pass through the origin, we get $F_4 = \alpha_3 F_3 + \alpha_2 F_2$ for some functions $\alpha_2(t)$ and $\alpha_3(t)$. We shall show that this implies $\Omega_4 = \sum_{i=1}^{3} \alpha_i \Omega_i$. To show it, we shall use real coordinates $x$ and $y$ in the plane and write

$$X' = \begin{pmatrix} 0, & 0 \\ 0, & \omega_1 \end{pmatrix} \begin{pmatrix} 1 \\ X \end{pmatrix}, \quad \Omega_i = \begin{pmatrix} 0, & 0 \\ \vartheta_i, & \Theta_i \end{pmatrix}, \quad i = 2, 3, 4.$$

If $Y = \omega_1 X$, $Y = (x, y)^{\mathsf{T}}$, we get

$$F_i = |Y, \vartheta_i + \Theta_i \omega_1^{-1}|\, Y = (x^2 + y^2)\, d_i + x b_i - a y_i$$

with

$$\Theta_i\omega_1^{-1} = \begin{pmatrix} c_i, & -d_i \\ d_i, & c_i \end{pmatrix}, \quad \vartheta_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{for} \quad i = 2, 3, 4 .$$

So $F_4 = \alpha_3 F_3 + \alpha_2 F_2$ implies $\Theta_4\omega_1^{-1} = \alpha_3\Theta_3\omega_1^{-1} + \alpha_2\Theta_2\omega_1^{-1} + \alpha_1 E$ for some function $\alpha_1(t)$ and $\vartheta_4 = \alpha_3\vartheta_3 + \alpha_2\vartheta_2$. This gives $\Theta_4 = \sum_{i=1}^{3} \alpha_i\Theta_i$ and therefore $\Omega_4 = \sum_{i=1}^{3} \alpha_i\Omega_i$. We complete the proof by applying Theorem 7.

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*Author's address:* 186 00 Praha 8, Sokolovská 83, Czechoslovakia (Matematicko-fyzikální fakulta UK).