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A REMARK ON THE π-REFLEXIVITY OF SOVA

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To Professor Samuel Verblunsky on his seventy-fifth birthday

There are several different types of systems of subsets of a locally convex space (l.c.s.) E which have been used to produce polar topologies on the continuous dual E'; for instance the $\mathscr{A}$ of [3] differs formally from the $\mathscr{G}$ of [2] which differs from the $\mathfrak{S}$ of [1]. In [4] Sova axiomatises a type of system (referred to as a dualising system) which differs from each of the above, and states that if χ is a dualising system on E then there is a (unique) topology on E' making it into a l.c.s. E^χ such that the neighbourhoods of zero are precisely those subsets which contain the polar of some member of χ. He introduces the term *reflexor* to describe a correspondence π which associates with each l.c.s. E a dualising system $\pi(E)$ of subsets of E; the space $E^{\pi(E)}$ is described as the π-dual of E.

A result important to the conclusions in [4] was stated therein as:

The inverse of the canonical transformation J transforms $E^{\cdot\cdot}$, $E^{\circ\circ}$, and $E^{\square\square}$ onto E in a continuous fashion.

The dualising systems $\cdot(E)$, $\bigcirc(E)$ and $\square(E)$ are made up as follows:

$\cdot(E)$ – finite subsets of E,

$\bigcirc(E)$ – subsets of E which have a convex relatively compact hull.

$\square(E)$ – subsets of E which have a convex relatively weakly compact hull,

where $\cdot$, $\bigcirc$ and $\square$ are respectively known as the weak, compact and weakly compact reflexors.

It is readily confirmed that the result is indeed true for the bidual spaces $E^{\circ\circ}$ and $E^{\square\square}$. Since J is a bijective transformation between E and $(E^{\cdot})'$ $(= (E^{\circ})' = (E^{\square})')$ then J^{-1} will be continuous if J is open. Let N be a neighbourhood of the origin in E; we show that $J(N)$ is a neighbourhood of the origin in $E^{\circ\circ}$ (whence certainly in $E^{\square\square}$ since it is straightforward to show that a subset B' of E' is weakly compact in $E^{\square}$ if, and only if, it is compact in $E^{\cdot}$). $N^{\times}$ (the polar of N) is compact in E°, and absolutely convex, so its polar in $(E^{\circ})'$ is from [4] a basic neighbourhood in $E^{\circ\circ}$. Now $\hat{x}$ is in the polar of $N^{\times}$ if, and only if $|f(x)| \leqq 1$ for all $f \in N^{\times}$ if, and only if, $x \in N^{\times\times} = N$ if, and only if, $\hat{x} = J(x) \in J(N)$, and the result follows.

The result fails to be true for the bidual $E^{\bullet\bullet}$ since it is not generally true that $N^{\times\times}$ is a basic neighbourhood in $E^{\bullet\bullet}$. This will be shown in the proof of the theorem we establish below which reinforces the failure of the above result and furthermore constrains the generality of the following postulations in [4].

(i) *If E is π-semireflexive then the mapping inverse to the canonical transformation is continuous as a mapping from $E^{\pi\pi}$ to E.*

(ii) *If E is π-reflexive, the canonical transformation is an isomorphism between E and $E^{\pi\pi}$.*

We examine (i) since its failure in general implies a fortiori the failure of (ii).

Suppose that N is a closed absolutely convex neighbourhood in E; then it must be shown that $J(N)$ is a neighbourhood in $E^{\pi\pi}$. $N^{\times}$ is compact in E° and in $E^{\bullet}$ and therefore weakly compact in $E^{\square}$. Since E is semireflexive if, and only if, $\pi(E) \subseteq \square(E)$ then $N^{\times}$ is weakly compact in E^{π}. If now the polar of $N^{\times}$ is taken in $(E^{\pi})'$ it follows that $N^{\times\times}$ is a basic neighbourhood in $E^{\pi\square}$ but not necessarily in $E^{\pi\pi}$.

Theorem. *The inverse transformation $J^{-1}: E^{\bullet\bullet} \to E$ is continuous if, and only if, the original topology on E is the weak topology.*

Proof. It is easy to show that if E has the weak topology then $J: E \to E^{\bullet\bullet}$ is open and therefore J^{-1} is continuous. Conversely suppose that $J: E \to E^{\bullet\bullet}$ is open. Let N be any neighbourhood of the origin in E; then $J(N)$ is a neighbourhood of the origin in $E^{\bullet\bullet}$. That is, $J(N)$ contains the polar of a finite set $\{f_1, f_2, \ldots, f_n\} \subseteq E^{\bullet}$. Thus, whenever $\hat{x} \in \{f_1, f_2, \ldots, f_n\}^{\times}$, we have $\hat{x} \in J(N)$. That is $\bigcap_{i}^{n} f_i^{-1}([-1, 1]) \subseteq N$.

Thus N contains a weak neighbourhood of the origin and therefore is itself a weak neighbourhood of the origin. If ξ is the original topology on E then $\xi \leqq \sigma$ (the weak topology). Therefore $\xi = \sigma$ as required.

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