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EXPONENTIAL DICHOTOMY OF EVOLUTIONARY PROCESSES IN BANACH SPACES

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1. INTRODUCTION

In this paper we study the uniform asymptotical behaviour of linear evolutionary processes defined on Banach spaces, the main restrictions on these processes being that their norms can increase not faster than an exponential function.

An operator characterization of the uniform exponential dichotomy (Theorem 3.1) is proved. This theorem is an extension of a similar result given by Massera and Schäffer in [6], Theorem 42B, C.

A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Datko in [4]. In this paper (Theorem 3.2) we extend Datko's theorem to the general case of uniform exponential dichotomy. Finally, an integral characterization (with respect to the second argument of the process) for the uniform exponential dichotomy is also obtained (Theorem 2.3). This theorem is a generalization of well-known results about the uniform asymptotical behaviour of the evolutionary processes generated by ordinary differential equations (considered by Barbašin [1], Coppel [3] and Lovelady [5]).

The particular case when the process is a strongly continuous semigroup of operators on a Banach space has been considered in the papers [7] and [8] of the authors. Thus this paper is in a sense a continuation to [7] and [8].

The purpose of this paper is to give a characterization for the uniform exponential dichotomy property of linear evolutionary processes which are defined in general Banach spaces and whose norms can increase not faster than an exponential. The results obtained are generalizations of well known theorems about the uniform exponential stability.

2. EVOLUTIONARY PROCESSES

Let X be a real or complex Banach space. The norm on X and on the space $L(X)$ of all bounded linear operators from X into itself will be denoted by $\|\cdot\|$. Let T be the set defined by

$$T = \{(t, s): 0 \leqq s \leqq t < \infty\}.$$

Definition 2.1. A mapping $P(\cdot, \cdot): T \to L(X)$ will be called an *evolutionary process* iff

(i) $P(t, s) P(s, t_0) = P(t, t_0)$ for $0 \leqq t_0 \leqq s < t$;
(ii) $P(t, t) x = x$ for all $x \in X$ and $t \geqq 0$;
(iii) $P(t, s)$ is strongly continuous in s on $[0, t]$ and in t on $[s, \infty)$;
(iv) there is a nondecreasing function $p: \mathbb{R}_+ = [0, \infty) \to \mathbb{R}_+$ such that

$$\|P(t, s)\| \leqq p(t - s) \quad \text{for all} \quad (t, s) \in T.$$

If in addition $P(\cdot, \cdot)$ satisfies the condition

(v) $P(t, s) = P(t - s, 0)$ for all $(t, s) \in T$

then $P(\cdot)$ is called a *semigroup of class* C_0.

Remark 2.1. If $P(\cdot, \cdot)$ is an evolutionary process then its norm can increase not faster than an exponential, i.e. we can suppose in (iv) that

$$p(t) = M e^{\omega t}$$

where $M \geqq 1$ and $\omega > 0$ are independent of t.

Indeed, if $M = p(1)$, $\omega \geqq \ln M$ and n is the positive integer such that $n \leqq t - s < n + 1$, then

$$\|P(t, s)\| \leqq \|P(t, s + n)\| \, \|P(s + n, s + n - 1)\| \ldots \|P(s + 1, s)\| \leqq$$
$$\leqq M e^{n\omega} \leqq M e^{\omega(t-s)}.$$

Definition 2.2. A mapping

$$P: \mathbb{R}_+^2 = [0, \infty) \times [0, \infty) \to L(X)$$

is called a *reversible evolutionary process* iff

(j) $P(t, r) P(r, s) = P(t, s)$ for $r, s, t \geqq 0$;
(jj) $P(t, t) x = x$ for all $t \geqq 0$ and $x \in X$;
(jjj) $P(t, s)$ is strongly continuous in t and in s on $\mathbb{R}_+$;
(jv) there is $M \geqq 1$ and $\omega > 0$ such that

$$\|P(t, s)\| \leqq M e^{\omega|t-s|} \quad \text{for all} \quad t, s \geqq 0.$$

Remark 2.2. If $P(\cdot, \cdot)$ is a reversible evolutionary process then there exists

$$P(t, s)^{-1} = P(s, t) \quad \text{for every} \quad t, s \geqq 0.$$

Example 2.1. Let $A: \mathbb{R}_+ \to L(X)$ be a strongly measurable function such that

$$\sup_{t \geqq 0} \int_t^{t+1} \|A(s)\| \, da < \infty.$$

Then the unique solution $P(\cdot)$ of the Cauchy problem

$$P'(t) = A(t)\,P(t)\,, \quad P(0) = I$$

(where I denotes the identity operator on X) has the property (see [6]) that

$$P(t, s) = P(t)\,P^{-1}(s)$$

is a reversible evolutionary process.

Throughout this paper we suppose that for every $t_0 \geqq 0$ the set

$$X_1(t_0) = \{x_0 \in X\colon P(\cdot, t_0)\,x_0 \in L^\infty_{t_0}(X)\}$$

is a closed complemented subspace of X. Here $L^\infty_{t_0}(X)$ denotes the Banach space of X-valued functions f defined a.e. on $[t_0, \infty)$, such that f is strongly measurable and essentially bounded. If $X_2(t_0)$ is a complementary subspace of $X_1(t_0)$ then we denote by $P_1(t_0)$ the projection along $X_2(t_0)$ (that is $P_1(t_0) \in L(X)$, $P_1^2(t_0) = P_1(t_0)$, $\operatorname{Ker} P_1(t_0) = X_2(t_0)$) and by $P_2(t_0) = I - P_1(t_0)$ the projection along $X_1(t_0)$.

We shall also denote

$$P_1(t, t_0) = P(t, t_0)\,P_1(t_0) \quad \text{and} \quad P_2(t, t_0) = P(t, t_0)\,P_2(t_0)\,.$$

Remark 2.3. If $P(\cdot, \cdot)$ is a reversible evolutionary process, $P(t) = P(t, 0)$, $X_k = X_k(0)$, $P_k = P_k(0)$ then it is easy to see that for all $k = 1, 2$ and $t \geqq s \geqq t_0 \geqq 0$ the following equalities hold:

(i) $P(t, t_0) = P(t)\,P^{-1}(t_0)$;

(ii) $P_k(t, t_0) = P(t)\,P_k P^{-1}(t_0)$;

(iii) $P_k(t, t_0) = P_k(t, s)\,P_k(s, t_0)$;

(iv) $X_k(t_0) = P(t_0)\,X_k$;

(v) $P_1(t_0) = P(t_0)\,P_1 P^{-1}(t_0)$ and $P_2(t_0) = P(t_0)\,P_2 P^{-1}(t_0)$ are projections along $X_2(t_0)$ and $X_1(t_0)$, respectively.

Now we present three helpful results.

Lemma 2.1. *Let $P(\cdot, \cdot)$ be an evolutionary process. Then*

(i) *$P(t, t_0)\,X_1(t_0) \subset X_1(t)$ for all $(t, t_0) \in T$;*

(ii) *$P_1(t, t_0) = P_1(t, s)\,P_1(s, t_0)$ for $t \geqq s \geqq t_0$;*

(iii) *there are $M > 1$ and $\omega > 0$ such that*

$$\|P(t + 1, t_0)\,x_0\| \leqq M e^{\omega} \int_t^{t+1} \|P(s, t_0)\,x_0\|\,ds$$

for all $(t, t_0) \in T$ and $x_0 \in X$;

(iv) *if $P_1(t, t_0)\,x_0 \neq 0$ for every $t \geqq t_0$ then $P(t, t_0)\,x_0 \neq 0$ for all $t \geqq t_0$;*

(v) *$P_2(t, t_0)\,x_0 \neq 0$ for all $(t, t_0) \in T$ and $x_0 \notin X_1(t_0)$.*

Proof. (i) If $x_0 \in X_1(t_0)$ then by definition of $X_1(t_0)$ we have that

$$P(\cdot, t)\,P(t, t_0)\,x_0 = P(\cdot, t_0)\,x_0 \in L^\infty_{t_0}(X) \subset L^\infty_t(X)$$

and hence $P(t, t_0)\,x_0 \in X_1(t)$.

(ii) From (i) it follows that $P_1(t)\, P_1(t, t_0) = P_1(t, t_0)$ and hence

$$P_1(t, s)\, P_1(s, t_0) = P(t, s)\, P_1(s)\, P_1(s, t_0) = P(t, s)\, P_1(s, t_0) =$$
$$= P(t, s)\, P(s, t_0)\, P_1(t_0) = P(t, t_0)\, P_1(t_0) = P_1(t, t_0)\,.$$

(iii) The equality follows from

$$\|P(t + 1, t_0)\, x_0\| \leqq \|P(t + 1, s)\|\, \|P(s, t_0)\, x_0\| \leqq M\mathrm{e}^{\omega}\|P(s, t_0)\, x_0\|$$

(where M and ω are given by Remark 2.1) by integration on $[t, t + 1]$.

(iv) follows from the equality

$$X_1(t_0) \cap X_2(t_0) = \{0\}\,.$$

(v) If there are $s \geqq t_0$ and $x_0 \notin X_1(t_0)$ such that $P_2(s, t_0)\, x_0 = 0$ then

$$P(t, t_0) = P(t, s)\, P(s, t_0)\, x_0 = P_1(t, t_0)\, x_0 \quad \text{for all} \quad t \geqq s$$

and $P_1(\cdot, t_0)\, x_0 \in L^{\infty}_{t_0}(X)$ imply that $x_0 \in X_1(t_0)$, which is a contradiction.

Lemma 2.2. *If there exists $N > 0$ such that*

$$\int_{t_0}^{\infty} \|P_1(t, t_0)\, x_0\|\, \mathrm{d}t \leqq N\|x_0\| \quad \textit{for every} \quad x_0 \in X\,,$$

then there is $N_1 > 0$ such that

$$\|P_1(t, t_0)\| \leqq N_1 \quad \textit{for all} \quad (t, t_0) \in T\,.$$

Proof. Assume that $P_1(\cdot, \cdot)$ is not uniformly bounded. Then for each natural number n there exists $(t_n, t_n^0) \in T$ such that $\|P_1(t_n, t_n^0)\| \geqq n$. This implies that there is $x_n \in X$ with $\|x_n\| = 1$ and $\|P_1(t_n, t_n^0)\, x_n\| \geqq n$.

Hence if $s \in [t_n^0, t_n]$ we can write

$$n \leqq \|P_1(t_n, t_n^0)\, x_n\| \leqq \|P(t_n, s)\|\, \|P_1(s, t_n^0)\, x_n\| \leqq M\mathrm{e}^{\omega(t_n - s)}\|P_1(s, t_n^0)\, x_n\|$$

and by integration on $[t_n^0, t_n]$ we obtain

$$n \leqq MN\mathrm{e}^{\omega(t_n - t_n{}^0)}\,;$$

which implies that $t_n - t_n^0 \to \infty$.

Thus we have

$$\frac{n}{\omega}\left(1 - \mathrm{e}^{-\omega(t_n - t_n{}^0)}\right) = n \int_{t_n{}^0}^{t_n} \mathrm{e}^{-\omega(t_n - s)}\, \mathrm{d}s \leqq M \int_{t_n{}^0}^{t_n} \|P_1(s, t_n^0)\, x_n\|\, \mathrm{d}s \leqq MN,$$

which is a contradiction in view of

$$\lim_{n \to \infty} \frac{n}{\omega}\left(1 - \mathrm{e}^{-\omega(t_n - t_n{}^0)}\right) = \infty\,.$$

This completes the proof of Lemma 2.2.

Lemma 2.3. *If there is $N > 0$ such that*

$$(t - t_0) \|P_1(t, t_0)\| \leqq N \quad \textit{for all} \quad (t, t_0) \in T$$

then there exist $N_1, v_1 > 0$ such that

$$\|P_1(t, t_0)\| \leqq N_1 e^{-v_1(t-t_0)} \quad \textit{for all} \quad (t, t_0) \in T .$$

Proof. Let $(t, t_0) \in T$. Then $t - t_0 = 2Nn + r$, where n is positive integer and $r \in [0, 2N)$.

Then

$$\|P_1(t, t_0)\| \leqq \|P_1(t_0 + 2Nn + r, t_0 + 2Nn)\| \, . \, \|P_1(t_0 + 2Nn, t_0 + 2N(n + 1))\| \ldots$$
$$\ldots \|P_1(t_0 + 2N, t_0)\| \leqq Me^{\omega r}(\tfrac{1}{2})^n \leqq Me^{2M\omega}(\tfrac{1}{2})^n = Me^{2M\omega}e^{-2vNn} \leqq$$
$$\leqq Me^{2M(\omega + v)}e^{-v(t-t_0)} = N_1 e^{-v(t-t_0)} ,$$

where $N_1 = Me^{2M(\omega+v)}$ and $v = \ln 2/2N$.

3. UNIFORM EXPONENTIAL DICHOTOMY

In this section we give necessary and sufficient conditions for the uniform exponential dichotomy of linear evolutionary processes in Banach spaces. As a particular case we can obtain a characterization of the uniform exponential stability of such processes. The uniform exponential stability and the dichotomy property are introduced by

Definition 3.1. An evolutionary process $P(\cdot, \cdot)$ is said to be

(i) *uniformly exponentially stable* (and we write u.e.s.) iff there exist constants $N \geqq 1$ and $v > 0$ such that

$$\|P(t, t_0)\| \leqq Ne^{-v(t-t_0)} \quad \text{for all} \quad (t, t_0) \in T ;$$

(ii) *uniformly exponentially dichotomic* (and we write u.e.d.) iff there are $N_1, N_2, v > 0$ such that

$$\|P_1(t, t_0)\, x_0\| \leqq N_1 e^{-v(t-s)} \|P_1(s, t_0)\, x_0\| \tag{3.1}$$

and

$$\|P_2(t, t_0)\, x_0\| \geqq N_2 e^{v(t-s)} \|P_2(s, t_0)\, x_0\| \tag{3.2}$$

for all $t \geqq s \geqq t_0 \geqq 0$ and $x \in X$.

Remark 3.1. Clearly, the uniform exponential stability is a particular case (when $X_1(t_0) = X$) of the uniform exponential dichotomy.

Remark 3.2. $P(\cdot, \cdot)$ is u.e.d. if and only if the inequalities (3.1) and (3.2) from Definition 3.1 hold for all $t \geqq s + 1 \geqq s \geqq t_0 \geqq 0$. Indeed, if $t_0 \leqq s \leqq t \leqq s + 1$

then

$$\|P_1(t, t_0)\, x_0\| \leqq \|P_1(t, s)\| \; \|P_1(s, t_0)\, x_0\| \leqq Me^{\omega}\|P_1(s, t_0)\, x_0\| \leqq$$
$$\leqq Me^{\omega+\nu}e^{-\nu(t-s} \,.\, \|P_1(s, t_0)\, x_0\|$$

and

$$Ne^{\nu}\|P_2(s, t_0)\, x_0\| \leqq \|P_2(s + 1, t_0)\, x_0)\| \leqq \|P(s + 1. t)\| \,.\, \|P_2(t, t_0)\, x_0\| \leqq$$
$$\leqq Me^{\omega}\|P_2(t, t_0)\, x_0\| \leqq Me^{\omega+\nu}e^{-\nu(t-s)}\|P_2(t, t_0)\, x_0\| \,.$$

A necessary and sufficient condition for u.e.d. is given in

Theorem 3.1. *An evolutionary process $P(\cdot, \cdot)$ is u.e.d. if and only if there are $N, \nu > 0$ such that*

$$(3.1)' \qquad \|P_1(t, t_0)\, x_0\| \leqq Ne^{-\nu(t-s)}\|P(s, t_0)\, x_0\|$$

and

$$(3.2)' \qquad \|P_2(s, t_0)\, x_0\| \leqq {}^{-\nu(t-s)}\|P(t, t_0)\, x_0\|$$

for all $t \geqq s \geqq t_0 \geqq 0$ and $x_0 \in X$.

Proof. Necessity. The inequality

$$\left\|\frac{u}{\|u\|} + \frac{v}{\|v\|}\right\| \max \|u\|, \|v\| \leqq 2\|u + v\|$$

(see [6], Theorem 11.A) after putting $u = P_1(t, t_0)\, x_0$ and $v = P_2(t, t_0)\, x_0$ with $x_0 \neq 0$, $P_1(t, t_0)\, x_0 \neq 0$ and $P_2(t, t_0)\, x_0 \neq 0$ for every $t \geqq t_0 \geqq 0$ yields

$$2\|P(t, t_0)\, x_0\| \geqq \left\|\frac{u}{\|u\|} + \frac{v}{\|v\|}\right\| \max \{\|u\|, \|v\|\} \,.$$

This inequality together with

$$Me^{\omega}\|P(t, t_0)\, y\| \geqq \|P(t + 1, t_0)\, y\|$$

for

$$y = \frac{P_1(t_0)\, x_0}{\|u\|} + \frac{P_2(t_0)\, x_0}{\|v\|},$$

(3.1) and (3.2) implies

$$2Me^{\omega}\|P(t, t_0)\, x_0\| \geqq Me^{\omega}\|P(t, t_0)\, y\| \max \{\|u\|, \|v\|\} \geqq$$
$$\geqq \|P(t + 1, t_0)\, y\| \max \{\|u\|, \|v\|\} \geqq$$
$$\geqq (\|P_2(t + 1, t_0)\, y\| - \|P_1(t + 1, t_0)\, y\|) \max \{\|u\|, \|v\|\} \geqq$$
$$\geqq \left(\frac{\|P_2(t + 1, t_0)\, x_0\|}{\|P_2(t, t_0)\, x_0\|} - \frac{\|P_1(t + 1, t_0)\, x_0\|}{\|P_1(t, t_0)\, x_0\|}\right) \max \{\|u\|, \|v\|\} \geqq$$
$$\geqq (N_2e^{\nu} - N_1e^{-\nu}) \max \{\|u\|, \|v\|\} = N_3 \max \{\|P_1(t, t_0)\, x_0\|, \|P_2(t_1, t_0)\, x_0\|\} \,.$$

Hence

$$\|P_1(t, t_0)\, x_0\| \leqq N_1 e^{-\nu(t-s)} \|P_1(s, t_0)\, x_0\| \leqq 2 \frac{MN_1 e^{\omega}}{N_3} e^{-\nu(t-s)} \|P(s, t_0)\, x_0\|$$

and

$$\|P_2(s, t_0)\, x_0\| \leqq \frac{e^{-\nu(t-s)}}{N_2} \|P_2(t, t_0)\, x_0\| \leqq \frac{2Me^{\omega}}{N_2 N_3} e^{-\nu(t-s)} \|P(t, t_0)\, x_0\|$$

for all $t \geqq s \geqq t_0 \geqq 0$ and $x_0 \in X$.

The sufficiency is trivial.

Corollary 3.1. *If $P(\cdot, \cdot)$ is u.e.d. then there exists $N > 0$ such that*

$$\|P_1(t)\| \leqq N \quad \text{and} \quad \|P_2(t)\| \leqq N \quad \text{for all} \quad t \geqq 0\,. \tag{3.3}$$

Proof. If we put $t = s = t_0$ in (3.1)′ and (3.2)′ then we obtain (3.3).

In the particular case of reversible evolutionary processes we obtain

Corollary 3.2. *The reversible evolutionary process $P(\cdot, \cdot)$ is u.e.d. if and only if there is $N, \nu > 0$ such that*

$$\|P_1(t, s)\| \leqq N e^{-\nu(t-s)} \tag{3.4}$$

and

$$\|P_2(s, t)\| \leqq N e^{-\nu(t-s)} \tag{3.5}$$

for all $t \geqq s \geqq 0$.

Proof. Suppose that $P(\cdot, \cdot)$ is u.e.d. and let $x \in X$ and $t \geqq s \geqq t_0 \geqq 0$.

The identity $x_0 = P(t_0, s)\, x$, the inequality (3.1)′ and the preceding theorem yield

$$\|P_1(t, s)\, x\| = \|P(t)\, P_1 P^{-1}(t_0)\, x_0\| = \|P_1(t, t_0)\, x_0\| \leqq$$
$$\leqq N e^{-\nu(t-s)} \|P(s, t_0)\, x_0\| = N e^{-\nu(t-s)} \|x_0\|\,,$$

which implies (3.4).

Similarly, the inequality (3.2)′ for $x_0 = P(t_0, t)\, x$ shows that $P(\cdot, \cdot)$ satisfies (3.5).

Conversely, if $P(\cdot, \cdot)$ verifies the inequalities (3.4) and (3.5) then

$$\|P_1(t, s)\, x\| \leqq N e^{-\nu(t-s)} \|x\|$$

and

$$\|P_2(s, t)\, y\| \leqq N e^{-\nu(t-s)} \|y\|$$

for all $(t, s) \in T$ and $x, y \in X$.

In particular, for $x = P(s, t_0)\, x_0$ and $y = P(t, t_0)\, x_0$ we obtain the inequalities (3.1)′ and (3.2)′, respectively. Theorem 3.1 implies that $P(\cdot, \cdot)$ is u.e.d.

Remark 3.3. The above characterization of the uniform exponential dichotomy is a generalization of a similar result for the case when $P(\cdot, \cdot)$ is an evolutionary process generated by a differential equation, given by Massera and Schäffer in [6], Theorem 42 B, C.

Theorem 3.2. *The evolutionary process is u.e.d. if and only if there exist $m, N > 0$ such that*

(3.6)
$$\int_{t_0}^{\infty} \|P_1(t, t_0)\, x_0\|\, \mathrm{d}t \leqq N\|x_0\|\,,$$

(3.7)
$$\int_{t_0}^{t} \|P_2(s, t_0)\, x_0\|\, \mathrm{d}t \leqq N\|P_2(t, t_0)\, x_0\|\,,$$

(3.8)
$$\|P_2(t + 1, t_0)\, x_0\| \geqq m\|P_2(t, t_0)\, x_0\|$$

for all $(t, t_0) \in T$ and $x_0 \in X$.

Proof. The necessity is simply verified. Now we prove the sufficiency. Let $t_0 \geqq 0$, $x_0 \in X$ be fixed.

From Lemmas 2.1 and 2.2 we have

$$\|P_1(t, t_0)\, x_0\| \leqq \|P_1(t, s)\|\, \|P_1(s, t_0)\, x_0\| \leqq N_1\|P_1(s, t_0)\, x_0\|$$

and integration on $[t_0, t]$ yields

$$(t - t_0)\, \|P_1(t, t_0)\, x_0\| \leqq NN_1\|x_0\|\,.$$

On the basis of Lemma 2.3 we have that there are $N_2 > 0$ and $\nu > 0$ such that

$$\|P_1(t, s)\, x\| \leqq N_2 \mathrm{e}^{-\nu(t-s)}\|x\|$$

for all $(t, s) \in T$ and $x \in X$.

In particular, for $x = P_1(s, t_0)\, x_0$ we obtain the inequality (3.1) from Definition 3.1.

Now we consider the function

$$f\colon [t_0, \infty) \to \mathbb{R}\,, \quad f(t) = \int_{t_0}^{t} \|P_2(s, t_0)\, x_0\|\, \mathrm{d}s\,.$$

We observe that

$$f(t) \leqq N f'(t) = N\|P_2(t, t_0)\, x_0\|$$

and by integration on $[s + 1, t]$ we obtain

$$f(s + 1)\, \mathrm{e}^{(t-s-1)/N} \leqq f(t) \leqq N\|P_2(t, t_0)\, x_0\|\,,$$

i.e.

(3.10)
$$\int_{t_0}^{s+1} \|P_2(u, t_0)\, x_0\|\, \mathrm{d}u\, \mathrm{e}^{(t-s-1)/N} \leqq N\|P_2(t, t_0)\, x_0\|$$

for all $t \geqq s + 1 \geqq t_0$.

On the basis of (3.8), Lemma 2.1 and (3.10) we have

$$\|P_2(s, t_0)\, x_0\|\, \mathrm{e}^{(t-s)/N} \leqq \frac{1}{m}\, \|P_2(s + 1, t_0)\, x_0\|\, \mathrm{e}^{(t-s)/N} \leqq$$

$$\leqq \frac{Me^{\omega}}{m} e^{(t-s)/N} \int_s^{s+1} \|P_2(u, t_0) x_0\| du \leqq \frac{Me^{\omega}}{m} e^{(t-s)/N} f(s+1) \leqq$$
$$\leqq \frac{MNe^{(\omega+1)N}}{m} \|P_2(t, t_0) x_0\|$$

for all $t \geqq s + 1 \geqq s \geqq t_0$. Remark 3.2 implies that $P(\cdot, \cdot)$ is u.e.d.

Remark 3.4. Theorem 3.2 is an extension of Corollary from [8] where the particular case of a semigroup of operators is considered.

Remark 3.5. In the particular case when $P_2(t_0) = I$ the preceding theorem yields Datko's characterization of the uniform exponential stability given by Theorem 1.1 from [4]. Thus Theorem 3.2 is a generalization of Datko's result. We remark that our proof of characterization of the uniform exponential dichotomy is not a generalization of Datko's proof of characterization of the uniform exponential stability.

As a consequence of the preceding theorem we obtain another characterization of the uniform exponential dichotomy of reversible evolutionary processes.

Corollary 3.3. *Let $P(\cdot, \cdot)$ be a reversible evolutionary process. Then it is u.e.d. if and only if there exists $N > 0$ such that*

$$\text{(3.11)} \qquad \int_{t_0}^{\infty} \|P_1(t, t_0) x\| dt + \int_0^{t_0} \|P_2(s, t_0) x\| ds \leqq N\|x\|$$

for all $t_0 \geqq 0$ and $x \in X$.

Proof. Necessity is obvious from Corollary 3.2. Conversely, if $P(\cdot, \cdot)$ satisfies (3.11) then

$$\int_{t_0}^{\infty} \|P_1(t, t_0) x_0\| dt \leqq N\|x_0\|$$

and

$$\int_{t_0}^{t} \|P_2(s, t_0) x_0\| ds = \int_{t_0}^{t} \|P_2(s, t) P_2(t, t_0) x_0\| ds \leqq$$
$$\leqq \int_0^t \|P_2(s, t) P_2(t, t_0) x_0\| ds \leqq N\|P_2(t, t_0) x_0\|$$

for all $(t, t_0) \in T$ and $x_0 \in X$.

Since in the case of reversible evolutionary processes the inequality (3.8) is satisfied, we obtain by Theorem 3.2 that $P(\cdot, \cdot)$ is u.e.d.

Theorem 3.3. *A reversible evolutionary process $P(\cdot, \cdot)$ is u.e.d. if and only if there is $N > 0$ such that*

$$\text{(3.13)} \qquad \int_{t_0}^{t} \|P_1(t, s)\| ds + \int_t^{\infty} \|P_2(t, s)\| ds \leqq N$$

for all $t \geqq t_0 \geqq 0$.

Proof. The necessity is trivial.

For the sufficiency part, we consider $t \geqq s \geqq t_0$ and $x \in X - \{0\}$. Then the function

$$f: [s, t] \to \mathbb{R}, \quad f(u) = \int_s^u \frac{\mathrm{d}r}{\|P_1(r, t_0)\, x\|}$$

satisfies the inequality

$$\|P_1(t, t_0)\, x\|\, f(t) \leqq \int_{t_0}^t \|P_1(t, u)\|\, \mathrm{d}u \leqq N$$

and hence

$$f(t) \leqq N f'(t)\,. \tag{3.14}$$

By integration on $[s + 1, t]$ we obtain

$$f(t) \geqq \mathrm{e}^{(t-s-1)/N} f(s + 1)\,. \tag{3.15}$$

On the other hand, for $u \in [s, s + 1]$ we have

$$\|P_1(u, t_0)\, x\| \leqq \|P_1(u, s)\|\, \|P_1(s, t_0)\, x\| \leqq M\mathrm{e}^{\omega} \|P_1(s, t_0)\, x\|$$

and by integration on $[s, s + 1]$ we deduce

$$f(s + 1) \geqq \frac{1}{M\mathrm{e}^{\omega}\|P_1(s, t_0)\, x\|}\,.$$

This inequality together with (3.14) and (3.15) implies

$$\mathrm{e}^{(t-s-1)/N} \frac{1}{\|P_1(s, t_0)\, x\|} \leqq M\mathrm{e}^{\omega} f(t) \leqq MN\mathrm{e}^{\omega} f'(t) = \frac{MN\mathrm{e}^{\omega}}{\|P_1(t, t_0)\, x\|}\,,$$

i.e.

$$\|P_1(t, t_0)\, x\| \leqq N_1 \mathrm{e}^{-\nu(t-s)} \|P_1(s, t_0)\, x\| \quad \text{for all} \quad t \geqq s + 1 \geqq s \geqq t_0 \tag{3.16}$$

where $N_1 = MN\mathrm{e}^{\omega + 1/N}$, $\nu = 1/N$.

This proves the inequality (3.1) from Definition 3.1. For the second inequality we consider the function

$$g: [s, \infty) \to \mathbb{R}, \quad g(t) = \int_t^{\infty} \frac{\mathrm{d}u}{\|P_2(u, t_0)\, x\|}\,.$$

Then

$$\|P_2(s, t_0)\, x\|\; g(t) \leqq \int_t^{\infty} \|P_2(s, u)\|\, \mathrm{d}u \leqq N$$

implies

$$g(t) \leqq -Ng'(t)\,,$$

which by integration on $[s, t]$ yields

$$\mathrm{e}^{(t-s)/N} g(t) \leqq (gs)\,.$$

Hence

$$\int_t^{t+1} \frac{\mathrm{d}u}{\|P_2(u, t_0)\, x\|}\, \mathrm{e}^{(t-s)/N} \leqq \mathrm{e}^{(t-s)/N}\, g(t) \leqq g(s) \leqq \frac{N}{\|P_2(s, t_0)\, x\|}\,.$$

For $u \in [t, t + 1]$ we have

$$\|P_2(u, t_0)\, x\| \leqq M\mathrm{e}^{\omega}\|P_2(t, t_0)\, x\|\,,$$

hence the preceding inequality implies

$$\frac{\mathrm{e}^{(t-s)/N}}{\|P_2(t, t_0)\, x\|} \leqq \frac{MN\mathrm{e}^{\omega}}{\|P_2(s, t_0)\, x\|}\,,$$

i.e.

$$\|P_2(s, t_0)\, x\| \leqq MN\mathrm{e}^{\omega}\mathrm{e}^{-(t-s)/N}\|P_2(t, t_0)\, x\| \tag{3.17}$$

for all $t \geqq s \geqq t_0 \geqq 0$.

This completes the proof.

Remark 3.6. The above characterization of the uniform exponential dichotomy for the case of reversible evolutionary processes generated by ordinary differential equations has been given by Coppel in [3]. This characterization for the particular case of the uniform exponential stability has been considered by Barbašin [1] and Lovelady [5]. For the finite dimensional case R. Conti has proved that the inequality (3.13) is a characterization for the admissibility of the pair (L^∞, L^∞) (see [2]).

Remark 3.7. Theorems 3.2, 3.3 and Corollary 3.3 remain valid if the power 1 from (3.6), (3.7), (3.11) and (3.13) is replaced by any $p \in [1, \infty)$, i.e. the inequalities (3.6), (3.7), (3.11) and (3.13) can be replaced, respectively, by

$$\int_{t_0}^{\infty} \|P_1(t, t_0)\, x_0\|^p\, \mathrm{d}t \leqq N\|x_0\|^p\,, \tag{3.6$'$}$$

$$\int_{t_0}^{t} \|P_2(s, t_0)\, x_0\|^p\, \mathrm{d}s \leqq N\|P_2(t, t_0)\, x_0\|^p\,, \tag{3.7$'$}$$

$$\int_{t_0}^{\infty} \|P_1(t, t_0)\, x\|^p\, \mathrm{d}t + \int_0^{t_0} \|P_2(s, t_0)\, x\|^p\, \mathrm{d}s \leqq N\|x\|^p \tag{3.11$'$}$$

and

$$\int_{t_0}^{t} \|P_1(t, s)\|^p\, \mathrm{d}s + \int_t^{\infty} \|P_2(t, s)\|^p\, \mathrm{d}s \leqq N\,. \tag{3.13$'$}$$

The proofs follow almost verbatim those given in the case $p = 1$.

In the light of the above considerations and Theorem 3.3 the following open question seems to be natural.

Question 3.1. Does the condition:

$$\int_{t_0}^{t} \|P_1(t, s)\, x\|\, ds + \int_{t}^{\infty} \|P_2(t, s)\, x\|\, ds \leqq N\|x\|\,, \quad \text{for all} \quad (t, t_0) \in T$$

and $x \in X$, (where N is independent of t, t_0 and x) imply the uniform exponential dichotomy of $P(\cdot, \cdot)$?

References

[1] *E. A. Barbašin:* Introduction in the theory of stability, Izd. Nauka, Moscow, 1967, (Russian).

[2] *R. Conti:* On the boundedness of solutions of ordinary differential equations, Funkcial. Ekvac. *9* (1966), 23—26.

[3] *W. A. Coppel:* Stability and asymptotic behaviour of differential equations, D.C. Heath, Boston, 1965.

[4] *R. Datko:* Uniform asymptotic stability of evolutionary processes in a Banach space, Siam J. Math. Anal., *3* (1973), 428—445.

[5] *D. L. Lovelady:* Boundedness properties for linear ordinary differential equations, Proc. Amer. Math. Soc., *41* (1973), 193—196.

[6] *J. L. Massera, J. J. Schäffer:* Linear differential equations and function spaces, Academic Press, New York and London, 1966.

[7] *M. Megan, P. Preda:* On exponential dichotomy in Banach spaces, Bull. Austral. Math. Soc., *23* (1981), 293—306.

[8] *P. Preda, M. Megan:* An extension of a theorem of R. Datko, preprint in Seminarul de Teoria Structurilor, Univ. Timişoara, *32* (1982), 1—12.

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