Werner J. Ricker

Spectral representation of local semigroups in locally convex spaces

Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 2, 248–259

Persistent URL: http://dml.cz/dmlcz/102013

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

SPECTRAL REPRESENTATION OF LOCAL SEMIGROUPS IN LOCALLY CONVEX SPACES

WERNER RICKER, Bedford Park

(Received June 21, 1983)

1. INTRODUCTION

A classical problem in the theory of semigroups of continuous linear operators acting in a Hilbert space is to determine when the operators have a joint spectral integral representation. Devinatz [2] and Nussbaum [10], [11] extended the notion of semigroup to include certain one-parameter families of unbounded (symmetric or self-adjoint) operators acting in a Hilbert space, such as the Riesz potential operators in $L^2(\mathbb{R}^n)$ [11], which have the semigroup property and are weakly continuous on a suitable subspace. Their results yield various integral representations of such a one-parameter family; see also the recent paper [7].

Examples of one-parameter families of unbounded linear operators which have the semigroup property are also encountered in spaces other than Hilbert space. A classical example is the Riemann-Liouville fractional integral in $L^p((0, \infty))$, $1 < p < \infty$, [5]. Accordingly, criteria which yield integral representations of more general one-parameter families of operators are of interest. Such a criterion was recently established by Kantorovitz and Hughes [6] for one-parameter families acting in a reflexive Banach space.

The purpose of this note is to reformulate the criterion of Kantorovitz and Hughes so that it applies to one-parameter families of operators acting in more general spaces. This so extended criterion is based on a characterization of Fourier-Stieltjes transforms of vector measures analogous to the well known Bochner-Schoenberg test.

More precisely, let X be a locally convex space. If D is a dense subpsace of X, denote by $\Pi(D)$ the algebra of all linear transformations with domain D and range contained in D. Let $\Delta = [0, \alpha)$, where $0 < \alpha \leqq \infty$. The system $\{T; D; \Delta\}$ is called a *local semigroup* on Δ if $T\colon \Delta \to \Pi(D)$ is a map such that $T(0)$ is the identity operator on D, $T(s + t) = T(s)\, T(t)$ whenever $s, t, s + t \in \Delta$, and $T(\cdot)(x)$ is a weakly continuous, X-valued function on Δ, for each $x \in D$. This is essentially the definition given in [6].

A characterization will be presented of those local semigroups $\{T; D; \Delta\}$ for which there exists an equicontinuous spectral measure P, defined on the Borel σ-algebra $\mathscr{B}$

of the real line $\mathbb{R}$, such that for each $x \in D$, the X-valued measure $E \mapsto P(E)(x)$, $E \in \mathscr{B}$, has compact support and

$$(1) \qquad T(t)(x) = \int_{\mathbb{R}} e^{-ts} \, dP(s)(x), \quad t \in \Delta.$$

The author wishes to thank Professor I. Kluvánek for valuable discussions.

2. PRELIMINARIES AND NOTATION

Throughout this note X will denote a quasi-complete, locally convex Hausdorff space. If A is a subset of X, then $\mathrm{bco}(A)$ denotes the convex, balanced hull of A. Its closure is denoted by $\overline{\mathrm{bco}}(A)$. The space of all continuous linear functionals on X is denoted by X'.

Let $\mathbb{C}$ denote the complex number field. An entire function $f: \mathbb{C} \to X$ is said to be of *exponential type* if there exists $\beta > 0$ such that for every $\varepsilon > 0$ the set

$$\{e^{-(\beta+\varepsilon)|z|} f(z);\ z \in \mathbb{C}\},$$

is bounded. If $\Delta = [0, \alpha)$, where $0 < \alpha \leqq \infty$, then a function $f: \Delta \to X$ is said to be entire of exponential type if it can be extended to an X-valued, entire function of exponential type.

By a vector measure in X is meant a σ-additive map $\mu: \mathscr{B} \to X$. For each $x' \in X'$, the complex-valued measure $E \mapsto \langle \mu(E), x' \rangle$, $E \in \mathscr{B}$, is denoted by $\langle \mu, x' \rangle$.

A complex-valued, $\mathscr{B}$-measurable function f on $\mathbb{R}$ is said to be *μ-integrable* if it is integrable with respect to every measure $\langle \mu, x' \rangle$, $x' \in X'$, and if, for every $E \in \mathscr{B}$, there exists an element $\int_E f \, d\mu$ of X such that

$$\left\langle \int_E f \, d\mu, x' \right\rangle = \int_E f \, d\langle \mu, x' \rangle,$$

for each $x' \in X'$. Bounded measurable functions are always μ-integrable [9; II Lemma 3.1]. Hence, the Fourier-Stieltjes transform, $\hat{\mu}$, of any vector measure $\mu: \mathscr{B} \to X$ can be defined by

$$\hat{\mu}(s) = \int_{\mathbb{R}} \exp(-ist) \, d\mu(t), \quad s \in \mathbb{R}.$$

Let $\mathscr{M}_d$ denote the linear space of all complex-valued measures on $\mathscr{B}$ with finite supports. The set of all measures $\omega \in \mathscr{M}_d$ such that $\|\hat{\omega}\|_\infty \leqq 1$ is denoted by Ω ($\|\cdot\|_\infty$ denotes the supremum norm).

The following result is a vector version of the Bochner-Schoenberg test. It is well known for Banach spaces [8]; its extension to more general spaces presents no difficulties.

Bochner-Schoenberg Criterion. *Let $f: \mathbb{R} \to X$ be a bounded, weakly continuous function. Then there exists a (unique) vector measure $\mu: \mathscr{B} \to X$ such that $f = \hat{\mu}$,* *if and only if,*

$$\left\{\int_{\mathbb{R}} f(t)\,\mathrm{d}\omega(t);\ \omega \in \Omega\right\}$$

is a relatively weakly compact subset of X.

Let $L(X)$ denote the space of all continuous linear operators on X, equipped with the topology of pointwise convergence. The space $L(X)$ may not be quasi-complete. The identity operator is denoted by I.

A map $P: \mathscr{B} \to L(X)$ is called a *spectral measure* if it is σ-additive, multiplicative and $P(\mathbb{R}) = I$. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)\,P(F)$, for every $E \in \mathscr{B}$ and $F \in \mathscr{B}$. The spectral measure P is said to be equicontinuous if its range $P(\mathscr{B}) = \{P(E);\ E \in \mathscr{B}\}$ is an equicontinuous part of $L(X)$. For such spectral measures, every bounded measurable function is P-integrable [13]. For each $x \in X$, denote by $P(\cdot)(x)$ the X-valued measure $E \mapsto P(E)(x)$, $E \in \mathscr{B}$.

Let $\Lambda = \{T; D; \Delta\}$ be a local semigroup. Then Λ is said to be *spectral* if there exists an equicontinuous spectral measure $P: \mathscr{B} \to L(X)$ such that for each $x \in D$, each of the functions $e^{-t(\cdot)}$, $t \in \Delta$, is $P(\cdot)(x)$-integrable and the identity (1) is valid. If, in addition, each measure $P(\cdot)(x)$, $x \in D$, has compact support, then Λ is said to be *of bounded type*. This is equivalent to the existence of an increasing sequence of bounded Borel sets E_k, $k = 1, 2, \ldots$, with $E_k \uparrow \mathbb{R}$, such that $D \subseteq \bigcup_{k=1}^{\infty} P(E_k)(X)$.

Let $P: \mathscr{B} \to L(X)$ be an equicontinuous spectral measure and $\Delta = [0, \alpha)$, where $0 < \alpha \leqq \infty$. Then $D_0 = \bigcup_{k=1}^{\infty} P([-k, k])(X)$ is a dense subspace of X such that for each $x \in D_0$, each of the functions $e^{-t(\cdot)}$, $t \in \Delta$, is $P(\cdot)(x)$-integrable. Accordingly, for each $t \in \Delta$, an element $T(t)$ of $\Pi(D_0)$ can be defined by the formula (1). In fact, for any dense subspace D of X, contained in D_0, which is invariant for each of the operators $T(t)$, $t \in \Delta$, the so constructed system $\{T; D; \Delta\}$ is a spectral local semigroup of bounded type. It will be said to *correspond to* (P, D, Δ).

3. STATEMENT OF RESULTS

Let $\Lambda = \{T; D; \Delta\}$ be a local semigroup. Let $\mathscr{N}$ be a family of continuous seminorms determining the topology of X. If c is a positive number belonging to Δ, then define for each $x \in D$ and $q \in \mathscr{N}$ the quantities

$$r_k(x, q, c) = \limsup_{n \to \infty} q([T(c/k) - I]^n (x))^{1/n}, \quad k = 1, 2, \ldots.$$

An element $x \in D$ is said to be a *binomial vector for Λ with respect to c*, if there exists a positive integer $k(x, c)$ such that

$$r_k(x, q, c) < 1, \quad q \in \mathscr{N}, \quad k \geqq k(x, c).$$

It is tacitly assumed that $k(x, c)$ is the minimal positive integer specified by these inequalities.

If $\omega \in \mathcal{M}_d$ is given by

(2) $$\omega = \sum_{k=1}^{N} c_k \delta_{t_k},$$

where $t_k \in \mathbb{R}$ and $c_k \in \mathbb{C}$, for each $k = 1, 2, \ldots, N$, and δ_t denotes the Dirac point mass at $t \in \mathbb{R}$, let

$$\omega_n = \sum_{j=1}^{N} c_j \binom{it_j}{n},$$

for each $n = 0, 1, 2, \ldots$, where $i = \sqrt{-1}$.

Let $x \in D$ be a binomial vector for Λ with respect to c. Since $\limsup_{n\to\infty} \left|\binom{z}{n}\right|^{1/n} \leqq 1$ for each complex number z, the series

$$b(x, c, \omega, k) = \sum_{n=0}^{\infty} \omega_n [T(c/k) - I]^n (x),$$

is absolutely convergent for each $\omega \in \mathcal{M}_d$ and each $k \geqq k(x, c)$. Accordingly, a subset $B(x, c)$ of X can be defined by

$$B(x, c) = \{b(x, c, \omega, k);\ \omega \in \Omega,\ k \geqq k(x, c)\}.$$

The main result can now be stated. It will be proved, along with the other results of this section, in § 4.

Theorem 1. *A local semigroup $\Lambda = \{T; D; \Delta\}$ is spectral and of bounded type if and only if for each $x \in D$, the function $T(\cdot)(x)$ is entire of exponential type and there exists a positive rational number $c \in \Delta$ such that the following conditions are satisfied.*

(i) *Every $x \in D$ is a binomial vector for Λ with respect to c.*

(ii) *For each $x \in D$, the set $B(x, c)$ is relatively weakly compact.*

(iii) *For each $q \in \mathcal{N}$ there exists a positive number $\alpha = \alpha(q)$ and seminorms $q_1, \ldots, q_r$ in $\mathcal{N}$ such that for each $x \in D$,*

$$q(\xi) \leqq \alpha \max\{q_j(x);\ 1 \leqq j \leqq r\}, \quad \xi \in B(x, c).$$

If the space X in Theorem 1 is a Banach space, then the hypothesis that $T(\cdot)(x)$ is entire of exponential type for each $x \in D$, can be omitted. This follows already from the conditions (i)–(iii) of the theorem (see the proof of Theorem 2). However, for non-normable spaces this is no longer the case. For example, let X denote the space of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$, equipped with the topology of pointwise convergence. Let $\Delta = [0, \infty)$ and $D = X$. For each $t \in \Delta$, define a continuous linear operator $T(t)$ by $T(t)(x) = y$, $x \in X$, where $y_n = e^{-tn} x_n$, for each $n = 1, 2, \ldots$. Then $\{T; D; \Delta\}$ is a local semigroup such that for any $c > 0$ the conditions (i)–(iii) of Theorem 1 are satisfied. However, there exist vectors $x \in D$ for which $T(\cdot)(x)$ has no entire extension of exponential type.

Theorem 2. *Let X be a Banach space and $\Lambda = \{T; D; \Delta\}$ a local semigroup. Then Λ is spectral and of bounded type if and only if there exists a positive rational number $c \in \Delta$, for which the conditions* (i) *and* (ii) *of Theorem* 1 *are satisfied, such that*

$$b(T) = \sup \{\|\xi\|;\ \xi \in B(x, c),\ x \in D,\ \|x\| \leqq 1\} < \infty . \tag{3}$$

In a reflexive Banach space a set is relatively weakly compact if and only if it is bounded. Hence, for reflexive spaces, the relative weak compactness of the sets in condition (ii) of Theorem 1 can be replaced by their boundedness. But, the boundedness of each of the sets $B(x, c)$, $x \in D$, follows from (3). Hence, Theorem 2 implies the following result due to Kantorovitz and Hughes [6; Theorem 3.3].

Corollary. *Let X be a reflexive Banach space and $\Lambda = \{T; D; \Delta\}$ a local semigroup. Then Λ is spectral and of bounded type if and only if there exists a positive rational number $c \in \Delta$ such that each $x \in D$ is a binomial vector for Λ with respect to c and* (3) *holds.*

There is a class of spaces, including many non-normable ones, for which the conditions (i)–(iii) of Theorem 1 suffice to guarantee that a given local semigroup in such a space is spectral, but not necessarily of bounded type.

A locally convex space X is said to be *weakly Σ-complete* if every sequence $\{x_n\}_{n=1}^{\infty}$ of its elements such that $\{\langle x_n, x'\rangle\}_{n=1}^{\infty}$ is absolutely summable for each $x' \in X'$, is itself summable to an element of X. In [9], such a space is said to have the B-P property. Weakly sequentially complete spaces, in particular reflexive spaces, are weakly Σ-complete. According to a theorem of Tumarkin [12], generalizing the well known result of Bessaga and Pełczyński, a space is weakly Σ-complete if and only if it does not contain an isomorphic copy of the space c_0.

Theorem 3. *Let X be a weakly Σ-complete space and $\Lambda = \{T; D; \Delta\}$ be a local semigroup. If there exists a positive rational number $c \in \Delta$ for which the conditions* (i)–(iii) *of Theorem* 1 *are satisfied, then Λ is a spectral local semigroup.*

4. PROOFS OF RESULTS

To prove the necessity of the conditions in Theorem 1, let $P: \mathscr{B} \to L(X)$ be an equicontinuous spectral measure and $\Lambda = \{T; D; \Delta\}$ be a spectral local semigroup of bounded type corresponding to (P, D, Δ).

Let $x \in D$. Then there exists a positive integer $m = m(x)$ such that $x \in P([-m, m])(X)$. Define an entire function with values in X by

$$z \mapsto \int_{-m}^{m} e^{-zs}\, dP(s)(x) = \int_{\mathbb{R}} e^{-zs}\, dP(s)(x), \quad z \in \mathbb{C} .$$

This function agrees with $T(\cdot)(x)$ on Δ (cf. (1)). It is again denoted by $T(\cdot)(x)$. It follows that for each $x' \in X'$ and $\varepsilon > 0$ the inequalities

$$\left|\mathrm{e}^{-(m+\varepsilon)|z|}\langle T(z)(x), x'\rangle\right| \leqq \mathrm{e}^{-\varepsilon|z|}\left|\langle P(\cdot)(x), x'\rangle\right|([-m, m]), \quad z \in \mathbb{C},$$

are valid. This shows that $T(\cdot)(x)$ is entire of exponential type.

Let c be any positive number in Δ. For each $t \in \mathbb{R}$, consider the series

$$v \mapsto \sum_{n=0}^{\infty} \binom{it}{n} (\mathrm{e}^{-cv/k} - 1)^n, \quad |v| \leqq m. \tag{4}$$

It follows, from the ratio test for example, that if $k(x, c)$ is chosen to be the smallest integer k satisfying $k > cm/\ln 2$, then the series (4) is absolutely convergent for all $t \in \mathbb{R}$ and all $k \geqq k(x, c)$, to the function

$$v \mapsto [1 + (\mathrm{e}^{-cv/k} - 1)]^{it} = \exp(-ictv/k), \quad |v| \leqq m. \tag{5}$$

Let $q \in \mathcal{N}$. If U_q^0 denotes the polar of $q^{-1}([0, 1])$, then

$$q(y) = \sup\{|\langle y, x'\rangle|;\ x' \in U_q^0\}, \quad y \in X. \tag{6}$$

Since the identity

$$[T(c/k) - I]^n(x) = \int_{-m}^{m} (\mathrm{e}^{-cv/k} - 1)^n \, \mathrm{d}P(v)(x), \tag{7}$$

is valid for each $k = 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$, it follows from (6) and (7) that for each $k = 1, 2, \ldots$, the inequality

$$q([T(c/k) - I]^n(x)) \leqq \gamma(x, q)(\mathrm{e}^{cm/k} - 1)^n,$$

is valid, where $\gamma(x, q) = \sup\{|\langle P(\cdot)(x), x'\rangle|(\mathbb{R});\ x' \in U_q^0\}$ is finite [9; II Lemma 1.1]. Accordingly,

$$r_k(x, q, c) = \limsup_{n \to \infty} q([T(c/k) - I]^n(x))^{1/n} \leqq (\mathrm{e}^{cm/k} - 1) < 1,$$

for all $k \geqq k(x, c)$. Since $q \in \mathcal{N}$ was arbitrary, this shows that x is a binomial vector for Λ with respect to c.

If $t \in \mathbb{R}$, then it follows for each $k \geqq k(x, c)$, the partial sums of the series (4) are uniformly bounded. Accordingly, if $\omega \in \Omega$ is given by (2), then the identities (4) and (5) and the Dominated Convergence Theorem for vector measures [9; II Theorem 4.2] imply that

$$\int_{-m}^{m} \sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} \binom{it_j}{n} (\mathrm{e}^{-cv/k} - 1)^n \, \mathrm{d}P(v)(x) = \int_{-m}^{m} \sum_{j=1}^{N} c_j \exp(-icvt_j/k) \, \mathrm{d}P(v)(x), \tag{8}$$

for each $k \geqq k(x, c)$. However, for each $k = 1, 2, \ldots$, we also have

$$\sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} \binom{it_j}{n} (\mathrm{e}^{-cv/k} - 1)^n = \sum_{n=0}^{\infty} \omega_n (\mathrm{e}^{-cv/k} - 1)^n, \quad |v| \leqq m.$$

Again by the Dominated Convergence Theorem and (7) it follows that

$$\int_{-m}^{m} \sum_{j=1}^{N} c_j \sum_{n=0}^{\infty} \binom{it_j}{n} (e^{-cv/k} - 1)^n \, dP(v)(x) = \sum_{n=0}^{\infty} \omega_n [T(c/k) - I]^n (x), \tag{9}$$

for all $k \geqq k(x, c)$. The identities (8) and (9) imply that

$$b(x, c, \omega, k) = \int_{\mathbb{R}} (\chi_{[-m,m]}(v) \sum_{j=1}^{N} c_j \exp(-icvt_j/k)) \, dP(v)(x), \tag{10}$$

for all $k \geqq k(x, c)$. Furthermore, the inequality $\|\hat{\omega}\|_\infty \leqq 1$ implies that the supremum norm of the integrand in (10) does not exceed 1. It follows [9; IV Lemma 6.1] that

$$B(x, c) \subseteq \overline{\text{bco}}(P(\cdot)(x))(\mathscr{B}),$$

and hence, that $B(x, c)$ is relatively weakly compact [9; IV Theorem 6.1].

To verify condition (iii) in Theorem 1, let

$$\mathscr{A} = \left\{ \int_{\mathbb{R}} f \, dP; \; \|f\|_\infty \leqq 1, \, f \text{ measurable} \right\}.$$

Then $\mathscr{A}$ is an equicontinuous part of $L(X)$, [13; Proposition 2.1]. Hence, if $q \in \mathscr{N}$, then there exists $\alpha = \alpha(q) > 0$ and seminorms $q_1, \ldots, q_r$ in $\mathscr{N}$ such that

$$q(S(x)) \leqq \alpha \max \{q_j(x); \; 1 \leqq j \leqq r\}, \quad x \in X, \tag{11}$$

for all $S \in \mathscr{A}$. Fix $x \in D$. If $\xi \in B(x, c)$, then it was noted (cf. (10)) that there exists a measurable function f with $\|f\|_\infty \leqq 1$ such that

$$\xi = \int_{\mathbb{R}} f(v) \, dP(v)(x) = \left(\int_{\mathbb{R}} f \, dP \right)(x). \tag{12}$$

It follows from (11), (12) and the definition of $\mathscr{A}$ that $q(\xi) \leqq \alpha \max \{q_j(x); \; 1 \leqq \leqq j \leqq r\}$. Hence, condition (iii) is verified. This completes the proof of necessity. □

The proof of the sufficiency of the conditions in Theorem 1 is based on the following lemma. Its proof is a combination of the Bochner-Schoenberg Criterion and the proof of Lemma 3.6 of [6]. Even though some of the arguments and calculations are identical to those in the proof of Lemma 3.6 in [6], they are included for completeness and ease of reading. Firstly however, some notation.

If $\Lambda = \{T; D; \Delta\}$ is a local semigroup and c is a positive rational number in Δ, then for each binomial vector x of Λ with respect to c we can define an entire, X-valued function $T_k(\cdot)(x)$, $k \geqq k(x, c)$, by

$$T_k(z)(x) = \sum_{n=0}^{\infty} \binom{z}{n} [T(c/k) - I]^n (x), \quad z \in \mathbb{C}. \tag{13}$$

Furthermore, if $k \geqq k(x, c)$ is fixed, then for each $q \in \mathscr{N}$ there exists a positive number $\beta = \beta(q)$ such that for each $\varepsilon > 0$ the set of numbers

$$\{e^{-(\beta+\varepsilon)|z|}q(T_k(z)(x));\ z \in \mathbb{C}\}, \tag{14}$$

is bounded. However, there may not exist a single number $\beta > 0$ such that (14) is bounded for all $q \in \mathcal{N}$ (cf. example in § 3). Accordingly, $T_k(\cdot)(x)$ may not be of exponential type, unless X is a Banach space.

Lemma 1. *Let $\Lambda = \{T; D; \Delta\}$ be a local semigroup and c a positive rational number in Δ. If $x \in D$ is a binomial vector for Λ with respect to c such that the set $B(x, c)$ is relatively weakly compact, then the function $T(\cdot)(x)$ has an entire extension and there exists a unique vector measure μ_x: $\mathcal{B} \to X$, such that each measure $\langle \mu_x, x' \rangle$, $x' \in X'$, has compact support and*

$$\langle T(z)(x), x' \rangle = \int_{\mathbb{R}} e^{-zs}\, d\langle \mu_x(s), x' \rangle, \quad z \in \mathbb{C}, \tag{15}$$

for each $x' \in X'$.

Proof. Let $c = d/e$. It will be shown that for each $k \geqq k(x, c)$, the function $z \mapsto T_{kd}(ekz)(x)$, $z \in \mathbb{C}$, is independent of k. Accordingly, if $T(\cdot)(x)$ is defined on $\mathbb{C}$ by

$$T(z)(x) = T_{kd}(ekz)(x), \quad z \in \mathbb{C}, \tag{16}$$

for any $k \geqq k(x, c)$, then $T(\cdot)(x)$ is entire and has the desired properties.

Let $\omega \in \Omega$ be given by (2). Then it follows from (13) that for each $k \geqq k(x, c)$,

$$\int_{\mathbb{R}} T_k(iv)(x)\, d\omega(v) = \sum_{j=1}^{N} c_j T_k(it_j)(x) = \sum_{n=0}^{\infty} \omega_n [T(c/k) - I]^n (x) \in B(x, c).$$

Accordingly, for each $k \geqq k(x, c)$ the set $\{\int_{\mathbb{R}} T_k(iv)(x)\, d\omega(v);\ \omega \in \Omega\}$ is relatively weakly compact. Furthermore, since the function $v \mapsto T_k(vi)(x)$, $v \in \mathbb{R}$, is bounded and weakly continuous it follows from the Bochner-Schoenberg Criterion that there exists a unique measure $v_x(k)$: $\mathcal{B} \to X$, with range contained in $\overline{\text{bco}}\, B(x, c)$, such that

$$T_k(is)(x) = \int_{\mathbb{R}} e^{-isv}\, d\mu_x(k)(v), \quad s \in \mathbb{R},$$

for each $k \geqq k(x, c)$. Furthermore, for each $x' \in X'$, $\langle \mu_x(k)(\cdot), x' \rangle$ is the unique Borel measure on $\mathbb{R}$ such that

$$\langle T_k(is)(x), x' \rangle = \int_{\mathbb{R}} e^{-isv}\, d\langle \mu_x(k)(v), x' \rangle, \quad s \in \mathbb{R}, \tag{17}$$

for each $k \geqq k(x, c)$.

Since the function $\langle T_k(i\cdot)(x), x' \rangle$ is entire of exponential type (cf. (14)) and is bounded on the real line, the Paley-Wiener-Schwartz theorem [3; Ch. 6, Theorem 5] implies that its Fourier transform (which is $2\pi\langle \mu_x(k), x' \rangle$ by (17)) has compact support. The bilateral Laplace transform

$$\int_{\mathbb{R}} e^{-zv}\, d\langle \mu_x(k)(v), x' \rangle, \quad z \in \mathbb{C},$$

is therefore well defined, entire and coincides with $\langle T_k(\cdot)(x), x'\rangle$ on the imaginary axis (by (17)). Hence,

$$\langle T_k(z)(x), x'\rangle = \int_{\mathbb{R}} e^{-zv}\, d\langle \mu_x(k)(v), x'\rangle \,, \tag{18}$$

for every $z \in \mathbb{C}$.

If N is a positive integer, then for each $k \geqq k(x, c)$,

$$T_k(N)(x) = \sum_{n=0}^{N} \binom{N}{n} [T(c/k) - I]^n (x) = T(c/k)^N (x)\,. \tag{19}$$

Since $c = d/e$ and $1/e$ belong to Δ, it follows that

$$T_{kd}(ekN)(x) = T(1/ek)^{ekN}(x) = [T(1/ek)^k]^{eN}(x) = [T(1/e)]^{eN}(x)\,, \tag{20}$$

for each $k \geqq k(x, c)$ and each positive integer N.

Fix $x' \in X'$ and $k, l \geqq k(x, c)$. The function

$$f(z) = \langle T_{kd}(ekz)(x), x'\rangle - \langle T_{ld}(elz)(x), x'\rangle\,, \quad z \in \mathbb{C}\,,$$

is a Laplace-Stieltjes transform (by (18)) which vanishes for positive integral values of z (cf. (20)). It follows from Lerch's theorem that $f(z) = 0$ for all $z \in \mathbb{C}$ [4; Theorem 6.2.2]. Accordingly, $T(\cdot)(x)$ is well defined by (16) and is entire.

If $r \in \Delta$ is a positive rational number, we may write $r = f/k$, where f and k are positive integers and $k \geqq k(x, c)$. Since $ef/ek = r \in \Delta$, it follows from (19) that

$$T_{kd}(ef)(x) = T(1/ek)^{ef}(x) = T(r)(x)\,.$$

The weak continuity of $T(\cdot)(x)$ on Δ then implies that it agrees with (16) on Δ. Hence, $T(\cdot)(x)$ has an entire extension.

Since $T_{kd}(ekz)(x) = T_{ld}(elz)(x)$, for all $k, l \geqq k(x, c)$ and all $z \in \mathbb{C}$, it follows from the identity

$$T_{kd}(ekis)(x) = \int_{\mathbb{R}} e^{-isekv}\, d\mu_x(kd)(v) = \int_{\mathbb{R}} e^{-is\xi}\, d\mu_x(kd)(\xi/ek)\,, \tag{21}$$

valid for all $s \in \mathbb{R}$ and $k \geqq k(x, c)$, and the uniqueness of Fourier-Stieltjes transforms that we may define a vector measure $\mu_x\colon \mathscr{B} \to X$ by

$$\mu_x(E) = \mu_x(kd)(E/ke)\,, \quad E \in \mathscr{B}\,,$$

for any $k \geqq k(x, c)$. It is clear from (16) and (21) that (15) is satisfied. This completes the proof of the lemma. □

We now prove the sufficiency of the conditions in Theorem 1. So, let $\Lambda = \{T; D; \Delta\}$ be a local semigroup and c be a positive rational number in Δ for which the conditions of Theorem 1 are satisfied.

For each $x \in D$, let $T(\cdot)(x)$ be the entire function and μ_x the X-valued measure as constructed in Lemma 1. Then for each $E \in \mathscr{B}$, define a map $P(E)\colon D \to X$ by

$$P(E)(x) = \mu_x(E), \quad x \in D. \tag{22}$$

Since for each complex number z the map $x \mapsto T(z)(x)$, $x \in D$, is linear (cf. (16)), it follows from (15) and the uniqueness of Laplace-Stieltjes transforms that the map $P(E)$ is linear.

Let $q \in \mathcal{N}$. Let $\alpha = \alpha(q) > 0$ and $q_1, \ldots, q_r \in \mathcal{N}$ be as given by condition (iii). For each $E \in \mathscr{B}$ it follows that

$$q(P(E)(x)) = q(\mu_x(E)) \leqq \alpha \max\{q_j(x);\ 1 \leqq j \leqq r\}, \quad x \in D,$$

since $\mu_x(E) \in \overline{\mathrm{bco}}\, B(x, c)$. Hence, each operator $P(E)$, $E \in \mathscr{B}$, is continuous on D and so can be extended uniquely to a continuous operator on all of X, still denoted by $P(E)$, which satisfies

$$q(P(E)(x)) \leqq \alpha \max\{q_j(x);\ 1 \leqq j \leqq r\}, \quad x \in X.$$

Accordingly, $P(\mathscr{B}) = \{P(E);\ E \in \mathscr{B}\}$ is an equicontinuous part of $L(X)$. Since $P(\cdot)(x)$ is σ-additive for each x in a dense subspace of X, it follows that $E \mapsto P(E)$, $E \in \mathscr{B}$, is an $L(X)$-valued measure.

If $x \in D$, then it follows from (15) and (22) that

$$\langle x, x' \rangle = \langle T(0)(x), x' \rangle = \langle P(\mathbb{R})(x), x' \rangle, \quad x' \in X'.$$

Accordingly, $P(\mathbb{R}) = I$. The next step is to show that P is multiplicative. Since $\|\mathrm{e}^{-\mathrm{i}s(\cdot)}\|_\infty \leqq 1$ for each $s \in \mathbb{R}$, it follows from (15) that for each $x \in D$ and $s \in \mathbb{R}$,

$$T(\mathrm{i}s)(x) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sv}\, \mathrm{d}P(v)(x) \in \overline{\mathrm{bco}}\,(P(\cdot)(x))(\mathscr{B}) \subseteq \overline{\mathrm{bco}}\, B(x, c). \tag{23}$$

Hence, if $q \in \mathcal{N}$, then by condition (iii) there is $\alpha > 0$ and seminorms $q_1, \ldots, q_r$ in $\mathcal{N}$ such that

$$q(T(\mathrm{i}s)(x)) \leqq \alpha \max\{q_j(x);\ 1 \leqq j \leqq r\}, \quad s \in \mathbb{R}, \tag{24}$$

for each $x \in D$. Since D is dense in X, each operator $T(\mathrm{i}s)$, $s \in \mathbb{R}$, has a unique continuous extension to all of X, again denoted by $T(\mathrm{i}s)$, such that (24) is valid for all $x \in X$. Hence, $\{T(\mathrm{i}s);\ s \in \mathbb{R}\}$ is an equicontinuous part of $L(X)$ and it follows from (23) and the uniqueness of continuous extension that

$$T(\mathrm{i}s)(x) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sv}\, \mathrm{d}P(v)(x), \quad s \in \mathbb{R}, \tag{25}$$

for each $x \in X$. Arguing as in the proof of Theorem 3.3 in [6] it follows that $T(\mathrm{i}\cdot)\colon \mathbb{R} \to L(X)$ is an equicontinuous group. It then follows from the group property and (25) that P is necessarily multiplicative.

It remains to show that for each $x \in D$, the measure $P(\cdot)(x) = \mu_x$ has compact support. Fix $x \in D$. By hypothesis, the entire extension of $T(\cdot)(x)$ (as constructed in Lemma 1), is of exponential type. Hence, there exists a positive number $\beta = \beta(x)$

such that for each $\varepsilon > 0$ and $x' \in X'$ there is a number $M = M(x, \varepsilon, x')$ such that

$$|\langle T(z)(x), x' \rangle| \leqq M e^{(\beta+\varepsilon)|z|}, \quad z \in \mathbb{C}. \tag{26}$$

Since for each $x' \in X'$, the function $\langle T(\mathrm{i}\cdot)(x), x' \rangle$ is entire of exponential type and is bounded on $\mathbb{R}$, it follows from (26) and the Paley-Wiener-Schwartz theorem that its Fourier transform, which is $2\pi\langle \mu_x, x' \rangle$ by (15), has support contained in $[-\beta, \beta]$. Since β is independent of x', it follows that μ_x has compact support. Hence, Λ is a spectral local semigroup of bounded type. This completes the proof of Theorem 1. □

Proof of Theorem 2. Suppose that Λ is a spectral local semigroup of bounded type corresponding to (P, D, Δ), where $P: \mathscr{B} \to L(X)$ is a spectral measure. Let c be an arbitrary positive number in Δ. Theorem 1 implies that the conditions (i)–(iii) are satisfied. Furthermore, by condition (iii) there is $\alpha > 0$ such that for each $x \in D$,

$$\|\xi\| \leqq \alpha \|\xi\|, \quad \xi \in B(x, c).$$

It follows easily that $b(T) \leqq \alpha < \infty$. Hence, (3) is valid.

Conversely, suppose that there is a positive rational number $c \in \Delta$ for which the stated requirements of Theorem 2 are satisfied. If $x \in D$, then it is easily verified that for each $\beta > 0$,

$$r_k(\beta x, \|\cdot\|, c) = r_k(x, \|\cdot\|, c), \quad k = 1, 2, \ldots$$

and $k(\beta x, c) = k(x, c)$. It follows that if $\xi \in B(x, c)$, then $\beta\xi \in B(\beta x, c)$. Hence, fix $x \in D$. If $\xi \in B(x, c)$, then $\xi/\|x\|$ belongs to $B(x/\|x\|, c)$. It follows from (3) that $\|\xi\|/\|x\| \leqq b(T)$. That is, for each $x \in D$,

$$\|\xi\| \leqq b(T)\, \|x\|, \quad \xi \in B(x, c).$$

Hence, conditions (i)–(iii) of Theorem 1 are satisfied.

It follows (cf. proof of Theorem 1) that there exists a spectral measure $P: \mathscr{B} \to L(X)$ such that for each $x \in D$ and $x' \in X'$ the measure $\langle P(\cdot)(x), x' \rangle$ has compact support and satisfies

$$\langle T(t)(x), x' \rangle = \int_{\mathbb{R}} \mathrm{e}^{-ts} \, \mathrm{d}\langle P(s)(x), x' \rangle, \quad t \in \Delta. \tag{27}$$

Fix $x \in D$. It follows from Rybakov's theorem [9; VI Theorem 3.2] (or from a well known result of W. Bade [1; Theorem 3.1]) that there exists $x' \in X'$ such that $P(\cdot)(x)$ is absolutely continuous with respect to $\langle P(\cdot)(x), x' \rangle$. Hence, $P(\cdot)(x)$ has compact support. Then each of the functions $\mathrm{e}^{-t(\cdot)}$, $t \in \Delta$, is $P(\cdot)(x)$-integrable and it follows from (27) that (1) is valid. Hence, Λ is a spectral local semigroup of bounded type. □

Proof of Theorem 3. As noted above, it follows from the conditions (i)–(iii) of Theorem 1 that there exists an equicontinuous spectral measure $P: \mathscr{B} \to L(X)$ such that for each $x \in D$ and $x' \in X'$ the measure $\langle P(\cdot)(x), x' \rangle$ has compact support and satisfies (27). That is, if $x \in D$, then for each $t \in \Delta$ the function $\mathrm{e}^{-t(\cdot)}$ is integrable with respect to each of the measures $\langle P(\cdot)(x), x'\rangle$, $x' \in X'$. Hence, each function $e^{-t(\cdot)}$, $t \in \Delta$, is actually $P(\cdot)(x)$-integrable [9; II Theorem 5.1]. It then follows from (27) that the identity (1) is valid for each $x \in D$ and $t \in \Delta$, that is, Λ is spectral. □

References

[1] *Bade, W.:* On Boolean algebras of projections and algebras of operators. Trans. Amer. Soc. *80*, 345—359 (1955).

[2] *Devinatz, A.:* A note on semigroups of unbounded self-adjoint operators. Proc. Amer. Math. Soc. *5*, 101—103 (1954).

[3] *Friedman, A.:* Generalized functions and partial differential equations. Englewood Cliffs: Prentice Hall 1963.

[4] *Hille, E., Phillips, R. S.:* Functional analysis and semigroups. Rev. ed. Amer. Math. Soc. Colloq. Publ., Vol. 31. Providence: Amer. Math. Soc. 1957.

[5] *Hughes, R. J.:* Semigroups of unbounded linear operators in Banach space. Trans. Amer. Math. Soc. *230*, 113—145 (1977).

[6] *Kantorovitz, S., Hughes, R. J.:* Spectral representation of local semigroups. Math. Ann. *259*, 455—470 (1982).

[7] *Klein, A., Landau, L.:* Construction of a unique self-adjoint generator for a symmetric local semigroup. J. Funct. Analysis *44*, 121—137 (1981).

[8] *Kluvánek, I.:* Characterization of Fourier-Stieltjes transforms of vector and operator valued measures. Czechoslovak Math. *J.* 17 (92), 261—276 (1967).

[9] *Kluvánek, I., Knowles, K.:* Vector measures and control systems. Amsterdam: North Holland 1976.

[10] *Nussbaum, A. E.:* Integral representation of semigroups of unbounded self-adjoint operators. Ann. of Math. *69*, 133—141 (1959).

[11] *Nussbaum, A. E.:* Spectral representation of certain one-parametric families of symmetric operators in Hilbert space. Trans. Amer. Math. Soc. *152*, 419—429 (1970).

[12] *Tumarkin, Ju. B.:* On locally convex spaces with basis. Dokl. Akad. Nauk SSSR, *195*, 1278—1281; Soviet Math. Dokl. 11, 1672—1675 (1970).

[13] *Walsh, B.:* Structure of spectral measures on locally convex spaces. Trans. Amer. Math. Soc. *120*, 295—326 (1965).

Author's address: School of Mathematical Sciences, The Flinders University of South Australia, Bedford Park 5042, Australia.

Current address: Department of Mathematies I. A. S., Australian National University, Canberra 2600, Australia.