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UNIVERSAL CYCLICALLY ORDERED SETS

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Let $\mathscr{C}$ be a class of structures and m a cardinal. A structure $\boldsymbol{Q} \in \mathscr{C}$ is an m-universal element in the class $\mathscr{C}$ iff for any structure $\boldsymbol{G} \in \mathscr{C}$ with card $G \leqq m$ there exists a substructure $\boldsymbol{G}' \subseteqq \boldsymbol{Q}$ isomorphic with $\boldsymbol{G}$. So, for instance, the ordinal power ${}^{\omega_i}\mathbf{2}$, i.e. the set of all sequences of 0's and 1's with length ω_i, ordered by the principle of the first difference, is an ω_i-universal linearly ordered set ([8], Théorème I). The cardinal power of type $\mathbf{2}^m$, i.e. the set of all mappings of a set M of cardinality m into $\{0, 1\}$ ordered by $f \leqq g \Leftrightarrow f(x) \leqq g(x)$ for all $x \in M$ is an m-universal ordered set ([7], Theorem 1). A set of type $F(\omega_i, \aleph_i)$, i.e. a set of all sequences of type ω_i composed from elements of a set of cardinality $\aleph_i$ with the relation $(a_k;\ k < \omega_i) \leqq (b_k;\ k < \omega_i)$ iff $(a_k;\ k < \omega_i)$ is a subsequence of $(b_k;\ k < \omega_i)$ is an $\aleph_i$-universal quasi-ordered set ([4], Theorem 2 and [3]). The aim of this paper is a construction of an m-universal cyclically ordered set. The universality is here meant in a weaker sense: to any cyclically ordered set $\boldsymbol{G} = (G, C)$ with card $G = m$ there exists a subset $\boldsymbol{G}'$ of the constructed m-universal cyclically ordered set such that $\boldsymbol{G}$ is a strongly homomorphic image of $\boldsymbol{G}'$.

1. Basic notions. A *cyclic order* on a set G is a ternary relation C on G which is

(i) *asymmetric*, i.e. $(x, y, z) \in C \Rightarrow (z, y, x) \bar{\in} C$,

(ii) *cyclic*, i.e. $(x, y, z) \in C \Rightarrow (y, z, x) \in C$,

(iii) *transitive*, i.e. $(x, y, z) \in C,\ (x, z, u) \in C \Rightarrow (x, y, u) \in C$.

If G is a set and C a cyclic order on G, then the pair $\boldsymbol{G} = (G, C)$ is called a *cyclically ordered set*. If, moreover, card $G \geqq 3$ and C is

(iv) *linear*, i.e. $x, y, z \in G,\ x \neq y \neq z \neq x \Rightarrow$ either $(x, y, z) \in C$ or $(z, y, x) \in C$,

then $\boldsymbol{G} = (G, C)$ is called a *linearly cyclically ordered set* or a *cycle*. If $C = \emptyset$, then $\boldsymbol{G} = (G, \emptyset)$ is called a *discrete cyclically ordered set*. Sometimes, for a cyclically ordered set $\boldsymbol{G} = (G, C)$ we denote by $\mathfrak{R}(\boldsymbol{G})$ the relation of $\boldsymbol{G}$, i.e. $\mathfrak{R}(\boldsymbol{G}) = C$. An element $x \in G$, where $\boldsymbol{G} = (G, C)$ is a cyclically ordered set, is called *isolated*, iff there exist no $y, z \in G$ with $(x, y, z) \in C$.

2. Homomorphism. Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ by cyclically ordered sets. A mapping $f\colon G \to H$ is called a *homomorphism* of $\boldsymbol{G}$ into $\boldsymbol{H}$ iff it has property

$$x, y, z \in G\,, \quad (x, y, z) \in C \Rightarrow (f(x), f(y), f(z)) \in D\,.$$

We denote by $\operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$ the set of all homomorphisms of $\boldsymbol{G}$ into $\boldsymbol{H}$. A homomorphism f of $\boldsymbol{G} = (G, C)$ into $\boldsymbol{H} = (H, D)$ is called *strong* iff it is surjective and has the property $u, v, w \in H$, $(u, v, w) \in D \Rightarrow$ there exist $x \in f^{-1}(u)$, $y \in f^{-1}(v)$, $z \in f^{-1}(w)$ with $(x, y, z) \in C$.

3. Power of cyclically ordered sets. Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be cyclically ordered sets. A *power* $\boldsymbol{G}^{\boldsymbol{H}}$ is a cyclically ordered set $\boldsymbol{K} = (K, E)$ where $K = \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$ and for $f, g, h \in K$ we have $(f, g, h) \in E \Leftrightarrow (f(x), g(x), h(x)) \in C$ for all $x \in H$.

It is easy to see that the relation E just defined is asymmetric, cyclic and transitive so that $\boldsymbol{G}^{\boldsymbol{H}}$ is in fact a cyclically ordered set.

Let $\mathbf{3}$ be a 3-element cycle, i.e. $\mathbf{3} = (\{0, 1, 2\}, \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\})$. One can expect – as an analogue to the class of ordered sets – that a power with base $\mathbf{3}$ can serve as a universal cyclically ordered set. But the following example shows that this is not the case.

4. Example. *Let $\boldsymbol{H} = (H, D)$ be any cyclically ordered set. Then the power $\mathbf{3}^{\boldsymbol{H}}$ contains no 4-element cycle.*

Proof. Assume $f, g, h, k \in \operatorname{Hom}(\boldsymbol{H}, \mathbf{3})$ and $(f, g, h) \in \mathfrak{R}(\mathbf{3}^{\boldsymbol{H}})$, $(f, h, k) \in \mathfrak{R}(\mathbf{3}^{\boldsymbol{H}})$. Let $x \in H$ be any element. If $f(x) = 0$, then $(f, g, h) \in \mathfrak{R}(\mathbf{3}^{\boldsymbol{H}})$ implies $g(x) = 1$, $h(x) = 2$ and then $(f(x), h(x), k(x)) \in \mathfrak{R}(\mathbf{3})$ never holds. Analogously we obtain a contradiction if $f(x) = 1$ and if $f(x) = 2$.

Denote by $2\;\mathbf{3}$ the type of a cyclically ordered set which is a direct sum of two 3-element cycles, i.e. $2\;\mathbf{3} = (\{0, 1, 2, 0', 1', 2'\}, \{(0, 1, 2), (1, 2, 0), (2, 0, 1), (0', 1', 2'), (1', 2', 0'), (2', 0', 1')\})$, and for any cardinal m let $\boldsymbol{m}$ be the type of a discrete cyclically ordered set with cardinality m.

5. Main theorem. *Let m be any cardinal. Then for any cyclically ordered set $\boldsymbol{G} = (G, C)$ with* card $G = m$ *there exists in a cyclically ordered set of type $(2\;\mathbf{3})^{\boldsymbol{m}}$ a subset $\boldsymbol{G}'$ such that $\boldsymbol{G}$ is a strong homomorphic image of $\boldsymbol{G}'$.*

Proof. Let M be any set with card $M = m$ and let $\boldsymbol{M} = (M, \emptyset)$ be a discrete cyclically ordered set. Note that $\operatorname{Hom}(\boldsymbol{M}, 2\;\mathbf{3})$ contains all mappings $f\colon M \to \{0, 1, 2, 0', 1', 2'\}$. Let $i\colon G \to M$ be a bijection. Let us assign to any element $x \in G$ a subset $U(x) \subseteq \operatorname{Hom}(\boldsymbol{M}, 2\;\mathbf{3})$ by the following rule:

(1) If x is not isolated, then $U(x)$ is the set of all $f \in \operatorname{Hom}(\boldsymbol{M}, 2\;\mathbf{3})$ with the following properties:

   (i) There exist $y, z \in G - \{x\}$ such that $(z, y, x) \in C$ and $f(i(x)) = 0$, $f(i(y)) = 1$, $f(i(z)) = 2$;

(ii) f is a constant mapping on $M - \{i(x), i(y), i(z)\}$ with the value in the set $\{0', 1', 2'\}$.

(2) If x is isolated, then $U(x) = \{f\}$ where $f(i(x)) = 0$ and $f(t) = 0'$ for any $t \in M - \{i(x)\}$.

We show first that $x, y \in G$, $x \neq y$ implies $U(x) \cap U(y) = \emptyset$. Indeed, suppose the existence of an $f \in U(x) \cap U(y)$. By definition we have $f \in U(x) \Rightarrow f(i(x)) = 0$ and $f(t) \neq 0$ for any $t \in M - \{i(x)\}$, so that $i(x)$ is the only element of the set M for which f takes the value 0. The same holds for the set $U(y)$ and thus we have $i(x) = i(y)$. As i is a bijection, we have $x = y$. Hence $x \neq y$ implies $U(x) \cap U(y) = \emptyset$. Now, put $G' = \bigcup_{x \in G} U(x)$. As $G' \subseteq \mathrm{Hom}\,(\boldsymbol{M}, 2\,3)$, the structure $\boldsymbol{G}' = (G', \mathfrak{R}((2\,3)^{\boldsymbol{M}}) \cap G'^3)$ is a cyclically ordered set which is a substructure of $(2\,3)^{\boldsymbol{M}}$. According to the preceding note $\{U(x);\, x \in G\}$ is a decomposition of the set G' so that there exists an equivalence Θ on G' such that $G'/\Theta = \{U(x);\, x \in G\}$. For any $U_1, U_2, U_3 \in G'/\Theta$ put $(U_1, U_2, U_3) \in S$ iff there exist $f \in U_1$, $g \in U_2$, $h \in U_3$ with $(f, g, h) \in \mathfrak{R}((2\,3)^{\boldsymbol{M}})$. Then S is a ternary relation on G'/Θ and we show that U is an isomorphism of $\boldsymbol{G}$ onto $(G'/\Theta, S)$. Trivially, U is a bijection of G onto G'/Θ. Let $x, y, z \in G$, $(x, y, z) \in C$. Let us define mappings $f, g, h\colon M \to \{0, 1, 2, 0', 1', 2'\}$ as follows:

$f(i(x)) = 0,\ f(i(y)) = 2,\ f(i(z)) = 1,\ f(t) = 0'$ for any $t \in M - \{i(x), i(y), i(z)\}$;
$g(i(y)) = 0,\ g(i(z)) = 2,\ g(i(x)) = 1,\ g(t) = 1'$ for any $t \in M - \{i(x), i(y), i(z)\}$;
$h(i(z)) = 0,\ h(i(x)) = 2,\ h(i(y)) = 1,\ h(t) = 2'$ for any $t \in M - \{i(x), i(y), i(z)\}$.

We see that $(f(t), g(t), h(t)) \in \mathfrak{R}(2\,3)$ for any $t \in M$, i.e. $(f, g, h) \in \mathfrak{R}((2\,3)^{\boldsymbol{M}})$ and $f \in U(x)$, $g \in U(y)$, $h \in U(z)$. Thus, $(U(x), U(y), U(z)) \in S$. Conversely, let $x, y, z \in G$ and $(U(x), U(y), U(z)) \in S$. Then there exist $f \in U(x)$, $g \in U(y)$, $h \in U(z)$ with $(f, g, h) \in \mathfrak{R}((2\,3)^{\boldsymbol{M}})$. Then $f(i(x)) = 0$, $g(i(y)) = 0$, $h(i(z)) = 0$ and $(f(t), g(t), h(t)) \in \mathfrak{R}(2\,3)$ for any $t \in M$. Therefore necessarily $g(i(x)) = 1$, $h(i(x)) = 2$, $f(i(y)) = 2$, $h(i(y)) = 1$, $f(i(z)) = 1$, $g(i(z)) = 2$. As $\{f(i(x)), f(i(y)), f(i(z))\} = \{0, 1, 2\}$ and $f \in U(x)$, by condition (i) in the definition of set $U(x)$, we have $(y, z, x) \in C$ and also $(x, y, z) \in C$. Thus, U is an isomorphism of $\boldsymbol{G}$ onto $(G'/\Theta, S)$; this yields simultaneously that $(G'/\Theta, S)$ is a cyclically ordered set. Now, we show that the natural projection nat Θ is a strong homomorphism of a cyclically ordered set $\boldsymbol{G}'$ onto a cyclically ordered set $(G'/\Theta, S)$. Let $f, g, h \in G'$, $(f, g, h) \in \mathfrak{R}((2\,3)^{\boldsymbol{M}})$. By definition of the set G' there exist elements $x, y, z \in G$ with $f \in U(x)$, $g \in U(y)$, $h \in U(z)$ so that $(U(x), U(y), U(z)) \in S$. But $\mathrm{nat}\,\Theta(f) = U(x)$, $\mathrm{nat}\,\Theta(g) = U(y)$, $\mathrm{nat}\,\Theta(h) = U(z)$, thus $(\mathrm{nat}\,\Theta(f), \mathrm{nat}\,\Theta(g), \mathrm{nat}\,\Theta(h)) \in S$ and $\mathrm{nat}\,\Theta\colon G' \to G'/\Theta$ is a homomorphism of $\boldsymbol{G}'$ into $G'/\Theta, S)$. We immediately see that this homomorphism is surjective. Let $U_1, U_2, U_3 \in G'/\Theta$ and $(U_1, U_2, U_3) \in S$. By definition of the relation S, there exist $f \in U_1$, $g \in U_2$, $h \in U_3$ such that $(f, g, h) \in \mathfrak{R}((2\,3)^{\boldsymbol{M}})$ and, trivially, $f \in (\mathrm{nat}\,\Theta)^{-1}(U_1)$, $g \in (\mathrm{nat}\,\Theta)^{-1}(U_2)$, $h \in (\mathrm{nat}\,\Theta)^{-1}(U_3)$. Hence nat Θ is a strong homomorphism of $\boldsymbol{G}'$ onto $(G'/\Theta, S)$ and hence the composition $U^{-1} \circ \mathrm{nat}\, \Theta$ is a strong homomorphism of a cyclically ordered set $\boldsymbol{G}' \subseteq (2\ 3)^M$ onto a cyclically ordered set $\boldsymbol{G}$.

6. Remark. A cyclically ordered set of type $(2\ 3)^m$ has cardinality 6^m and is "*m*-universal" in the following weaker sense: To obtain all cyclically ordered sets of cardinality m up to isomorphisms, it suffices to take all subsets of a cyclically ordered set of type $(2\ 3)^m$ and all their strong homomorphic images.

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