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A NON ABSOLUTELY CONVERGENT INTEGRAL WHICH ADMITS TRANSFORMATION AND CAN BE USED FOR INTEGRATION ON MANIFOLDS

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"Which is the most profound and most difficult mathematical theorem that admits a concrete and unquestionable physical interpretation?
For me, Stokes' theorem is the number one candidate ..."

R. Thom, in: La Science malgré tout, Encyclopaedia Universalis Organum, p. 7

0. INTRODUCTION

J. Mawhin in [1] introduced the notion of the generalized Perron integral (GP-integral) in the n-dimensional Euclidean space $\boldsymbol{R}^n$, using a Riemann-type definition but restricting the class of "admissible" partitions of the domain of integration. He proved that the divergence theorem holds for the GP-integral for any differentiable function provided the integration domain is an interval. Modifying his results a little, we can assert the following two properties of the GP-integral:

1. Let $L = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \boldsymbol{R}^n$, $L_1 = [a_2, b_2] \times \ldots \times [a_n, b_n] \subset \boldsymbol{R}^{n-1}$, let $g: L \to \boldsymbol{R}^1$ be differentiable on L. Then $\partial g/\partial x_1$ is GP-integrable on L and

$$(\mathrm{GP})\int_L \frac{\partial g}{\partial x_1}\,\mathrm{d}x = \int_{L_1} [g(b_1, x_2, \ldots, x_n) - g(a_1, x_2, \ldots, x_n)]\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\,.$$

2. If, moreover, $\omega = g\,\mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$ then

$$(\mathrm{GP})\int_L \mathrm{d}\omega = \int_{\partial L} \omega\,,$$

where ∂L stands for the boundary of L.

On the other hand, the GP-integral has some not so nice properties. Firstly, it is not additive with respect to the integration domain, in the following sense:

If $L^1, L^2 \subset \boldsymbol{R}^n$, $n \geqq 2$, are compact non-overlapping intervals, $L = L^1 \cup L^2$ an interval, $f: L \to \boldsymbol{R}^1$ GP-integrable on L^i, $i = 1, 2$, then f need not be GP-integrable on L (for a counterexample, see [2]).

Secondly, the GP-integral (like the Perron integral) strongly depends on the coordinate system, so that no comprehensive transformation theorem is available. For example, if f is GP-integrable and φ is a rotation of the coordinate system, then $f \circ \varphi = h$ need not be GP-integrable. (Cf. [3], 11.4 with $f = \zeta$.)

The first disadvantage was removed in [2] by modifying the class of admissible partitions (moreover, a dominated convergence theorem was established). The other drawback was dealt with in [4] for $n = 2$.

In the present paper we give a definition of an integral in $\boldsymbol{R}^n$ based on partition of unity. We shall establish a transformation theorem and a divergence theorem. Further, we shall prove that the integral is an extension of the Lebesgue integral. Since the divergence theorem implies that partial derivatives of differentiable functions are integrable, our integral exists for some non absolutely integrable functions, thus being a true extension of the Lebesgue integral. As concerns the relationship between our and the Perron integral, let us mention that a partial derivative of a differentiable function need not be Perron integrable, while the function ζ from [3], 11.4 is Perron integrable but our integral of ζ does not exist.

1. PU-PARTITIONS

Let $\boldsymbol{R}^n$ denote the n-dimensional Euclidean space, $\boldsymbol{R}^1 = \boldsymbol{R}$, $\boldsymbol{R}^+ = (0, +\infty)$. For $M \subset \boldsymbol{R}^n$, the symbols $\operatorname{Cl} M$, $\operatorname{Int} M$, ∂M stand for the closure, interior and boundary of M, respectively. If M is Lebesgue measurable, then $m_n(M)$ denotes its n-dimensional measure (the index is omitted if there is no danger of misunderstanding). If $y \in \boldsymbol{R}^n$, $r > 0$, we denote

$$\begin{aligned}
\operatorname{diam} M &= \sup \{\|y - x\|;\ x, y \in M\}\,,\\
\operatorname{dist}(y, M) &= \inf \{\|y - x\|;\ x \in M\}\,,\\
B(y, r) &= \{x \in \boldsymbol{R}^n;\ \|y - x\| < r\}\,,\\
\Omega(M, r) &= \{x \in \boldsymbol{R}^n;\ \operatorname{dist}(x, M) < r\}\,.
\end{aligned}$$

For a function $f: M \to \boldsymbol{R}$ we denote

$$\operatorname{supp} f = \operatorname{Cl}\{x \in \boldsymbol{R}^n;\ f(x) \neq 0\}$$

(the support of f). Any function $\delta: M \to \boldsymbol{R}^+$ is called *a gauge* (on M).

Definition 1.1. Let $M \subset \boldsymbol{R}^n$ be bounded. A family

$$\Delta = \{(t^j, \vartheta_j);\ j = 1, \ldots, k\}\,, \tag{1.1}$$

where k is a positive integer, $t^j \in M$, $\vartheta_j: \boldsymbol{R}^n \to [0, 1]$ are C^1 functions with compact supports,

$$(1.2) \qquad 0 \leqq \sum_{j=1}^{k} \vartheta_j(x) \leqq 1 \quad \text{for} \quad x \in \boldsymbol{R}^n ,$$

$$(1.3) \qquad \operatorname{Int} \{x \in \boldsymbol{R}^n;\ \sum_{j=1}^{k} \vartheta_j(x) = 1\} \supset \operatorname{Cl} M ,$$

is called *a PU-partition of M* (the letters PU stand for "partition of unity").

If δ is a gauge on M, then a PU-partition (1.1) of M is said to be *δ-fine* if

$$(1.4) \qquad \operatorname{supp} \vartheta_j \subset B(t^j, \delta(t^j)) , \quad j = 1, \ldots, k .$$

For any PU-partition (1.1) of M we define

$$(1.5) \qquad \Sigma(\Delta) = \sum_{j=1}^{k} \int_{\boldsymbol{R}^n} \|x - t^j\| \sum_{i=1}^{n} \left|\frac{\partial \vartheta_j}{\partial x_i}\right| \mathrm{d}x .$$

(Notice that in fact we integrate over supp ϑ_j. In similar situations we shall often omit the index indicating the integration domain.)

Proposition 1.1. *Let $M \subset \boldsymbol{R}^n$ be compact. Then there exists $K > 0$ such that for every gauge δ on M there is a δ-fine PU-partition Δ of M with $\Sigma(\Delta) \leqq K$.*

Proof. Let us introduce an auxiliary function $\omega: \boldsymbol{R} \to \boldsymbol{R}$ of class C^∞ such that $\omega(s) = 0$ for $s \leqq -1$, $\omega(s) = 1$ for $s \geqq 1$, $\mathrm{d}\omega/\mathrm{d}s \geqq 0$, $\omega(-s) + \omega(s) = 1$ for $s \in \boldsymbol{R}$. Set

$$(1.6) \qquad h_{a,b,\sigma}(s) = \omega\left(\frac{s-a}{\sigma}\right) \omega\left(\frac{b-s}{\sigma}\right)$$

provided $a < b$. If $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ is a compact n-dimensional interval, put

$$\chi_{J,\sigma}(x) = \prod_{i=1}^{n} h_{a_i,b_i,\sigma}(x_i) ,$$

where $x \in \boldsymbol{R}^n$, $x = (x_1, \ldots, x_n)$. Then we have, for $t \in J$:

$$\lim_{\sigma \to 0+} \sum_{i=1}^{n} \int_{\boldsymbol{R}^n} \|x - t\| \left|\frac{\partial \chi_{J,\sigma}}{\partial x_i}\right| \mathrm{d}x = \int_{\partial J} \|x - t\| \, \mathrm{d}S_{n-1} \leqq \operatorname{diam} J \, \mathscr{m}_{n-1}(\partial J) .$$

Let us proceed to the proof proper of our proposition. Let $I \subset \boldsymbol{R}^n$ be a compact interval, $M \subset \operatorname{Int} I$. Given a gauge δ on M, we can put $\delta(x) = \frac{1}{2} \operatorname{dist}(x, M)$ for $x \in I \setminus M$, thus extending δ to the whole of I. It was shown in [2] that there exists $K_0 > 0$ (generally depending on I) such that for every gauge δ_0 on I there is a δ_0-fine P-partition Π (i.e. a family $\Pi = \{(t^j, J^j);\ j = 1, \ldots, k\}$ such that J^j are compact intervals, $t^j \in J^j$, $J^j \subset B(t^j, \delta_0(t^j))$) with $\Sigma_0(\Pi) = \sum_{j=1}^{k} \operatorname{diam} J^j \, \mathscr{m}_{n-1}(\partial J^j) \leqq K_0$. (Such a P-partition is constructed by repeatedly halving the edges of I, thus obtaining — after a finite number of steps — subintervals with the required properties. For a more precise result see Lemma 6.1 below.)

For the given gauge δ (extended to I in the way mentioned above) construct a $(\delta/2)$-fine P-partition $\Pi = \{(t^j, J^j); j = 1, \ldots, k\}$ with $\Sigma_0(\Pi) \leqq K_0$. Then

$$\Delta = \{(t^j, \chi_{J^j,\sigma}); j = 1, \ldots, k, \ t^j \in M\}$$

is a PU-partition of M if σ is sufficiently small; moreover, it is δ-fine provided $\sigma \leqq \frac{1}{2} \min \{\delta(t^j); j = 1, \ldots, k\}$. If we take σ so small that

$$\sum_{i=1}^{n} \int_{R^n} \|x - t^j\| \left|\frac{\partial \chi_{J^j,\sigma}}{\partial x_i}\right| \mathrm{d}x \leqq \operatorname{diam} J^j \, m_{n-1}(\partial J^j) + \frac{1}{k},$$

we obtain

$$\Sigma(\Delta) \leqq \sum_{j=1}^{k} \left[\operatorname{diam} J^j \, m_{n-1}(\partial J^j) + \frac{1}{k}\right] \leqq K_0 + 1 .$$

Moreover, the PU-partition Δ fulfils $\sum_{j=1}^{k} \chi_{J^j,\sigma}(x) \leqq 1$ for $x \in R^n$, which proves our proposition.

Remark 1.1. It is clear from the above proof that the constant K from the proposition can be chosen the same for all sets M which are subsets of a fixed compact interval.

2. DEFINITION AND ELEMENTARY PROPERTIES OF THE PU-INTEGRAL

Definition 2.1. Let $f: R^n \to R$ be a function with compact support. For a PU-partition Δ of $\operatorname{supp} f$ defined by (1.1) put

$$S(f, \Delta) = \sum_{j=1}^{k} f(t^j) \int_{R^n} \vartheta_j(x) \, \mathrm{d}x . \tag{2.1}$$

Let $\gamma \in R$ satisfy the following condition:

for every $\varepsilon > 0$, $K > 0$ there is a gauge δ on $\operatorname{supp} f$ such that

$$|\gamma - S(f, \Delta)| \leqq \varepsilon \tag{2.2}$$

provided Δ is a δ-fine PU-partition of $\operatorname{supp} f$ with $\Sigma(\Delta) \leqq K$.

Then f is said to be *PU-integrable* and we write

$$\gamma = (\mathrm{PU}) \int f(x) \, \mathrm{d}x ; \tag{2.3}$$

γ is called *the PU-integral of f*.

Remark 2.1. The above definition is a modification of the definitions of the Riemann and Perron integrals (cf. [3]). If we delete the requirement on $\Sigma(\Delta)$ and also that on the smoothness of the functions ϑ_j, then, taking ϑ_j to be the characteristic functions of (semiclosed) intervals $J^j = (a^j, b^j] = (a_1^j, b_1^j] \times \ldots \times (a_n^j, b_n^j]$ and assuming $t^j \in \operatorname{Cl} J^j$, we obtain the Perron integral. If, moreover, we admit only constant gauges δ, we arrive at the Riemann integral. (Evidently, $S(f, \Delta) = \sum_{j=1}^{k} f(t^j)\, m(J^j)$.)

Let us notice that the PU-integral has the elementary properties usually required of the concept of an integral, namely, monotonicity and linearity. Monotonicity is immediately seen from the definition if we observe that for a nonnegative function the integral sums, and hence also the integral itself, are nonnegative. The property of linearity can be formulated as follows: If $f_i\colon \boldsymbol{R}^n \to \boldsymbol{R}$ are PU-integrable, $c_i \in \boldsymbol{R}$, $i = 1, 2$, then $f\colon x \mapsto c_1 f_1(x) + c_2 f_2(x)$ is PU-integrable and

$$(\mathrm{PU})\int f\,\mathrm{d}x = c_1\,(\mathrm{PU})\int f_1\,\mathrm{d}x + c_2\,(\mathrm{PU})\int f_2\,\mathrm{d}x.$$

Indeed, it is evident that if f_i is PU-integrable and $c_i \in \boldsymbol{R}$, then $c_i f_i$ is PU-integrable and

$$(\mathrm{PU})\int c_i f_i\,\mathrm{d}x = c_i\,(\mathrm{PU})\int f_i\,\mathrm{d}x.$$

The additivity of the PU-integral immediately follows from an alternative equivalent definition of the PU-integral:

Definition 2.2. Let I be a compact interval in $\boldsymbol{R}^n$, let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$, $\operatorname{supp} f \subset I$. Let $\gamma \in \boldsymbol{R}$ satisfy the following condition:

for every $\varepsilon > 0$, $K > 0$ there is a gauge δ on I such that (2.2) holds provided Δ is a δ-fine PU-partition of I with $\Sigma(\Delta) \leqq K$.

Then f is said to be *PUI-integrable*, γ is its *PUI-integral* and we write

$$\gamma = (\mathrm{PUI})\int f\,\mathrm{d}x.$$

Theorem 2.1. *Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ have a compact support, let $I \subset \boldsymbol{R}^n$ be a compact interval, $I \supset \operatorname{supp} f$. Then f is PU-integrable if and only if it is PUI-integrable; the identity*

$$(\mathrm{PU})\int f\,\mathrm{d}x = (\mathrm{PUI})\int f\,\mathrm{d}x \tag{2.4}$$

holds provided one of the integrals exists.

Proof. 1. Let f be PU-integrable, $\varepsilon > 0$, $K > 0$. Find a gauge δ on $\operatorname{supp} f$ such that

$$\left|S(f, \Delta) - (\mathrm{PU})\int f\,\mathrm{d}x\right| \leqq \varepsilon$$

whenever Δ is a δ-fine PU-partition of $\operatorname{supp} f$ with $\Sigma(\Delta) \leqq K$. Let δ_0 be a gauge on I satisfying

(i) $\delta_0(x) = \delta(x)$ for $x \in \operatorname{supp} f$;

(ii) $\delta_0(x) = \frac{1}{2} \operatorname{dist}(x, \operatorname{supp} f)$ for $x \in I \setminus \operatorname{supp} f$. Let

$$\Delta_0 = \{(\tau^j, \zeta_j);\ j = 1, \dots, k\}$$

be a δ_0-fine PU-partition of I with $\Sigma(\Delta_0) \leqq K$ and denote

$$Q = \{j;\ \tau^j \in \operatorname{supp} f\},$$
$$\Delta = \{(\tau^j, \zeta_j);\ j \in Q\}.$$

It follows from (ii) that Δ is a PU-partition of $\operatorname{supp} f$ (in particular, $\sum_{j\in Q} \zeta_j(x) = 1$ for $x \in \Omega(\operatorname{supp} f, \eta) \cap \{x; \sum_{j=1}^{k} \zeta_j(x) = 1\}$ where $\eta = \min\{\frac{1}{2} \operatorname{dist}(\tau^j, \operatorname{supp} f);\ j = 1, \dots, k, j \notin Q_j\}$). Further, (i) implies that Δ is δ-fine and, finally, $\Sigma(\Delta) \leqq \Sigma(\Delta_0) \leqq K$. Evidently,

$$S(f, \Delta_0) = S(f, \Delta),$$

which proves that $(\mathrm{PUI}) \int f \, dx$ exists and (2.4) holds.

2. Let f be PUI-integrable, $\varepsilon > 0$, $K > 0$. Find a gauge δ on I such that

$$\left| S(f, \Delta) - (\mathrm{PUI}) \int f \, dx \right| \leqq \varepsilon \tag{2.5}$$

for every δ-fine PU-partition Δ of I with $\Sigma(\Delta) \leqq K + K_0 + 1$, where K_0 is such a constant that for any gauge $\tilde{\delta}$ on I there exists a $\tilde{\delta}$-fine PU-partition $\tilde{\Delta}$ of I with $\Sigma(\tilde{\Delta}) \leqq K_0$ (cf. Proposition 1.1). Let

$$\Delta = \{(t^j, \vartheta_j);\ j = 1, \dots, k\}$$

be a δ-fine (more precisely, $\delta|_{\operatorname{supp} f}$-fine) PU-partition of $\operatorname{supp} f$ with $\Sigma(\Delta) \leqq K$, and set $\vartheta(x) = \sum_{j=1}^{k} \vartheta_j(x)$ for $x \in \boldsymbol{R}^n$. Then $\vartheta\colon \boldsymbol{R}^n \to [0, 1]$ is of class C^1, has a compact support, $\operatorname{Int}\{x; \vartheta(x) = 1\} \supset \operatorname{supp} f$ and $|\partial\vartheta/\partial x_l| \leqq C$ for $l = 1, \dots, n$, where $C > 0$ is a constant.

Let δ^* be a gauge on I fulfilling

(i) $\delta^*(x) \leqq \delta(x)$ for $x \in I$;
(ii) $B(x, 2\delta^*(x)) \subset \operatorname{Int}\{x; \vartheta(x) = 1\}$ for $x \in \operatorname{supp} f$;
(iii) $\delta^*(x) \leqq [2\, m(I)\, Cn]^{-1}$ for $x \in I$;
(iv) $m(\bigcup_{x\in I} B(x, \delta^*(x))) \leqq 2\, m(I)$.

Let $Z = \{(\tau^i, \zeta_i);\ i = 1, \dots, s\}$ be a δ^*-fine PU-partition of I with $\Sigma(Z) \leqq K_0$. Put

$$\Delta^* = \Delta \cup \{(\tau^i, \zeta_i(1 - \vartheta));\ i = 1, \dots, s\}.$$

Then, as a consequence of the identity $\sum_{i=1}^{s} \zeta_i(x) = 1$ that holds in a neighbourhood of I, Δ^* is a δ-fine PU-partition of I since

$$\sum_{j=1}^{k} \vartheta_j(x) + \sum_{i=1}^{s} (1 - \vartheta(x))\, \zeta_i(x) = \vartheta(x) + (1 - \vartheta(x)) \sum_{i=1}^{s} \zeta_i(x)$$

for $x \in \boldsymbol{R}^n$. (ii) implies that $\zeta_i(x)\,(1 - \vartheta(x)) = 0$ for $x \in \boldsymbol{R}^n$ if $\tau_i \in \operatorname{supp} f$, so that

$$S(f, \Delta^*) = S(f, \Delta)\,. \tag{2.6}$$

Moreover,

$$\Sigma(\Delta^*) = \Sigma(\Delta) + \sum_{i=1}^{s} \int \|x - \tau^i\| \sum_{l=1}^{n} \left| \frac{\partial}{\partial x_l} [(1 - \vartheta(x))\,\zeta_i(x)] \right| \mathrm{d}x \leqq$$

$$\leqq \Sigma(\Delta) + \Sigma(Z) + \sum_{i=1}^{s} \int \|x - \tau^i\|\, \zeta_i(x) \sum_{l=1}^{n} \left| \frac{\partial \vartheta(x)}{\partial x_l} \right| \mathrm{d}x\,.$$

Each summand in the last term is integrated over $\operatorname{supp} \zeta_i \subset B(\tau^i, \delta^*(\tau^i))$, hence $\|x - \tau^i\| \leqq [2\,m(I)\, Cn]^{-1}$ by (iii). Using, moreover, the estimate $|\partial\vartheta/\partial x_l| \leqq C$ we obtain

$$\sum_{i=1}^{s} \int \|x - \tau^i\|\, \zeta_i(x) \sum_{l=1}^{n} \left| \frac{\partial \vartheta(x)}{\partial x_l} \right| \mathrm{d}x \leqq \frac{1}{2\,m(I)} \sum_{i=1}^{s} \int \zeta_i(x)\, \mathrm{d}x =$$

$$= \frac{1}{2\,m(I)} \int \sum_{i=1}^{s} \zeta_i(x)\, \mathrm{d}x\,.$$

Taking into account that $\sum_{i=1}^{s} \zeta_i(x) \leqq 1$ for $x \in \boldsymbol{R}^n$, we conclude by virtue of (iv) that $\Sigma(\Delta^*) \leqq K + K_0 + 1$. Hence

$$\left| S(f, \Delta^*) - (\mathrm{PUI}) \int f\, \mathrm{d}x \right| \leqq \varepsilon$$

so that (2.6) implies (2.5). Our theorem is proved.

3. TRANSFORMATION THEOREM

Theorem 3.1. *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ with a compact support be PU-integrable, let $G \subset \boldsymbol{R}^n$ be an open bounded set. Let $\psi: G \to \psi(G)$ be a C^1-diffeomorphism,* $\operatorname{supp} f \subset \psi(G)$.
Then $(f \circ \psi)\, |\det D\psi|$ is PU-integrable and

$$(\mathrm{PU}) \int f\, \mathrm{d}x = (\mathrm{PU}) \int f(\psi(y))\, |\det D\,\psi(y)|\, \mathrm{d}y\,. \tag{3.1}$$

Proof proceeds analogously to that of Theorem 1, [4]. Let $\varepsilon > 0$, $K > 0$. Let c_1, c_2 be positive constants to be fixed later. Find a gauge δ' on $\operatorname{supp} f$ such that for every δ'-fine PU-partition Δ' of $\operatorname{supp} f$ with $\Sigma(\Delta') \leqq K' = Kc_1$ we have

$$\left| (\mathrm{PU}) \int f(x)\, \mathrm{d}x - S(f, \Delta') \right| < \tfrac{1}{2}\varepsilon\,. \tag{3.2}$$

Further, find a gauge δ on $\operatorname{supp} f \circ \psi = \psi^{-1}(\operatorname{supp} f)$ such that

$$\psi(B(y, \delta(y))) \subset B(\psi(y), \delta'(\psi(y)))\,, \tag{3.3}$$

$$(3.4) \qquad |\det D\,\psi(\eta) - \det D\,\psi(y)| \leqq \frac{\varepsilon}{2c_2[1 + |f(\psi(y))|]},$$

$$(3.5) \qquad \|\psi(\eta) - \psi(y)\| \leqq (1 + \|D\,\psi(y)\|)\,\|\eta - y\|$$

for $y, \eta \in \operatorname{supp} f \circ \psi$, $\eta \in B(y, \delta(y))$.

Now let $\Delta = \{(y^j, \zeta_j);\ j = 1, \ldots, k\}$ be a δ-fine PU-partition of $\operatorname{supp} f \circ \psi$ with $\Sigma(\Delta) \leqq K$. Without loss of generality we may suppose $\operatorname{supp} \zeta_j \subset G$. Put $x^j = \psi(y^j)$, $\vartheta_j = \zeta_j \circ \psi^{-1}$. Then

$$\Delta' = \{(x^j, \vartheta_j),\ j = 1, \ldots, k\}$$

is a δ'-fine PU-partition of $\operatorname{supp} f$. Indeed, denote

$$Z = \{y \in \boldsymbol{R}^n;\ \sum_{j=1}^{k} \zeta_j(y) = 1\}, \quad \Theta = \{x \in \boldsymbol{R}^n;\ \sum_{j=1}^{k} \vartheta_j(x) = 1\}.$$

Then $\Theta = \psi(Z)$ since $\vartheta_j = \zeta_j \circ \psi^{-1}$. Moreover, since $\operatorname{Int} Z \supset \operatorname{supp} f \circ \psi$ and ψ is a diffeomorphism, we have

$$\operatorname{Int} \Theta \supset \psi(\operatorname{supp} f \circ \psi) = \operatorname{supp} f.$$

Secondly, we have to prove $\operatorname{supp} \vartheta_j \subset B(x^j, \delta'(x^j))$. But if $x \in \operatorname{supp} \vartheta_j$ then $\psi^{-1}(x) \in \operatorname{supp} \zeta_j$, that is, $\psi^{-1}(x) \in B(y^j, \delta(y^j))$, and hence, in virtue of (3.3), $x \in \psi(B(y^j, \delta(y^j)) \subset B(\psi(y^j), \delta'(\psi(y^j))) = B(x^j, \delta'(x^j))$. Finally, for $\Sigma(\Delta')$ we have

$$\Sigma(\Delta') = \sum_{j=1}^{k} \int_{\boldsymbol{R}^n} \|x - x^j\| \sum_{i=1}^{n} \left|\frac{\partial \vartheta_j}{\partial x_i}\right| \mathrm{d}x \leqq$$

$$\leqq \sum_{j=1}^{k} \int_{\boldsymbol{R}^n} \|\psi(y) - \psi(y^j)\| \sum_{i=1}^{n} \sum_{s=1}^{n} \left|\frac{\partial \zeta_j}{\partial y_s} \frac{\partial \psi_s^{-1}}{\partial x_i}\right| |\det D\,\psi(y)|\, \mathrm{d}y \leqq$$

$$\leqq c \sum_{j=1}^{k} \int_{\boldsymbol{R}^n} \|\psi(y) - \psi(y^j)\|\, |\det D\,\psi(y)| \sum_{s=1}^{n} \left|\frac{\partial \zeta_j}{\partial y_s}\right| \mathrm{d}y,$$

where c is a positive constant. By (3.5) we obtain

$$\Sigma(\Delta') \leqq c \max_j [1 + \|D\,\psi(y^j)\|] \sup\{|\det D\,\psi(y)|;\ y \in G_0\}\,.$$

$$\cdot \sum_{j=1}^{k} \int_{\boldsymbol{R}^n} \|y - y^j\| \sum_{s=1}^{n} \left|\frac{\partial \zeta_j}{\partial y_s}\right| \mathrm{d}y =$$

$$= Kc \max_j [1 + \|D\,\psi(y^j)\|] \sup\{|\det D\,\psi(y)|;\ y \in G_0\},$$

where $G_0 = \bigcup_{j=1}^{k} \operatorname{supp} \zeta_j \subset G$ is a compact set.

Thus, choosing

$$c_1 = c \max_j [1 + \|D\,\psi(y^j)\|] \sup\{|\det D\,\psi(y)|;\ y \in G_0\},$$

we have $\Sigma(\Delta') \leqq Kc_1 = K'$.

Now we have to estimate the difference

$$\left|(\mathrm{PU})\int f(x)\,\mathrm{d}x - S(f\circ\psi|\det D\psi|, \varDelta)\right| \leqq$$

$$\leqq \left|(\mathrm{PU})\int f(x)\,\mathrm{d}x - S(f, \varDelta')\right| + |S(f, \varDelta') - S(f\circ\psi|\det D\psi|, \varDelta)| \leqq$$

$$\leqq \tfrac{1}{2}\varepsilon + |S(f, \varDelta') - S(f\circ\psi|\det D\psi|, \varDelta)|$$

(cf. (3.2)). The last term is estimated as follows:

$$|S(f, \varDelta') - S(f\circ\psi|\det D\psi|, \varDelta)| =$$

$$= \left|\sum_{j=1}^{k}\left[f(x^j)\int_{\boldsymbol{R}^n}\vartheta_j(x)\,\mathrm{d}x - f(\psi(y^j))\,|\det D\,\psi(y^j)|\int_{\boldsymbol{R}^n}\zeta_j(y)\,\mathrm{d}y\right]\right| \leqq$$

$$\leqq \sum_{j=1}^{k}|f(x^j)|\left|\int_{\boldsymbol{R}^n}\vartheta_j(x)\,\mathrm{d}x - |\det D\,\psi(y^j)|\int_{\boldsymbol{R}^n}\zeta_j(y)\,\mathrm{d}y\right| \leqq$$

$$\leqq \sum_{j=1}^{k}|f(x^j)|\left|\int_{\boldsymbol{R}^n}\vartheta_j(\psi(y))\,|\det D\,\psi(y)| - \zeta_j(y)\,|\det D\,\psi(y^j)|\,\mathrm{d}y\right| \leqq$$

$$\leqq \sum_{j=1}^{k}|f(x^j)|\int_{\boldsymbol{R}^n}\zeta_j(y)\,|\det D\,\psi(y) - \det D\,\psi(y^j)|\,\mathrm{d}y \leqq$$

$$\leqq \sum_{j=1}^{k}|f(x^j)|\frac{\varepsilon}{2c_2[1+|f(x^j)|]}\int_{\boldsymbol{R}^n}\zeta_j(y)\,\mathrm{d}y \leqq \tfrac{1}{2}\varepsilon c_2^{-1}\sum_{j=1}^{k}\int_{\boldsymbol{R}^n}\zeta_j(y)\,\mathrm{d}y$$

(cf. (3.4)). By choosing the constant c_2 suitably (notice that $0 \leqq \sum_{j=1}^{k}\zeta_j(y) \leqq 1$ and we have assumed $\operatorname{supp}\zeta_j \subset G$ for $j = 1, \ldots, k$) we conclude that the estimated difference is less than $\frac{1}{2}\varepsilon$. Consequently,

$$\left|(\mathrm{PU})\int f(x)\,\mathrm{d}x - S(f\circ\psi|\det D\psi|, \varDelta)\right| \leqq \varepsilon\,,$$

which proves (3.1).

4. STOKES' THEOREM

Theorem 4.1. *Let $g: \boldsymbol{R}^n \to \boldsymbol{R}$ with a compact support have the differential Dg at every $x \in \boldsymbol{R}^n$. Put $f_p = \partial g/\partial x_p$, $p = 1, \ldots, n$. Then f_p is PU-integrable and*

$$(4.1)\qquad (\mathrm{PU})\int f_p\,\mathrm{d}x = 0\,.$$

Proof. Let $\varepsilon > 0$, $K > 0$. For every $t \in \boldsymbol{R}^n$ find $\delta(t) > 0$ such that

$$(4.2)\qquad \|g(x) - g(t) - Dg(t)(x - t)\| \leqq \frac{\varepsilon}{K}\|x - t\|$$

for $x \in B(t, \delta(t))$. Then $\delta\colon t \mapsto \delta(t)$ is a gauge on $\boldsymbol{R}^n$. Let (1.1) be a δ-fine PU-partition of supp g with $\Sigma(\Delta) \leqq K$. Denote

$$q_t(x) = g(t) + Dg(t)(x - t). \tag{4.3}$$

Then

$$\left| \int \sum_{j=1}^{k} (g(x) - q_{t^j}(x)) \frac{\partial \vartheta_j}{\partial x_p} \,\mathrm{d}x \right| \leqq \frac{\varepsilon}{K} \int \sum_{j=1}^{k} \|x - t^j\| \left| \frac{\partial \vartheta_j}{\partial x_p} \right| \mathrm{d}x \leqq \varepsilon$$

since $\Sigma(\Delta) \leqq K$. Further, since $\sum_{j=1}^{k} \vartheta_j(x) = 1$ for $x \in \operatorname{supp} g$, we have

$$\sum_{j=1}^{k} \int g \frac{\partial \vartheta_j}{\partial x_p} \,\mathrm{d}x = 0,$$

hence

$$\left| \sum_{j=1}^{k} \int q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \,\mathrm{d}x \right| \leqq \varepsilon. \tag{4.4}$$

On the other hand, integration by parts (with respect to x_p) yields

$$\int_{\boldsymbol{R}} q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \,\mathrm{d}x_p = -\frac{\partial g}{\partial x_p}(t^j) \int_{\boldsymbol{R}} \vartheta_j(x) \,\mathrm{d}x_p, \tag{4.5}$$

hence

$$\int q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_p}(x) \,\mathrm{d}x = -\frac{\partial g}{\partial x_p}(t^j) \int \vartheta_j(x) \,\mathrm{d}x. \tag{4.6}$$

Combining (4.4) and (4.6) we conclude

$$\left| \sum_{j=1}^{k} f_p(t^j) \int \vartheta_j(x) \,\mathrm{d}x \right| \leqq \varepsilon,$$

which proves (4.1).

Corollary. *If g, f_p satisfy the assumptions of Theorem* 4.1 *and $h\colon \boldsymbol{R}^n \to \boldsymbol{R}$ is of class C^1, then* (PU) $\int h f_p \,\mathrm{d}x$ *exists, $p = 1, \ldots, n$.*

Proof. By Theorem 4.1 we have

$$0 = (\mathrm{PU}) \int \frac{\partial}{\partial x_p}(hg) \,\mathrm{d}x = \int \frac{\partial h}{\partial x_p} g \,\mathrm{d}x + (\mathrm{PU}) \int h f_p \,\mathrm{d}x$$

and the first integral on the right hand side exists since the integrand is continuous.

Theorem 4.2. *Let $g\colon \boldsymbol{R}^n \to \boldsymbol{R}$ with a compact support have the differential Dg at every $x \in \boldsymbol{R}^n$. Put again $f_p = \partial g / \partial x_p$, $p = 1, \ldots, n$, and*

$$\chi(x) = \begin{cases} 1 & \text{for } x_1 \geqq 0, \\ 0 & \text{for } x_1 < 0 \end{cases}$$

(*of course we write* $x = (x_1, \ldots, x_n)$ *for* $x \in \boldsymbol{R}^n$). *Then*

$$(\mathrm{PU})\int \chi f_p \,\mathrm{d}x = 0 \quad \textit{for} \quad p = 2, \ldots, n\,,$$

$$(\mathrm{PU})\int \chi f_1 \,\mathrm{d}x = -\int_{\boldsymbol{R}^{n-1}} g(0, x_2, \ldots, x_n)\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\,.$$

Proof. We proceed similarly as in the proof of Theorem 4.1. Let $0 < \varepsilon < 1$, $K > 0$, $\varrho > 1$, $\operatorname{supp} g \subset B(0, \varrho - 1)$. Let $\delta: \boldsymbol{R}^n \to \boldsymbol{R}^+$ be a gauge found in the same way as above so that (4.2) holds. Moreover, let us suppose that $\delta(t) \leqq |t_1|/2$ provided $t = (t_1, \ldots, t_n)$, $t_1 \neq 0$, and that the following condition holds:

(4.7) Let $t \in \operatorname{supp} g$, $t = (0, t_2, \ldots, t_n)$. If $\max\{|f_p(t)|;\ p = 1, 2, \ldots, n\} \leqq 1$, then $\delta(t) \leqq 2^{-2}\,\varepsilon(2\varrho)^{-n+1}$; if $2^{m-1} < \max\{|f_p(t)|;\ p = 1, 2, \ldots, n\} \leqq 2^m$, $m = 1, 2, \ldots$, then $\delta(t) \leqq 2^{-2m-2}\,\varepsilon(2\varrho)^{-n+1}$.

Let (1.1) be a δ-fine PU-partition of $\operatorname{supp} g$ with $\Sigma(\Delta) \leqq K$. Let us denote $Q_+ = \{j;\ t_1^j > 0\}$, $Q_0 = \{j;\ t_1^j = 0\}$, $Q = Q_+ \cup Q_0$. Then

$$S(\chi f_p, \Delta) = \sum_{j\in Q} f_p(t^j) \int_{\boldsymbol{R}^n} \vartheta_j \,\mathrm{d}x\,.$$

In the same way as in the proof of Theorem 4.1 we establish an estimate analogous to (4.4), namely,

$$\sum_{j\in Q} \left| \int_{\boldsymbol{R}^n} \chi q_{t^j} \frac{\partial \vartheta_j}{\partial x_p} \,\mathrm{d}x \right| \leqq \varepsilon\,. \tag{4.8}$$

If $p \neq 1$, we integrate by parts with respect to x_p, thus obtaining (4.6), which yields

$$\sum_{j\in Q} f_p(t^j) \int_{\boldsymbol{R}^n} \chi \vartheta_j \,\mathrm{d}x \leqq \varepsilon\,.$$

Since $\chi \vartheta_j = \vartheta_j$ for $j \in Q_+$, we have

$$S(\chi f_p, \Delta) = \sum_{j\in Q} f_p(t^j) \int_{\boldsymbol{R}^n} \chi \vartheta_j \,\mathrm{d}x + \sum_{j\in Q_0} f_p(t^j) \int_{\boldsymbol{R}^n} (1 - \chi)\, \vartheta_j \,\mathrm{d}x\,,$$

so that

$$S(\chi f_p, \Delta) \leqq \varepsilon + \sum_{j\in Q_0} |f_p(t^j)| \int_{\boldsymbol{R}^n} \vartheta_j \,\mathrm{d}x\,. \tag{4.9}$$

In order to estimate the sum on the right hand side, put

$$Q_{00} = \{j \in Q_0;\ \max\{|f_p(t^j)|;\ p = 1, 2, \ldots, n\} \leqq 1\}\,,$$

$$Q_{0m} = \{j \in Q_0;\ 2^{m-1} < \max\{|f_p(t^j)|;\ p = 1, 2, \ldots, n\} \leqq 2^m\}\,,\quad m = 1, 2, \ldots$$

Since

$$0 \leqq \sum_{j \in Q_{0m}} \vartheta_j(x) \leqq 1 ,$$

$$\operatorname{supp} \sum_{j \in Q_{0m}} \vartheta_j \subset \left[-2^{-2m-2} \varepsilon(2\varrho)^{-n+1}, 2^{-2m-2} \varepsilon(2\varrho)^{-n+1}\right] \times$$

$$\times \left[-\varrho, \varrho\right] \times \ldots \times \left[-\varrho, \varrho\right] \subset \boldsymbol{R}^n ,$$

we have

$$\sum_{j \in Q_0} \left|f_p(t^j)\right| \int_{\boldsymbol{R}^n} \vartheta_j \, \mathrm{d}x \leqq \sum_{m=0}^{\infty} 2^m \int_{\boldsymbol{R}^n} \sum_{j \in Q_{0m}} \vartheta_j \, \mathrm{d}x \leqq \sum_{m=0}^{\infty} 2^m \, . \, 2^{-2m-1} \varepsilon(2\varrho)^{-n+1} (2\varrho)^{n-1} = \varepsilon ,$$

which together with (4.9) yields

$$\left|S(\chi f_p, \Delta)\right| \leqq 2\varepsilon .$$

If $p = 1$, integration by parts yields again

$$\int_{\boldsymbol{R}} \chi(x) \, q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \, \mathrm{d}x_1 = -f_1(t_j) \int_{\boldsymbol{R}} \vartheta_j(x) \, \mathrm{d}x_1$$

if $j \in Q_+$ (recall that $\delta(t^j) \leqq \left|t_1^j\right|/2$ in this case), and

$$\int_{\boldsymbol{R}} \chi(x) \, q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \, \mathrm{d}x_1 =$$

$$= -q_{t^j}(0, x_2, \ldots, x_n) \, \vartheta_j(0, x_2, \ldots, x_n) - f_1(t^j) \int_{\boldsymbol{R}} \chi(x) \, \vartheta_j(x) \, \mathrm{d}x_1$$

if $j \in Q_0$. This yields

$$S(\chi f_1, \Delta) = \sum_{j \in Q} f_1(t^j) \int_{\boldsymbol{R}^n} \vartheta_j \, \mathrm{d}x =$$

$$= -\sum_{j \in Q} \int_{\boldsymbol{R}^n} \chi(x) \, q_{t^j}(x) \frac{\partial \vartheta_j}{\partial x_1} (x) \, \mathrm{d}x + \sum_{j \in Q_0} f_1(t_j) \int_{\boldsymbol{R}^n} (1 - \chi) \, \vartheta_j \, \mathrm{d}x \; \cdot$$

$$- \sum_{j \in Q_0} \int_{\boldsymbol{R}^{n-1}} q_{t^j}(0, x_2, \ldots, x_n) \, \vartheta_j(0, x_2, \ldots, x_n) \, \mathrm{d}x_2 \ldots \mathrm{d}x_n .$$

By (4.8), (4.2) and (4.3) we obtain

$$\left|S(\chi f_1, \Delta) + \int_{\boldsymbol{R}^{n-1}} g(0, x_2, \ldots, x_n) \, \mathrm{d}x_2 \ldots \mathrm{d}x_n\right| \leqq \tag{4.10}$$

$$\leqq \varepsilon + \sum_{j \in Q_0} \left|f_1(t^j)\right| \int_{\boldsymbol{R}^n} \vartheta_j \, \mathrm{d}x +$$

$$+ \left|\int_{\boldsymbol{R}^{n-1}} g(0, x_2, \ldots, x_n) \, \mathrm{d}x_2 \ldots \mathrm{d}x_n - \sum_{j \in Q_0} \int_{\boldsymbol{R}^{n-1}} q_{t^j}(0, x_2, \ldots, x_n) \, \vartheta_j(0, x_2, \ldots, x_n) \, \mathrm{d}x_2 \ldots \mathrm{d}x_n\right| .$$

In the same way as the sum on the right hand side of (4.9) was estimated we obtain that

$$\sum_{j\in Q_0} |f_1(t^j)| \int_{R^n} \vartheta_j \, dx \leqq \varepsilon .$$

We still have to estimate the last term on the right hand side of (4.10).

Since $\sum_{j\in Q_0} \vartheta_j(0, x_2, \ldots, x_n) = 1$ for $(0, x_2, \ldots, x_n) \in \operatorname{supp} g$, we have

$$\int_{R^{n-1}} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n = \sum_{j\in Q_0} \int_{R^{n-1}} g(0, x_2, \ldots, x_n) \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n .$$

It follows from (4.7) that $\max \{\delta(t^j); j \in Q_0\} \leqq 2^{-2} \varepsilon (2\varrho)^{-n+1}$ and that $|x_j| < \varrho$ for $j = 2, 3, \ldots, n$ provided $\sum_{j\in Q_0} \vartheta_j(0, x_2, \ldots, x_n) > 0$. Therefore

$$\Bigg| \int_{R^{n-1}} g(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n -$$

$$- \sum_{j\in Q_0} \int_{R^{n-1}} q_{t^j}(0, x_2, \ldots, x_n) \, \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \Bigg| \leqq$$

$$\leqq \sum_{j\in Q_0} \int_{R^{n-1}} |g(0, x_2, \ldots, x_n) - q_{t^j}(0, x_2, \ldots, x_n)| \, \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leqq$$

$$\leqq \sum_{j\in Q_0} \int_{R^{n-1}} \frac{\varepsilon}{K} \, \|(0, x_2, \ldots, x_n) - t^j\| \, \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leqq$$

$$\leqq \frac{\varepsilon}{K} \max \{\delta(t^j); j \in Q_0\} \int_{R^{n-1}} \sum_{j\in Q_0} \vartheta_j(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \leqq$$

$$\leqq \frac{\varepsilon}{K} \, 2^{-2} \varepsilon (2\varrho)^{-n+1} (2\varrho)^{n-1} = \varepsilon^2 (4K)^{-1} .$$

This completes the proof of Theorem 4.2.

Let N be an n-manifold of class C^1 (without boundary or with a boundary ∂N). Let $\{(U_\lambda, h_\lambda); \lambda \in \Lambda\}$ be a system of charts. Put $V_\lambda = h_\lambda(U_\lambda) \subset R^n$. It may be assumed that for $\lambda \in \Lambda$ one of the following conditions is fulfilled:

(4.11) V_λ is open in R^n ;

(4.12) V_λ is relatively open in the half-space $\{x \in R^n; x_1 \geqq 0\}$, the points $x = (0, x_2, \ldots, x_n) \in V_\lambda$ being images of points of the boundary ∂N.

Let ξ be a p-form on N. For $\lambda \in \Lambda$ let us represent $(h_\lambda)^* \xi$ in the form

$$(h_\lambda)^* \xi = \sum_{i_1 < i_2 < \ldots < i_p} a^{(\lambda)}_{i_1, \ldots, i_p}(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_p} .$$

ξ is called *differentiable* if $a^{(\lambda)}_{i_1, \ldots, i_p}$ is differentiable on V_λ for all $\lambda, i_1, \ldots, i_p$. Assume that ξ is an n-form with a compact support. Let the following conditions be fulfilled:

(4.13) if $\lambda \in \Lambda$ and if (4.11) holds, then (PU) $\int \psi a_{1,\dots,n} \, dx$ exists for every C^1 function $\psi: R^n \to R$ such that $\operatorname{supp} \psi \subset V_\lambda$;

(4.14) if $\lambda \in \Lambda$ and if (4.12) holds, then (PU) $\int \varrho \, dx$ exists provided ϱ is defined by

$$\varrho(x) = \begin{cases} \psi(x) \, a_{1,\dots,n}(x) & \text{for} \quad x \in V_\lambda, \\ 0 & \text{otherwise}, \end{cases}$$

$\psi: R^n \to R$ being such a C^1 function that

$$\operatorname{supp} \psi \cap \{x \in R^n;\ x_1 \geqq 0\} \subset V_\lambda .$$

Then ξ is called *PU-integrable.*

Let ξ be PU-integrable. Let $\varphi_i: N \to [0, 1]$, $i = 1, \dots, m$ be such C^1 functions that

$$\operatorname{supp} \xi \subset \operatorname{Int} \{y \in N;\ \sum_{i=1}^{m} \varphi_i(y) = 1\}$$

and that for every i there exists such a $\lambda_i \in \Lambda$ that $\operatorname{supp} \varphi_i \subset U_{\lambda_i}$. Put

$$\Gamma(\xi, \varphi_1, \dots, \varphi_m, \lambda_1, \dots, \lambda_m) = \sum_{i=1}^{m} (\mathrm{PU}) \int (h_{\lambda_i})^* (\varphi_i \xi) \, dx$$

(we put $(h_{\lambda_i})^* (\varphi_i \xi)(x) = 0$ for $x \in R^n \setminus V_{\lambda_i}$). It can be proved that Γ is independent of $\varphi_1, \dots, \varphi_m, \lambda_1, \dots, \lambda_m$, so that it is called *the PU-integral of* ξ and denoted by (PU) $\int_N \xi$.

Theorem 4.3. (Stokes). *Let η be a differentiable $(n-1)$-form on N with a compact support. Then $d\eta$ is a PU-integrable n-form and*

$$(\mathrm{PU}) \int_N d\eta = \int_{\partial N} \eta . \tag{4.15}$$

Theorem 4.3 can be proved in a standard manner from Theorems 4.1 and 4.2 and Corollary of Theorem 4.1. The identity (4.15) also holds if N is replaced by a singular chain (cf. [5], [6]).

5. RELATION TO THE LEBESGUE INTEGRAL

Theorem 5.1. *Let $f: R^n \to R$ with a compact support have a convergent Lebesgue integral. Then it is PU-integrable and*

$$(\mathrm{PU}) \int f \, dx = (\mathrm{L}) \int f \, dx .$$

Proof. Let $I \subset R^n$ be a compact interval, $\operatorname{Int} I \supset \operatorname{supp} f$. Suppose first that f is bounded. Then for every $\alpha > 0$ there exist functions $u, v: I \to R$ satisfying the following conditions:

(i) u is upper semicontinuous, v is lower semicontinuous;
(ii) $u(x) \leqq f(x) \leqq v(x)$ for $x \in \boldsymbol{R}^n$;
(iii) $\int u \, dx + \alpha \geqq \int f \, dx \geqq \int v \, dx - \alpha$.

Moreover, choosing $\eta > 0$ such that $B(x, 2\eta) \subset I$ for $x \in \operatorname{supp} f$, we may assume $u(x) = 0 = v(x)$ for $x \in \boldsymbol{R}^n \setminus \Omega(\operatorname{supp} f, \eta) \supset \boldsymbol{R}^n \setminus I$.

Find a gauge δ on I such that

$$\delta(x) \leqq \eta \quad \text{for} \quad x \in I\,,$$
$$u(x) \leqq u(t) + \alpha\,, \quad v(x) \geqq v(t) - \alpha \quad \text{for} \quad x \in B(t, \delta(t))\,.$$

Let $\Delta = \{(t^j, \vartheta_j);\ j = 1, \ldots, k\}$ be a δ-fine PU-partition of I. Denote $g(x) = \sum_{j=1}^{k} f(t^j)\, \vartheta_j(x)$. Then, applying the identities $\sum_{j=1}^{k} \vartheta_j(x) = 1$ for $x \in I$, $u(x) = v(x) = g(x) = 0$ for $x \in \boldsymbol{R}^n \setminus I$, we obtain

$$u(x) - g(x) = \sum_{j=1}^{k} (u(x) - f(t^j))\, \vartheta_j(x) \leqq$$
$$\leqq \sum_{j=1}^{k} (u(x) - u(t^j))\, \vartheta_j(x) \leqq \alpha \sum_{j=1}^{k} \vartheta_j(x) \leqq \alpha\,,$$
$$g(x) - v(x) = \sum_{j=1}^{k} (f(t^j) - v(x))\, \vartheta_j(x) \leqq$$
$$\leqq \sum_{j=1}^{k} (v(t^j) - v(x))\, \vartheta_j(x) \leqq \alpha \sum_{j=1}^{k} \vartheta_j(x) \leqq \alpha$$

for $x \in \boldsymbol{R}^n$. Consequently,

$$\int u \, dx - S(f, \Delta) \leqq \alpha\, m(I)\,, \quad S(f, \Delta) - \int v \, dx \leqq \alpha\, m(I)\,,$$

and (iii) implies

$$\int f \, dx - S(f, \Delta) \leqq \int u \, dx - S(f, \Delta) + \alpha \leqq \alpha(m(I) + 1)\,,$$
$$S(f, \Delta) - \int f \, dx \leqq S(f, \Delta) - \int v \, dx + \alpha \leqq \alpha(m(I) + 1)\,.$$

Thus, given $\varepsilon > 0$, we find a gauge δ and a constant α such that

$$\left|\int f \, dx - S(f, \Delta)\right| \leqq \varepsilon \tag{5.1}$$

holds for any δ-fine PU-partition Δ of I, which proves our theorem provided f is bounded.

Let us now suppose that f is not bounded. For simplicity, assume that $f\colon \boldsymbol{R}^n \to \{0\} \cup [1, \infty)$. (The general case then immediately follows by the linearity property of the integrals.) Let again $\alpha > 0$ and let η_l, $l = 1, 2, \ldots$ be a sequence of positive reals. Denote

$$E_l = \{x \in \boldsymbol{R}^n;\ (1 + \alpha)^{l-1} \leqq f(x) < (1 + \alpha)^l\}\,, \quad l = 1, 2, \dots,$$

$$E_0 = \operatorname{supp} f \setminus \bigcup_{l=1}^{\infty} E_l\,.$$

(Obviously $f(x) = 0$ for $x \in E_0$.) Since E_l are measurable, there exist open sets V_l, $V_l \supset E_l$ with $m(V_l \setminus E_l) \leqq \eta_l$, $l = 1, 2, \dots$. Let δ be a gauge on I such that

$$B(x, \delta(x)) \subset V_l \quad \text{for} \quad x \in E_l\,.$$

Let again $\Delta = \{(t^j, \vartheta_j);\ j = 1, \dots, k\}$ be a δ-fine PU-partition of I. Denote

$$g_l(x) = \begin{cases} \sum\limits_{t^j \in E_l} f(t^j)\, \vartheta_j(x) & \text{if} \quad \{t^j \in E_l\} \neq \emptyset\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$g(x) = \sum_{j=1}^{k} f(t^j)\, \vartheta_j(x) = \sum_{l=1}^{\infty} g_l(x)\,.$$

Evidently,

$$\int g_l \,\mathrm{d}x \leqq (1 + \alpha)^l\, m(V_l)\,,$$

$$\int_{E_l} f \,\mathrm{d}x \geqq (1 + \alpha)^{l-1}\, m(E_l)\,,$$

hence

$$\int g_l \,\mathrm{d}x \leqq (1 + \alpha) \int_{E_l} f \,\mathrm{d}x + (1 + \alpha)^l\, m(V_l \setminus E_l)\,,$$

$$S(f, \Delta) = \int g \,\mathrm{d}x \leqq (1 + \alpha) \int f \,\mathrm{d}x + \sum_{l=1}^{\infty} (1 + \alpha)^l\, \eta_l\,.$$

A suitable choice of α, η_l evidently yields

$$S(f, \Delta) \leqq \int f \,\mathrm{d}x + \varepsilon\,.$$

Since f is bounded from below, the other estimate is obtained as in the first part of the proof by approximating the function f from below by an upper semicontinuous function. Hence (5.1) again holds and the proof of Theorem 5.1 is complete.

Observe that the function Σ was not needed in the proof of Theorem 5.1. Therefore an analogous result (Theorem 5.2 below) can be proved in the same way for the following modification of the PU-integral:

Let $M \subset \boldsymbol{R}^n$ be bounded. A family (1.1) is called *a PUL-partition of M* if $t^j \in M$, $\vartheta_j\colon \boldsymbol{R}^n \to [0, 1]$ are measurable functions with compact supports and if (1.2), (1.3) hold. If δ is a gauge on M, then a PUL-partition of M is said to be *δ-fine* if (1.4) holds.

Definition 5.1. Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ have a compact support. Let $\gamma \in \boldsymbol{R}$ satisfy the following condition:

for every $\varepsilon > 0$ there is a gauge δ on $\operatorname{supp} f$ such that (2.2) holds provided Δ is a δ-fine PUL-partition of $\operatorname{supp} f$. Then f is said to be *PUL-integrable* and we write

$$\gamma = (\mathrm{PUL}) \int f \,\mathrm{d}x .$$

Theorem 5.2. *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ with a compact support be Lebesgue integrable with a convergent integral. Then it is PUL-integrable and*

$$(\mathrm{PUL}) \int f \,\mathrm{d}x = (\mathrm{L}) \int f \,\mathrm{d}x .$$

Let us introduce SL-partitions, SL-integrable functions and the SL-integral in the analogous way as the PUL-partitions etc. with the only change that ϑ_j are characteristic functions of semiclosed intervals (cf. Remark 2.1).

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ have a compact support, let $I \subset \boldsymbol{R}^n$ be a compact interval, $\operatorname{Int} I \supset \supset \operatorname{supp} f$. The integral $(\mathrm{S}_{\mathscr{B}}) \int f \,\mathrm{d}A$ was introduced in [3], 7.4 (put $A(J) = (b_1 - a_1) . (b_2 - a_2) \dots (b_n - a_n)$ for $J = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$). It was proved in [3], Theorem 7.6 that $(\mathrm{S}_{\mathscr{B}}) \int_I f \,\mathrm{d}A$ exists if and only if $(\mathrm{L}) \int_I f \,\mathrm{d}x$ exists and is finite, and that both the integrals coincide in this case. It is easy to prove that $(\mathrm{SL}) \int f \,\mathrm{d}x$ exists if and only if $(\mathrm{S}_{\mathscr{B}}) \int_I f \,\mathrm{d}A$ exists and that both the integrals coincide in this case. Obviously, $(\mathrm{SL}) \int f \,\mathrm{d}x$ exists and is equal to $(\mathrm{PUL}) \int f \,\mathrm{d}x$ if the latter exists. We conclude:

Theorem 5.3. *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ have a compact support. Then the following conditions are equivalent:*

(i) *f is PUL-integrable;*

(ii) *f is SL-integrable;*

(iii) *f is Lebesgue integrable, with a finite integral.*

If one of the above conditions is fulfilled, then

$$(\mathrm{PUL}) \int f \,\mathrm{d}x = (\mathrm{SL}) \int f \,\mathrm{d}x = (\mathrm{L}) \int f \,\mathrm{d}x .$$

6. SOME RESULTS ON THE M_2-INTEGRAL

The M_2-integral was introduced in [2]. In an equivalent manner it can be described as follows: Let $I \subset \boldsymbol{R}^n$ be a compact interval. A system

$$\Gamma = \{(t^i, J^i) \mid i = 1, 2, \dots, k\}$$

is called an *L-partition* of I, if J^i are nonoverlapping intervals, $t^i \in I$, $i = 1, 2, \dots, k$, $\bigcup_{i=1}^{k} J^i = I$. An L-partition Γ is called a *P-partition*, if $t^i \in J^i$ for $i = 1, 2, \dots, k$.

Let δ be a gauge on I. Γ is called *δ-fine*, if $J^i \subset B(t^i, \delta(t^i))$ for $i = 1, 2, \ldots, k$. Put

$$\Sigma(\Gamma) = \sum_{i=1}^{k} \int_{\partial J^i} \|t^i - x\| \, \mathrm{d}m_{n-1} ,$$

m_{n-1} being the $(n-1)$-dimensional Lebesgue measure. (If $n = 1$, $J^i = [a^i, b^i]$, then $\Sigma(\Gamma) = \sum_{i=1}^{k} (|t^i - a^i| + |t^i - b^i|)$.) For $f: I \to \boldsymbol{R}$ put

$$S(f, \Gamma) = \sum_{i=1}^{k} f(t^i) \, m_n(J^i) .$$

Definition 6.1. Let $\gamma \in \boldsymbol{R}$, $f: I \to \boldsymbol{R}$, $I \subset \boldsymbol{R}^n$ being a compact interval. Let the following condition be fulfilled:

for every $\varepsilon > 0$ and $K > 0$ there exists such a gauge δ on I that

$$|\gamma - S(f, \Gamma)| \leqq \varepsilon \tag{6.1}$$

provided Γ is a δ-fine L-partition of I with $\Sigma(\Gamma) \leqq K$.

Then f is said to be *M_2-integrable* and we write

$$\gamma = (\mathrm{M}_2) \int_I f \, \mathrm{d}x ; \tag{6.2}$$

γ is called *the M_2-integral of f*.

The equivalence of Definition 6.1 and Definition 4 of [2] follows immediately from Corollary of Theorem 8 of [2].

Theorem 6.1. *Let $I \subset \boldsymbol{R}^n$ be a compact interval, $f: \boldsymbol{R}^n \to \boldsymbol{R}$,* supp $f \subset I$. *Let f be PU-integrable. Then f is M_2-integrable and*

$$(\mathrm{M}_2) \int_I f \, \mathrm{d}x = (\mathrm{PU}) \int f \, \mathrm{d}x . \tag{6.3}$$

Proof. Let $I = [\alpha, \beta] = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]$. By Theorem 2.1 f is PUI-integrable. Let $\varepsilon > 0$, $K > 0$. By Definition 2.2 there is such a gauge δ on I and $\gamma \in \boldsymbol{R}$ that

$$|\gamma - S(f, \Delta)| \leqq \varepsilon$$

for every (2δ)-fine PU-partition Δ of I with $\Sigma(\Delta) \leqq K + 1$. Let Γ be a δ-fine L-partition of I with $\Sigma(\Gamma) \leqq K$,

$$\Gamma = \{(t^i, J^i) ; \quad i = 1, 2, \ldots, k\} .$$

For a fixed i write J^i in the form

$$J^i = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

and for $\sigma > 0$, $x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{R}^n$ put

$$\vartheta_\sigma^i(x) = \prod_{l=1}^{n} h_{c_l, d_l, \sigma}(x_l) \tag{6.4}$$

where

$$\begin{aligned}
c_l &= a_l && \text{if } a_l > \alpha_l\,,\\
c_l &= \alpha_l - 2\sigma && \text{if } a_l = \alpha_l\,,\\
d_l &= b_l && \text{if } b_l < \beta_l\,,\\
d_l &= \beta_l + 2\sigma && \text{if } b_l = \beta_l\,, \quad l = 1, 2, \ldots, n\,.
\end{aligned}$$

For the definition of $h_{a,b,\sigma}$, see (1.6) and above. Obviously $\Delta_\sigma = \{(t^j, \vartheta^j_\sigma);\ j = 1, 2, \ldots, k\}$ is a PU-partition of I for σ sufficiently small and

$$\lim_{\sigma \to 0} S(f, \Delta_\sigma) = S(f, \Gamma)\,. \tag{6.5}$$

Moreover,

$$\lim_{\sigma \to 0} \sum_{l=1}^{n} \int \|x - t^i\|\, |\partial \vartheta^i_\sigma / \partial x_l|\, \mathrm{d}x = \int_{\partial J^i} \|x - t^i\|\, \mathrm{d}\mathscr{m}_{n-1}\,,$$

so that

$$\lim_{\sigma \to 0} \Sigma(\Delta_\sigma) = \Sigma(\Gamma)\,. \tag{6.6}$$

If σ is sufficiently small, Δ_σ is 2δ-fine, $\Sigma(\Delta_\sigma) < K + 1$ so that

$$|\gamma - S(f, \Delta_\sigma)| \leqq \varepsilon$$

any by (6.5) we obtain

$$|\gamma - S(f, \Gamma)| \leqq \varepsilon\,,$$

which makes the proof complete.

Let $f: I \to \boldsymbol{R}$ be M_2-integrable. The following assertions are consequences of [2], Theorem 1, Corollary and Remark 5.

(6.7) If $H \subset I$ is an interval, then

$$(\mathrm{M}_2) \int_H f\,\mathrm{d}x \quad \text{exists}.$$

(6.8) Let $H, H^1, \ldots, H^s$ be intervals, $\bigcup_{i=1}^{s} H^i = H \subset I$ and let the intervals $H^1, H^2, \ldots$ $\ldots, H^s$ be nonoverlapping. Then

$$(\mathrm{M}_2) \int_H f\,\mathrm{d}x = \sum_{i=1}^{s} (\mathrm{M}_2) \int_{H^i} f\,\mathrm{d}x\,.$$

The following theorem is an analogue of the Saks-Henstock Lemma, which was proved in [2], p. 372.

Theorem 6.2. *Let $I \subset \boldsymbol{R}^n$ be a compact interval and let $f: I \to \boldsymbol{R}$ be M_2-integrable. Let $\gamma = (\mathrm{M}_2) \int_I f\,\mathrm{d}x$, $\varepsilon > 0$, $K > 0$ and let δ be such a gauge on I that*

$$|\gamma - S(f, \Gamma)| \leqq \varepsilon$$

for every δ-fine L-partition Γ of I with

$$\Sigma(\Gamma) \leqq K + 4n^2 m_n(I) .$$

Let L^s, $s = 1, 2, \ldots, r$ be nonoverlapping intervals, $t^s \in I$, $L^s \subset B(t^s, \delta(t^s))$ for $s = 1, 2, \ldots, r$,

$$\sum_{s=1}^{r} \int_{\partial L^s} \|x - t^s\| \, \mathrm{d}m_{n-1} \leqq K \tag{6.9}$$

(in case that $n = 1$, $L^s = [a^s, b^s]$ condition (6.9) reads

$$\sum_{s=1}^{r} \left(|t^s - a^s| + |t^s - b^s|\right) \leqq K) .$$

Then

$$\left|\sum_{s=1}^{r} \left(f(t^s)\, m_n(L^s) - (\mathrm{M}_2) \int_{L^s} f \,\mathrm{d}x\right)\right| \leqq \varepsilon . \tag{6.10}$$

The following lemma will be needed in the proof:

Lemma 6.1. *Let $I \subset \mathbf{R}^n$ be a compact interval and let δ be a gauge on I. Then there exists such a δ-fine P-partition $\Gamma = \{(t^i, J^i);\ i = 1, 2, \ldots, k\}$ of I that*

$$\Sigma_0(\Gamma) = \sum_{i=1}^{k} \operatorname{diam}(J^i)\, m_{n-1}(\partial J^i) \leqq 4n^2 m_n(I) . \tag{6.11}$$

Proof. Without loss of generality we may assume that

$$I = [0, b_1] \times [0, b_2] \times \ldots \times [0, b_n] , \tag{6.12}$$

$$0 < b_1 \leqq b_j \quad \text{for} \quad j = 2, 3, \ldots, n . \tag{6.13}$$

There are such nonnegative integers l_j that $2^{l_j} b_1 \leqq b_j < 2^{l_j + 1} b_1$ for $j = 2, 3, \ldots, n$. Let $\mathscr{J}$ be the set of such $j \in \{2, 3, \ldots, n\}$ that $l_j > 0$. If $\mathscr{J} \neq \emptyset$, cut I by the hyperplanes $x_j = p b_j 2^{-l_j}$, $p = 1, 2, \ldots, 2^{l_j} - 1$, $j \in \mathscr{J}$. Thus I is cut into intervals $I^1, I^2, \ldots, I^q$ with $q = \prod_{j=1}^{n} 2^{l_j}$ and the lengths of edges of each interval I^m are b_1, $b_2 2^{-l_2}$, $b_3 2^{-l_3}, \ldots, b_n 2^{-l_n}$, $m = 1, 2, \ldots, q$. If we find δ-fine P-partitions Γ_m of intervals I^m fulfilling $\Sigma_0(\Gamma_m) \leqq 4n^2 m_n(I^m)$, $m = 1, 2, \ldots, q$, we may put $\Gamma = \bigcup_{m=1}^{q} \Gamma_m$. Γ is a P-partition of I fulfilling (6.11).

Thus we may assume without loss of generality that

$$b_j < 2b_1 \quad \text{for} \quad j = 2, 3, \ldots, n . \tag{6.14}$$

If there is such a $t \in I$ that $I \subset B(t, \delta(t))$, we put $\Gamma = \{(t, I)\}$. Otherwise we cut I by hyperplanes $x_j = \frac{1}{2} b_j$, $j = 1, 2, \ldots, n$; thus I is cut into intervals I^p, $p = 1, 2, \ldots, 2^n$. Let P be the set of such p that there is $t^p \in I^p$ fulfilling $I^p \subset B(t^p, \delta(t^p))$. Couples (t^p, I^p), $p \in P$ are elements of Γ; intervals I^p, $p \notin P$ are cut in an analogous way etc. After a finite number of steps we obtain a δ-fine P-partition

$$\Gamma = \{(t^i, J^i) \mid i = 1, 2, \ldots, k\}$$

of I. Moreover, there exist such nonnegative integers r_i that the lengths of edges of J^i are $2^{-r_i} b_j$, $j = 1, 2, \ldots, n$. We have

$$m_{n-1}(\partial J^i) \operatorname{diam} J^i = 2^{1-nr_i} \sum_{j=1}^{n} b_1 \ldots b_{j-1} b_{j+1} \ldots b_n \sqrt{(b_1^2 + \ldots + b_n^2)} \leqq$$

$$\leqq 2 \,.\, 2^{-nr_i} b_1 b_2 \ldots b_n \sum_{j=1}^{n} \frac{1}{b_1} \sqrt{(b_1^2 + \ldots + b_n^2)} \leqq 2\, m_n(J^i)\, n \sqrt{(4n)} \leqq 4n^2\, m_n(J^i)\,.$$

Hence $\Sigma_0(\Gamma) \leqq 4n^2\, m_n(I)$ which proves the lemma.

Proof of Theorem 6.2. There are such intervals $G^1, G^2, \ldots, G^p \subset I$ that the in- intervals $G^1, G^2, \ldots, G^p, L^1, L^2, \ldots, L^r$ are nonoverlapping and

$$I = \left(\bigcup_{s=1}^{r} L^s\right) \cup \left(\bigcup_{i=1}^{p} G^i\right).$$

Let $\eta > 0$. There exist gauges δ_i on G^i such that $\left|S(f, \Gamma_i) - (\mathrm{M}_2) \int_{G^i} f \,\mathrm{d}x\right| \leqq \eta/p$ for every δ_i-fine L-partition Γ_i of G^i with $\Sigma(\Delta_i) \leqq 4n^2\, m_n(G^i)$, $i = 1, 2, \ldots, p$. Moreover, we can assume that $\delta_i(x) \leqq \delta(x)$ for $x \in G^i$. Let $\tilde{\Gamma}_i$ be δ_i-fine P-partitions of G^i with $\Sigma_0(\tilde{\Gamma}_i) \leqq 4n^2\, m_n(G^i)$ (cf. Lemma 6.1). Put

$$\Gamma = \bigcup_{s=1}^{r} \{(t^s, L^s)\} \cup \bigcup_{i=1}^{p} \tilde{\Gamma}_i\,.$$

As every P-partition is simultaneously an L-partition, Γ is a δ-fine L-partition of I. Moreover,

$$\Sigma(\Gamma) = \sum_{s=1}^{r} \int_{\partial L^s} \|x - t^s\| \,\mathrm{d}m_{n-1} + \sum_{i=1}^{p} \Sigma(\tilde{\Gamma}_i)\,.$$

Since $\tilde{\Gamma}_i$ are P-partitions of G^i, we have $\Sigma(\tilde{\Gamma}_i) \leqq \Sigma_0(\tilde{\Gamma}_i)$ for $i = 1, 2, \ldots, p$ (cf. (6.11)), so that

$$\Sigma(\Gamma) \leqq K + 4n^2 \sum_{i=1}^{p} m_n(G^i) \leqq K + 4n^2\, m_n(I)\,.$$

It follows that

$$\left|(\mathrm{M}_2) \int_I f \,\mathrm{d}x - S(f, \Gamma)\right| \leqq \varepsilon\,.$$

Simultaneously we have (cf. (6.8))

$$(\mathrm{M}_2) \int_I f \,\mathrm{d}x = \sum_{s=1}^{r} (\mathrm{M}_2) \int_{L^s} f \,\mathrm{d}x + \sum_{i=1}^{p} (\mathrm{M}_2) \int_{G^i} f \,\mathrm{d}x\,,$$

$$S(f, \Gamma) = \sum_{s=1}^{r} f(t^s)\, m_n(L^s) + \sum_{i=1}^{p} S(f, \tilde{\Gamma}_i)\,,$$

$$\left|(\mathrm{M}_2) \int_{G^i} f \,\mathrm{d}x - S(f, \tilde{\Gamma}_i)\right| \leqq \eta/p\,.$$

It follows that

$$\left|\sum_{s=1}^{r} (\mathrm{M}_2) \int_{L^s} f \,\mathrm{d}x - \sum_{s=1}^{r} f(t^s)\, m_n(L^s)\right| \leqq \varepsilon + \eta$$

and (6.10) holds, since $\eta > 0$ was arbitrary.

7. EXAMPLES

A. Let $n = 1$, $\alpha > 0$, $\beta > 0$. Let ω be defined as before (see (1.6) and above). Put

$$\xi(x) = \begin{cases} x^\alpha \cos \pi x^{-\beta} & \text{for} \quad x > 0\,, \\ 0 & \text{for} \quad x \leqq 0\,, \end{cases}$$

$\eta(x) = \xi(x)\, \omega(3 - x)$ for $x \in \boldsymbol{R}$. If $\alpha > 1$, then η is differentiable and (PU) $\int \eta' \,\mathrm{d}x$ exists by Theorem 4.1, η' being the derivative of η, $\eta'(x) = \alpha x^{\alpha-1} \cos \pi x^{-\beta} + \pi\beta x^{\alpha-\beta-1} \sin \pi x^{-\beta}$ for $0 < x \leqq 2$.

Let $\lambda > 0$. Put

$$v(x) = \begin{cases} x^{-\lambda} \sin \pi x^{-\beta} & \text{for} \quad 0 < x \leqq 1\,, \\ 0 & \text{for} \quad x \leqq 0 \quad \text{and for} \quad x > 1\,. \end{cases}$$

It is not difficult to conclude that

(7.1) $$(\mathrm{PU}) \int v \,\mathrm{d}x \quad \text{exists if} \quad 0 < \lambda < 1 \quad \text{or if} \quad 0 < \lambda < \beta\,.$$

Indeed, for $0 < \lambda < 1$, v is even Lebesgue integrable; the other part of (7.1) follows from the fact that the difference $v - (\pi\beta)^{-1} \eta'$ is absolutely integrable for $0 < \lambda < \beta$ if we put $\alpha = \beta - \lambda + 1$.

Our aim is to prove that

(7.2) $$(\mathrm{M}_2) \int_{[0,1]} v \,\mathrm{d}x \quad \text{does not exist if} \quad \lambda \geqq 1\,, \quad 0 < \beta \leqq \lambda\,.$$

Observe that (7.2) and Theorem 6.1 imply

(7.3) $$(\mathrm{PU}) \int v \,\mathrm{d}x \quad \text{does not exist if} \quad \lambda \geqq 1\,, \quad 0 < \beta \leqq \lambda\,.$$

The zeros σ_l of v in $(0, 1]$ are given by $\sigma_l = (l)^{-1/\beta}$, $l = 1, 2, \ldots$. Let p be a positive odd integer and for $s = 1, 2, \ldots$ put

(7.4) $$L_s = [\sigma_{p+2s+1}, \sigma_{p+2s}]\,.$$

We have $v(x) \geqq 0$ for $x \in L_s$, $s = 1, 2, \ldots$ and there is such a $c > 0$ that

(7.5) $$(\mathrm{M}_2) \int_{L_s} v \,\mathrm{d}x \geqq \frac{c}{\beta} (p + 2s)^{(\lambda-\beta-1)/\beta}\,, \quad s = 1, 2, \ldots\,.$$

In order to prove (7.2) assume that $\lambda \geqq 1$, $0 < \beta \leqq \lambda$ and that $(\mathrm{M}_2) \int_{[0,1]} v \,\mathrm{d}x$

exists. In Theorem 6.2 put $K = 1$, $\varepsilon = c/4\beta$, $\gamma = (\mathrm{M}_2) \int_{[0,1]} v \, \mathrm{d}x$ and let δ be such a gauge on $[0, 1]$ that $|\gamma - S(v, \Gamma)| \leqq c/4\beta$ for every δ-fine L-partition of $[0, 1]$ with $\Sigma(\Gamma) \leqq 5$.

Let p be so large that $\sigma_p < \delta(0)$, $\sigma_p < \frac{1}{4}$. Put $t_s = 0$ and define L_s by (7.4), $s = 1, 2, \ldots, r$, where r will be chosen later. Observe that (under our choice) the left hand side in (6.9) is $\sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1})$ and the left hand side in (6.10) is

$$\sum_{s=1}^{r} (\mathrm{M}_2) \int_{L_s} v \, \mathrm{d}x$$

since $f(t_s) = 0$ for $s = 1, 2, \ldots, r$.

Let two cases be treated separately:

1. $1 \leqq \beta \leqq \lambda$. Then by (7.5)

$$\sum_{s=1}^{r} (\mathrm{M}_2) \int_{L_s} v \, \mathrm{d}x \geqq \frac{c}{\beta} \sum_{s=1}^{r} (p + 2s)^{-1/\beta} \geqq \frac{c}{2\beta} \sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1}) .$$

Since $\sum_{s=1}^{\infty} (\sigma_{p+2s} + \sigma_{p+2s+1}) = \infty$, $\sigma_p < \frac{1}{4}$, the number r may be chosen in such a way that $\frac{1}{2} < \sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1}) \leqq 1$ so that $\sum_{s=1}^{r} (\mathrm{M}_2) \int_{L_s} v \, \mathrm{d}x > c/4\beta$. Thus (6.9) is fulfilled and (6.10) is not fulfilled, which is a contradiction.

2. $0 < \beta < 1 \leqq \lambda$. As $\sum_{s=1}^{\infty} (\sigma_{p+2s} + \sigma_{p+2s+1}) < \infty$, p can be chosen so large that

$$\sum_{s=1}^{r} (\sigma_{p+2s} + \sigma_{p+2s+1}) < 1 \quad \text{for} \quad r = 1, 2, \ldots .$$

On the other hand $(\mathrm{M}_2) \int_{L_s} v \, \mathrm{d}x \geqq (c/\beta)(p+2s)^{-1}$ by (7.5), so that $\sum_{s=1}^{\infty} (\mathrm{M}_2) \int_{L_s} v \, \mathrm{d}x = \infty$ and thus (6.10) does not hold for r sufficiently large.

Remark 7.1. Let $\lambda \geqq 1$. Since v is continuous on $(0, 1]$ and $\lim_{\tau \to 0+} \int_\tau^1 v(x) \, \mathrm{d}x$ exists iff $\beta > \lambda - 1$, we may conclude that the Perron integral (P) $\int_{[0,1]} v \, \mathrm{d}x$ exists iff $\beta > \lambda - 1$.

B. Let $n = 2$. Put

$$\zeta(x) = \zeta(x_1, x_2) = \zeta(x_1) \, \omega(x_2 + 1) \, \omega(2 - x_2) ,$$
$$\varrho(x) = \varrho(x_1, x_2) = v(x_1) \, \omega(x_2 + 1) \, \omega(2 - x_2) \quad \text{for} \quad x = (x_1, x_2) \in \boldsymbol{R}^2 .$$

If $\alpha > 1$, then ζ is differentiable and (PU) $\int (\partial\zeta/\partial x_1) \, \mathrm{d}x$ exists by Theorem 6.1. Again (cf. Part A) it is not difficult to conclude that

$$\text{(7.6)} \qquad (\mathrm{PU}) \int \varrho \, \mathrm{d}x \quad \text{exists if} \quad 0 < \lambda < 1 \quad \text{or if} \quad 0 < \lambda < \beta .$$

Analogously to Part A we have

$$\text{(7.7)} \qquad (\mathrm{M}_2) \int_I \varrho \, \mathrm{d}x \quad \text{does not exist, if} \quad \lambda \geqq 1 ,$$
$$0 < \beta \leqq \lambda , \quad I = [0, 1] \times [0, 1]$$

and (7.7) and Theorem 6.1 imply

$$(7.8) \qquad (\mathrm{PU}) \int \varrho \, dx \quad \text{does not exist, if} \quad \lambda \geqq 1\,, \quad 0 < \beta \leqq \lambda\,.$$

In order to indicate the proof of (7.7) let us start with the following lemma:

Lemma 7.1. *Let $\varkappa > 0$, $E \subset [0, 1]$, $m_e(E) > \frac{1}{2}$, $m_e(E)$ being the outer Lebesgue measure of E. Then there are $\tau_1, \tau_2, \ldots, \tau_q \in E$ such that $q \geqq 1/8\varkappa$ and that the intervals $[\tau_i - \varkappa, \tau_i + \varkappa]$, $i = 1, 2, \ldots, q$ are disjoint.*

Proof. $\tau_1 \in E$ can be chosen arbitrarily. If $\tau_i \in E$ are known, $i = 1, 2, \ldots, k$ such that $k < 1/8\varkappa$ and that the intervals $[\tau_i - \varkappa, \tau_i + \varkappa]$ are disjoint for $i = 1, 2, \ldots, k$, then

$$E \setminus \Big(\bigcup_{i=1}^{k} [\tau_i - 2\varkappa, \tau_i + 2\varkappa] \Big) \neq \emptyset\,, \quad \tau_{k+1} \in E \setminus \Big(\bigcup_{i=1}^{k} [\tau_i - 2\varkappa, \tau_i + 2\varkappa] \Big)$$

may be chosen arbitrarily and the intervals $[\tau_i - \varkappa, \tau_i + \varkappa]$, $i = 1, 2, \ldots, k + 1$ are disjoint.

(7.7) can be proved in an analogous way as (7.2). If δ is a gauge on I, put $Q(\varkappa) = \{x_2 \in [0, 1];\ \delta(0, x_2) \geqq 2\varkappa\}$. Since $Q(\varkappa_1) \supset Q(\varkappa_2)$ for $0 < \varkappa_1 < \varkappa_2$ and $\bigcup_{\varkappa > 0} Q(\varkappa) = [0, 1]$, there is $\varkappa_0 > 0$ such that $m_e(Q(\varkappa_0)) > \frac{1}{2}$. Let p be such an even positive integer that $\sigma_p < \varkappa_0$. For every $s = 1, 2, \ldots$ put $E = Q(\varkappa)$, $\varkappa = \sigma_{p+2s}$, and define $\tau_1^{(s)}, \tau_2^{(s)}, \ldots, \tau_{q_s}^{(s)}$ by Lemma 7.1. Theorem 6.2 can be applied in an analogous way, the role of couples (t_s, L_s), $s = 1, 2, \ldots, r$ being played by the couples

$$\big((0, \tau_i^{(s)}),\ [\sigma_{p+2s+1}, \sigma_{p+2s}] \times [\tau_i^{(s)} - \sigma_{p+2s}, \tau_i^{(s)} + \sigma_{p+2s}]\big)\,,$$

$i = 1, 2, \ldots, q_s$, $s = 1, 2, \ldots, r$ with a suitable choice of r (and possibly of p).

Remark 7.2. Again it can be proved that

$$(\mathrm{P}) \int_I \varrho \, dx \quad \text{exists if} \quad \lambda \geqq 1\,, \quad \beta > \lambda - 1\,.$$

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