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SEMIGROUPS WHOSE PROPER SUBSEMIGROUPS ARE DUO*)

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Duo semigroups have been studied in the last years by a number of mathematicians: Pondělíček in [10] and [11], Anjaneyulu in [1], [2] and [3] with particular attention to ideal theory, and the authors in [6], where duo semigroups whose congruences are pairwise permutable are completely described.

The idea of studying semigroups whose proper subsemigroups are duo (hereafter called subduo) was inspired by the fact that quasi hamiltonian semigroups, studied by the authors in [5], have this property (see Th. 1.9). We observe that problems of this nature have already been studied in several papers, for instance in [12], where Rédei and Trachtman described semigroups whose proper subsemigroups are commutative, and in [4] where the authors examined semigroups whose proper subsemigroups are quasicommutative.

The main results of this paper are a characterization of subduo semigroups in general (Th. 3.2), and further characterizations of the following particular cases: subduo semigroups without idempotents (Th. 3.3), archimedean subduo semigroups (Th. 2.10 and Th. 2.12) and, finally, subduo groups (Th. 2.6).

Terminology and notation not defined here may be found in [7] and [9].

1. PRELIMINARIES

Definition 1.1. A semigroup S is called *duo* if every one-sided ideal of S is two-sided (see [11], p. 20).

It is well-known that a semigroup S is duo if and only if, for every $a, b \in S$, there exist $x, y \in S^1$ such that $ab = bx = ya$.

Definition 1.2. A semigroup S is called *subduo* if every proper subsemigroup of S (or equivalently every proper subsemigroup of S generated by two elements) is duo.

It will be shown below (Remarks 2.8 and 2.10) that neither of the conditions "duo" and "subduo" implies the other.

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Definition 1.3. A semigroup S is *quasi hamiltonian* if for every $a, b \in S$ there exist two positive integers h, k such that $ab = b^h a^k$ (see [5], p. 131).

Definition 1.4. A semigroup S is *subcommutative* if S is not commutative and every proper subsemigroup of S is commutative (see [4], p. 35 and [12]).

At this point we list several properties of archimedean, duo, quasi hamiltonian, and subcommutative semigroups, which will be utilized in what follows, and which are either well-known or easily verifiable.

α) An archimedean semigroup whose idempotents are in the center is t-archimedean (i.e. left and right archimedean).
β) Let S be a t-archimedean semigroup, and a an element of S. If $a = xay$ for some $x, y \in S^1$ with $xy \in S$, then a lies in a subgroup of S. Consequently, S has an idempotent.
γ) A simple t-archimedean semigroup is a group.
δ) A semigroup is t-archimedean with idempotent if and only if it is an ideal extension of a group by a nilsemigroup.
ε) The idempotents of a duo semigroup (if there are any) are in the center.
ζ) An archimedean duo semigroup is t-archimedean.
η) A simple duo semigroup is a group.
θ) A duo semigroup is a semilattice of t-archimedean semigroups.
ι) A group is a duo semigroup.
κ) A subduo semigroup is either duo or is generated by two elements.
λ) Every homomorphic image of a duo semigroup is duo.
μ) Every homomorphic image of a subduo semigroup is subduo.
ν) A quasi hamiltonian semigroup is duo and subduo.
ξ) A subcommutative semigroup of order greater than two is a semilattice of t-archimedean semigroups (see [4], Lemma 2.4 and [12], p. 16).
o) A subcommutative semigroup is subduo but not necessarily duo (see [12]).

The aim of this section is to prove that subduo semigroups are semilattices of archimedean semigroups. To this end we need to establish certain preliminary facts.

Lemma 1.5. *The idempotents (if any) of a subduo semigroup S of order greater than two are permutable.*

Proof. Let e and f be two idempotents of S with $ef \neq fe$. Let us examine the subsemigroups $\langle e, ef\rangle$ and $\langle e, fe\rangle$. We may note that $\langle e, ef\rangle \subset S$ implies $ef = efe$, while $\langle e, ef\rangle = S$ implies $f = ex$ (for some $x \in S$), whence $ef = f$. Similarly, $\langle e, fe\rangle \subset S$ yields $fe = efe$, and $\langle e, fe\rangle = S$ yields $fe = f$. Since $ef \neq fe$, either $ef = f$ or $fe = f$. Now, exchanging e with f, we find that either $fe = e$ or $ef = e$. The possible cases are that either $ef = f$ and $fe = e$ or $ef = e$ and $fe = f$. But, since $\langle e, f\rangle = S$ by Proposition ε, in each of these cases we have $|S| = 2$, a contradiction.

Theorem 1.6. *The idempotents (if any) of a subduo semigroup which is not subcommutative are in the center.*

Proof. Let a be an element and e an idempotent of S with $ae \neq ea$. Then Proposition ε implies $\langle a, e\rangle = S$. Since $\langle e, ea\rangle \subset S$ implies $ea = eae$ and $\langle e, ae\rangle \subset S$ implies $ae = eae$, we may suppose $\langle e, ea\rangle = S$. Thus $a = ey$ for some $y \in S$, whence $ea = a$. Now consider the subsemigroup $\langle e, a^2\rangle$. If $\langle e, a^2\rangle \subset S$, then $a^2e = ea^2 = a^2$ and the elements of S are $e, a, a^2, a^3, \ldots, ae$. It is immediate that the only non permutable elements of S are e and a which generate S, thus we have that S is subcommutative, a contradiction. Therefore $\langle e, a^2\rangle = S$, which yields either $a = a^{2h}$ or $a = a^{2h}e$ (h a positive integer). But $a = a^{2h}e$ implies $ae = a$, a contradiction. Thus $a = a^{2h}$ and a^{2h-1} is an idempotent. Now, in view of Lemma 1.5 we have that $ae = a^{2h}e = a(a^{2h-1}e) = aea^{2h-1} = a^{2h} = a$, another contradiction. We conclude that $ae = ea$.

Lemma 1.7. *A simple subduo semigroup has at least one idempotent.*

Proof. Let S be a simple subduo semigroup. If S is left and right simple, it is a group, and the statement is true. Otherwise, there is an element $a \in S$ for which we may suppose $Sa \subset S$. $Sa^5S = S$ implies $a = xa^5y$ for some $x, y \in S$. Hence $a^2 = xaa^2a^2ya$ with xa, a^2, $ya \in Sa$. Since Sa is duo, we have $a^2 = a^2uva^2$ for some $u, v \in S^1$, and S has an idempotent.

Theorem 1.8. *A simple subduo semigroup S of order greater than two is a group.*

Proof. By Lemma 1.7, S has an idempotent. If all the idempotents of S are in the center, S is a group in view of Propositions α and γ. Otherwise, there exist an idempotent e and an element d of S such that $ed \neq de$, and therefore $\langle e, d\rangle = S$, by Proposition ε. Since $eS \subset S$ implies $ed = ede$ and $Se \subset S$ implies $de = ede$, we may suppose $Se = S$, whence

$$d = de . \tag{1}$$

Moreover, since S is simple, we have $e = xdy$ for some $x, y \in S$, and from (1) and the fact that $\langle e, d\rangle = S$, it follows that

$$e = ed^k \tag{2}$$

for some positive integer k.

If $k = 1$, we have $ed = e$, $d^2 = dede = de = d$, whence S has order two, a contradiction.

If $k > 1$, then (1) and (2) yield $d = de = ded^k = ded^{k-1}d$. Hence ded^{k-1} is an idempotent and, by Lemma 1.5, we have $d = ded^{k-1}d = ded^{k-1}ed = eded^{k-1}d = ed$, a contradiction. Thus S is a group.

Lemma 1.9. *An archimedean subduo semigroup S of order greater than two is t-archimedean.*

Proof. If S is simple, the statement follows from Theorem 1.8. Otherwise, S is a nilextension of a proper ideal I. Since I is an archimedean duo semigroup, it is t-archimedean by Proposition ζ. At this point it is easily verifiable that S itself is t-archimedean.

Remark. A left (right)-zero semigroup of order two is an archimedean subduo semigroup which is not t-archimedean.

Lemma 1.10. *A subduo semigroup S is a semilattice of archimedean semigroups.*

Proof. If S is subcommutative, the statement follows from Proposition ξ. Then we may suppose S not to be subcommutative. Let $a, b \in S$ with $a = xby$ for some $x, y \in S^1$. If $\langle b, yxbyx \rangle \subset S$, it results that $b(yxbyx) = zb$ for some $z \in S^1$, and therefore $a^3 = xb(yxbyx)\,by = xzb^2y$. If, on the contrary, $\langle b, yxbyx \rangle = S$, we have three different cases:

1) $yx \in bS^1 \cup S^1b$. In this case it is immediate that $b^2 \mid a^2$.
2) $yx = yxbyx$. Then yxb is idempotent and, by Theorem 1.6, we have $a^2 = xbyxby = xyxb^2y$.
3) $yx = yxbyxwyxbyx$ with $w \in S^1$. In this case, putting $v = yxbyxwyx$, the element vb is idempotent, ad we have $a^2 = xbyxby = xbvbyxby = xvb^2yxby$.

Thus, in any case, b^2 divides a power of a, and the statement is proved in view of the well-known characterization of a semilattice of archimedean semigroups due to Putcha.

Theorem 1.11. *A subduo semigroup S of order greater than two is a semilattice of t-archimedean semigroups.*

Proof. S is a semilattice of archimedean semigroups by Lemma 1.10. Now, if S itself is archimedean, the statement follows from Lemma 1.9; otherwise the statement follows from Proposition ζ.

We conclude this section with a property of duo semigroups which shall be utilized in the sequel.

Theorem 1.12. *Let $H = \langle a, b \rangle$ be a duo semigroup. If $ab \neq b^pa^q$ for every pair of positive integers p, q, then ba lies in a subgroup of H. Moreover, if H is not archimedean, ab also belongs to a subgroup of H.*

Proof. Since H is duo, we have $ab \in bH^1$, whence either

$$ab = b^r \tag{3}$$

for some positive integer r, or

$$ab = bax \neq b^r \tag{4}$$

for some $x \in H$.

Analogously we conclude that either

(5) $$ba = b^s$$

for some positive integer s, or

(6) $$ba = yab \ne b^s$$

for some $y \in H$.

The relations (3) and (6) lead to the contradiction $ba = yab = yb^r \in \langle b \rangle$. Analogously (4) and (5) lead to a contradiction. If (3) and (5) hold, then $r \geqq s$ cannot be valid since it should follow $ab = b^{r-s+1}a$, a contradiction. Then it must be either

(7) $$ab = b^r\,, \quad ba = b^s \quad \text{with} \quad r < s$$

or

(8) $$ab = bax\,, \quad ba = yab \quad \text{with} \quad x \notin \langle a \rangle\,.$$

Now it is convenient to distinguish two cases:

A. H is t-archimedean. Then (7) implies $ba = b^{s-r}ab = b^{s-r-1}(ba)\,b$. From (8) it follows that $ba = ybax$. In both cases ba lies in a subgroup in view of Proposition β.

B. H is not t-archimedean. In this case, we observe that from the relations $ab \in H^1a$ and $ba \in aH^1$ we may deduce in the same way as above that either

(9) $$ab = a^m\,, \quad ba = a^n \quad \text{with} \quad 0 < m < n$$

or

(10) $$ab = zba\,, \quad ba = abw \quad \text{with} \quad z, w \in H\,, \quad z \notin \langle b \rangle\,.$$

Consequently, the relations (7) together with (9) lead to a contradiction since H, being a semilattice of t-archimedean semigroups (Proposition θ), should be t-archimedean. From (7) and (10) it follows that

(11) $$ba = b^{s-r}ab = b^{s-r}zba$$

From (8) and (9) it follows that

(12) $$ba = aba^{n-m} = baxa^{n-m}\,.$$

Finally, suppose that (8) and (10) hold. Then $ba = yab = ybax$; since $x \notin \langle a \rangle$, there exists $v \in H^1$ such that $x = bv$. Hence, by (10),

(13) $$ba = ybabv = ybzbav\,.$$

At this point, using the fact that $x \notin \langle a \rangle$ and $z \notin \langle b \rangle$, we may deduce from each of (11), (12), (13) that $ba = cbad$ for some $c, d \in H^1$, with at least one of them lying in the t-archimedean component of H which contains ba. Hence it immediately follows that ba is in a group. Moreover, by Lemma 1.8 of [5], if H is not archimedean, ab also is in a group.

The following is an immediate consequence of Theorem 1.12.

Corollary 1.13. *Every proper subsemigroup of an idempotent-free subduo semigroup is quasi hamiltonian.*

2. STRUCTURE OF ARCHIMEDEAN SUBDUO SEMIGROUPS

In this section we will provide a description of archimedean subduo semigroups. To this aim we will preliminarily prove some propositions which will enable us, in particular, to present a characterization of subduo groups.

Lemma 2.1. *Let S be a t-archimedean semigroup. If S contains a non commutative duo subsemigroup $H = \langle a, b \rangle$, then S has an idempotent.*

Proof. By Theorem 1.12, if S is idempotent-free, there are four positive integers p, q, i, j such that $ab = b^p a^q$, $ba = a^i b^j$. Hence it follows that $ab = b^{p-1} a^{i-1} (ab) \,.\, b^{j-1} a^{q-1}$, and therefore, by Proposition β, we have $p = q = 1$, that is $ab = ba$, a contradiction.

Theorem 2.2. *An archimedean subduo semigroup S which is neither commutative nor subcommutative has an idempotent.*

Proof. S necessarily has order greater than two; consequently, by Lemma 1.9, it is t-archimedean. Moreover, S contains a proper non commutative subsemigroup $H = \langle a, b \rangle$. Since H is duo, the statement follows from Lemma 2.1.

Corollary 2.3. *An archimedean subduo semigroup S which is neither commutative nor subcommutative is an ideal extension of a subduo group G by a subduo nil-semigroup N. Moreover, N is duo if and only if S is duo.*

Proof. The fact that S is an ideal extension of a subduo group G by a subduo nilsemigroup N is true by Lemma 1.9, Theorem 2.2 and Propositions δ and μ. If S is duo, then N is duo as well by Proposition λ. Conversely, let us suppose that N is duo. If S is not duo, it is generated by two elements a, b (Proposition κ) which do not satisfy both the conditions $ab = bc$, $ab = da$ for any $c, d \in S^1$. Consequently, if $ab \in G$, then denoting by u the identity of G, we conclude that $ab = abu$. Since $au, bu \in G$, there exist two elements $g, g' \in G$ such that $aubu = bug = bg$ and $aubu = g'au = g'a$. Hence $ab = bg = g'a$, a contradiction. If $ab \in S \setminus G$, then denoting by $*$ the operation in N, we obviously have $N = \langle a, b, * \rangle$ with $a * b \neq 0$. Since N is duo, there exist $c, d \in N$ such that $a * b = b * c = d * a$, whence $ab = bc = da$, a contradiction. Thus S is duo.

Lemma 2.4. *Let G be a subduo group. Every subsemigroup $H = \langle a, b \rangle$ of G is either commutative or a group.*

Proof. Let us suppose $ab \neq ba$, and put $c = ab$, $d = aba$. It is immediate that $cd \neq dc$, and therefore the subsemigroup $K = \langle c, d \rangle$ is not commutative. Since K is duo, we have $cd = dz$ for some $z \in K \setminus \langle d \rangle$ ($z \in \langle d \rangle$ would imply $cd = d^t$ for a positive integer t, whence $c = d^{t-1}$ and $cd = dc$, a contradiction). Hence

$$cd = dcx \tag{14}$$

for some $x \in K$, and analogously

$$dc = cdy \tag{15}$$

for some $y \in K$. Relations (14) and (15) imply $cd = cdyx$, whence, denoting by u the identity of G, we have $u = yx \in K$. Hence $u = (ab)^{h_1} (aba)^{k_1} \dots (ab)^{h_r} (aba)^{k_r}$ with $h_i \geqq 0$, $k_i \geqq 0$, $\sum(h_i + k_i) > 0$. Now it is immediate that $a^{-1}, b^{-1} \in H$; consequently H is a group.

Lemma 2.5. *Let G be a subduo group and S a proper subsemigroup of G. If $c \in S$ and $c^{-1} \in G \setminus S$, then c is in the center of S.*

Proof. If c is not in the center of S, there exists $g \in S$ with $cg \neq gc$. Then the subsemigroup $L = \langle c, g \rangle$ is not commutative; consequently it is a group, by Lemma 2.4. Hence $c^{-1} \in L \subseteq S$, a contradiction.

Theorem 2.6. *A group G is subduo if and only if every subsemigroup of G is either commutative or a group.*

Proof. The "if part" of the theorem is obvious, so let us prove the "only if part".

Let S be a subsemigroup of G which is neither commutative nor a group. Then S contains two elements a, b with $ab \neq ba$ and an element c such that $c^{-1} \in G \setminus S$. By Lemma 2.5, c belongs to the center of S. Let us consider the element ac of S. If ac is in the center of S, it follows that $(ab)\, c = acb = bac = (ba)\, c$, whence $ab = ba$, a contradiction. If ac is not in the center, then $(ac)^{-1} \in S$ by Lemma 2.5. Hence $c^{-1} = c^{-1}a^{-1}a = (ac)^{-1}\, a \in S$, again a contradiction.

Remark 2.7. It is evident that every commutative as well as every torsion group is subduo. As regards non commutative or non torsion subduo groups their structure is unknown, and its determination seems to be a problem suggesting no easy solution. It must be observed that even the structure of infinite subcommutative groups is not yet known (see [12]). Nonetheless, their existence was recently assured by Ol'šanskiĭ in [8] where an infinite non abelian group whose proper subgroups are cyclic of prime orders, is constructed.

We give here some information about subduo groups which may be deduced as Corollaries of Theorem 2.6.

Corollary A. *Let G be a subduo group, and let T be the set of the eleements of G having finite orders. T is a proper subgroup of G if and only if $xa = ax$ for every $x \in T$, $a \in G \setminus T$.*

Proof. Suppose that T is a proper subgroup of G with $xa \neq ax$ for some $x \in T$, $a \in G \setminus T$. Then the subsemigroup $\langle x, a \rangle$ is a group (Theorem 2.6) and $a^{-1} = x^{h_1}a^{k_1}x^{h_2}a^{k_2} \dots$ (h_i, k_i non negative integers, $\sum(h_i + k_i) > 0$). Hence, since T is a normal subgroup of G, it follows that $a^{-1} = ya^{k_1+k_2+\dots}$ for some $y \in T$. Consequently $y^{-1} = a^{1+k_1+k_2+\dots}$, a contradiction.

Conversely, let $xa = ax$ for every $x \in T$, $a \in G \setminus T$. Let $y \in T$. If $xy \in G \setminus T$ we have $x(xy) = (xy)x$, whence $xy = yx$ which implies $xy \in T$, a contradiction.

Corollary B. *Let G be a non torsion subduo group and T the set of the elements of G having finite orders. T is a subgroup of G if and only if T is contained in the center of G.*

Proof. Let T be a subgroup, and $C(T)$ its centralizer. $C(T)$ is a group containing $G \setminus T$ (Corollary A). $C(T) \subset G$ implies that G is the set union of its proper subgroups T and $C(T)$, which is a contradiction. Hence $C(T) = G$ and T is in the center of G. The converse is immediate.

Corollary C. *Let G be a subduo group. If H is a normal non abelian subgroup of G, G/H is a torsion group.*

Proof. Let $g \in G$. The subsemigroup $\langle H, g \rangle$ is not commutative, so its a group (Theorem 2.6). Then $g^{-1} = ag^k$ for some $a \in H$ and some non negative integer k. Hence $g^{k+1} = a^{-1} \in H$.

Remark 2.8. The above Corollary C assures the existence of groups which are not subduo semigroups. Indeed, let us consider a non abelian torsion group A and a non torsion abelian group B. The direct product $G = A \times B$ is not subduo since $G/A \simeq B$ is not a torsion group. This example also shows that the direct product of two subduo groups is not necessarily subduo.

Lemma 2.9. *Let S be an ideal extension of a group G by an nilsemigroup. Let u be the identity of G and let a, b be two elements of S. If the subsemigroup $\langle au, bu \rangle$ is a group, then $\langle au, bu \rangle \subseteq \langle a, b \rangle$.*

Proof. Since $u \in \langle au, bu \rangle$, we have $u = (au)^{h_1}(bu)^{k_1} \dots (au)^{h_r}(bu)^{k_r} = a^{h_1}b^{k_1} \dots a^{h_r}b^{k_r}u$ for some h_i, $k_i \geqq 0$ with $\sum(h_i + k_i) > 0$. Let $c = a^{h_1}b^{k_1} \dots a^{h_r}b^{k_r}$. Since there exists a positive integer n such that $c^n \in G$, we have $u = cu = (cu)^n = c^n u = c^n \in \langle a, b \rangle$, which proves the statement.

Now we are able to state the following theorem, which provides a classification of archimedean subduo semigroups.

Theorem 2.10. *S is an archimedean subduo semigroup if and only if S satisfies one of the following conditions:*

i) *S is a subduo group;*

ii) *S is archimedean commutative or subcommutative;*

iii) *S is an ideal extension of a subduo group G by a subduo nilsemigroup N and, when denoting by $*$ the operation in N and by ϕ the G-endomorphism of S, one of the following conditions holds*:

A) *N is duo, and for every* $a, b \in N \setminus 0$, *the relation* $a * b \neq b * a$ *implies* $\phi(a) = [\phi(a)]^{t+1}$, $\phi(b) = [\phi(b)]^{t+1}$ *for some positive integer t*;

B) *N is not duo and, for every* $a, b \in N \setminus 0$, *the relation* $a * b \neq b * a$ *implies* $\phi(a) = [\phi(a)]^{t+1}$, $\phi(b) = [\phi(b)]^{t+1}$ *for some positive integer t when* $\langle a, b, * \rangle \subset N$, *and* $G = \langle \phi(a), \phi(b) \rangle$ *when* $\langle a, b, * \rangle = N$.

Proof. Let S be an archimedean subduo semigroup. If S is either commutative or subcommutative or is a subduo group, the statement is trivially true. Otherwise, S is an ideal extension of a subduo group G by a subduo nilsemigroup N, by Corollary 2.3. Let us first show that if $a, b \in N \setminus 0$ with $a * b \neq b * a$ and the subsemigroup $H = \langle a, b \rangle$ is duo, then there exists a positive integer t such that

$$\phi(a) = [\phi(a)]^{t+1}, \quad \phi(b) = [\phi(b)]^{t+1}. \tag{16}$$

In fact, by Theorem 1.12 it follows that either

$$ab = b^p a^q \tag{17}$$

for some positive integers p, q, or $ba \in G$. Analogously, either $ba = a^i b^j$ for some positive integers i, j, or $ab \in G$. Since $ab, ba \in G$ imply $a * b = b * a$, the relation (17) cannot hold together with $ba = a^i b^j$ (Proposition β). Thus we may suppose that (17) holds together with $ab \in G$. Since H is duo, we have $ba \in H^1 b \cap aH^1$; moreover, G is an ideal and contains ab; hence $ba \in \langle b \rangle \cap \langle a \rangle$, that is

$$ba = a^n = b^s \tag{18}$$

for some n, s positive integers. From (17) and (18) it follows that

$$ab = a^m = b^r \tag{19}$$

with m, r positive integers. Therefore $a^{m+1} = aba = a^{n+1}$ and $b^{r+1} = bab = b^{s+1}$. Since $a * b \neq b * a$ implies $m \neq n$, $r \neq s$, the elements a and b are periodic and have a power which is idempotent. Denoting by u the identity of G, we have $a^t = b^t = u$ for a suitable positive integer t. Hence $au = (au)^{t+1}$, $bu = (bu)^{t+1}$, that is, (16) are valid. Consequently, if N is duo, (16) hold for every $a, b \in N \setminus 0$ with $a * b \neq b * a$. Suppose, on the contrary, that N is not duo, and let $a, b \in N \setminus 0$. If $\langle a, b, * \rangle \subset N$, we have $H = \langle a, b \rangle \subset S$; so H is duo and $a * b \neq b * a$ implies (16). If $\langle a, b, * \rangle = N$, then $\langle a, b \rangle \supset S \setminus G$. Since S is subduo and not duo (Corollary 2.3), S is generated by two elements α, β (Proposition κ). Since $\alpha \in G$ ($\beta \in G$) implies either $a, b \in G$ or $ab = ba$, both contradictions, we necessarily have $\alpha, \beta \in S \setminus G$, whence $\alpha, \beta \in \langle a, b \rangle$. Thus $S = \langle \alpha, \beta \rangle = \langle a, b \rangle$. At this point it is immediate that $G = \langle au, bu \rangle = \langle \phi(a), \phi(b) \rangle$.

Conversely, the statement is trivially true when S satisfies conditions i) or ii). Now, let us suppose that S satisfies iii). It is well known that S is archimedean, so it remains to prove that it is subduo. Let a, b be two elements of S with $\langle a, b\rangle \subset S$ and $ab \neq ba$. We have the following two cases to examine:

1) $ab, ba \in G$;
2) $a, b \in S \setminus G$ with $a * b \neq b * a$.

In case 1) we have $ab = abu$, whence

$$ab = aubu\ . \tag{20}$$

If the subsemigroup $\langle au, bu\rangle$ of G is commutative, it follows from (20) that $ab = buau = bau = ba$, a contradiction. Thus $\langle au, bu\rangle$ is a group (Theorem 2.6) and (20) implies $ab = bux = yau = yua$ for some $x, y \in \langle au, bu\rangle$. Since $\langle au, bu\rangle \subseteq H = \langle a, b\rangle$ by Lemma 2.9, we may conclude that $ab \in bH \cap Ha$.

In case 2) we must observe that, if $N = \langle a, b, *\rangle$, then N is necessarily duo. Otherwise, condition B) of the statement implies that $G = \langle \phi(a), \phi(b)\rangle = \langle au, bu\rangle$, whence $G \subseteq \langle a, b\rangle$ by Lemma 2.9. Moreover, $N = \langle a, b, *\rangle$ implies $S \setminus G \subset \langle a, b\rangle$. Thus $S = \langle a, b\rangle$, a contradiction. Hence we may obviously suppose $b * a \neq 0$. By the hypotheses, there exist two elements $x, y \in \langle a, b, *\rangle \setminus 0$ such that $b * a = a * x = y * b$. This implies $ba = ax = yb$ with $x, y \in H$. If also $a * b \neq 0$, we conclude in the same way that $ab = \bar{x}a = b\bar{y}$ for some $\bar{x}, \bar{y} \in H$. If, on the contrary, $a * b = 0$, we have $ab \in G$, whence $ab = abu = aubu$. If $\langle au, bu\rangle$ is a subgroup of G, we conclude in the same way as in case 1) that $ab \in bH \cap Ha$. Otherwise, $\langle au, bu\rangle$ is commutative (Theorem 2.6) and we have $ab = buau$, whence

$$ab = bau\ . \tag{21}$$

By hypothesis, we have $\phi(a) = [\phi(a)]^{t+1}$, that is $au = (au)^{t+1}$, whence $u = (au)^t$. Moreover, there is a positive integer m such that $a^m \in G$. Therefore we have $u = (au)^t = (au)^{tm} = a^{tm}u = a^{tm}$. Now (21) implies $ab = ba^{tm+1}$ and the statement is proved.

Remark 2.11. The following examples show that there are actually non commutative nilsemigroups which are duo and subduo, as well as subduo nilsemigroups which are not duo; in fact, let $N_1 = \langle a, b\rangle$ with $ab = 0$, $a^2 = b^2 = ba$, and $N_2 = \langle a, b\rangle$ with $ab = a^2 = b^2 = 0$. It is immediate that their multiplication tables are respectively

	a	b	c	0
a	c	0	0	0
b	c	c	0	0
c	0	0	0	0
0	0	0	0	0

	a	b	c	0
a	0	0	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

and that N_1 and N_2 are subduo, but N_1 only is duo. Every nil extension of a torsion group by N_1 or N_2 provides an example of an archimedean subduo semigroup which is respectively duo or not duo.

A further characterization of t-archimedean subduo semigroups is provided by the following

Theorem 2.12. *S is a t-archimedean subduo semigroup if and only if it satisfies one of the following conditions*:

i) *S is archimedean commutative*;

ii) *S is archimedean subcommutative of order greater than two*;

iii) *S is an ideal extension of a group G by a nilsemigroup N, every proper non commutative subsemigroup $H = \langle a, b \rangle$ of S is t-archimedean with an idempotent and $ba \in S \setminus G$ implies $ba = a^h = b^k$ and $ab = (ab)^r$ for three integers $h, k, r > 1$.*

Proof. Let S be a t-archimedean subduo semigroup. The statement is obvious when S is either commutative or subcommutative. Otherwise, S has an idempotent u (Theorem 2.2), consequently it is an ideal extension of a subduo group G whose identity is u, by a nilsemigroup N (Proposition δ). Let $H = \langle a, b \rangle$ be a proper non commutative subsemigroup of S. Then H is duo. Let us suppose H to be non archimedean. Then necessarily $ab, ba \in G$: otherwise, by Theorem 1.12, we should have $ab = b^p a^q, ba = a^i b^j$ (p, q, i, j positive integers), whence $ab = b^{p-1}a^{i-1}abb^{j-1}a^{q-1}$. Since $ab \neq ba$, at least one of $b^{p-1}a^{i-1}$ and $b^{j-1}a^{q-1}$ is actually an element of $\langle a, b \rangle$. Thus, by Proposition β, we have $ab \in G$ and, G being an ideal of S, also $ba \in G$, a contradiction. Now if the subsemigroup $\langle au, bu \rangle$ of G is commutative, we find the contradiction $ab = abu = aubu = buau = ba$. If $\langle au, bu \rangle$ is a group (Theorem 2.6), then $au = buxu$ for some $x \in H$. Hence, by Lemma 2.9, $au \in H$. Let t be a positive integer such that $a^t \in G$. Then $(au)^t = a^t u = a^t$, whence $a^t \in bH$. Hence a and b are in the same archimedean component of H (Proposition θ) which implies that H is archimedean, a contradiction. Thus H is t-archimedean by Proposition ζ, and has an idempotent in view of Theorem 1.12. Now, let us suppose $ba \in S \setminus G$. By Theorem 1.12 we have $ab = b^p a^q (p, q > 0)$. If also $ab \in S \setminus G$, then analogously $ba = a^i b^j (i, j > 0)$, which implies as above that either $ab = ba$ or $ba \in G$, both contradictions. So necessarily $ab \in G$. Since H is duo, we have $ba \in H^1 b \cap aH^1$ and, G being an ideal of S, $ba = a^h = b^k$ for some $h, k > 1$. Since $ab \in H^1 a$, it follows that $ab = a^m$ for some positive integer $m \neq h$. Hence, $a^{m+1} = aba = a^{h+1}$, whence $a^n = u$ for some positive integer n. Now we have $(ab)^n = a^{mn} = u$, whence $ab = (ab)^{n+1}$.

Conversely, the statement is obvious when S satisfies condition i), and it immediately follows from Lemma 1.9 when S satisfies ii). Thus, let us suppose that S satisfies iii). S is t-archimedean by Proposition δ, so it remains to prove that S is subduo. Let $a, b \in S$ with $ab \neq ba$ and $H = \langle a, b \rangle \subset S$. Then H is t-archimedean with an idempotent which coincides with the identity u of G. Therefore $H \cap G$ is a subgroup of G, since it is t-archimedean as an ideal of H, and contains an identity. We distinguish two cases:

A) $ab, ba \in G$. In this case, since $ua, bu \in H \cap G$, it follows that $ab = uabu = bux = yua$ with $x, y \in H \cap G$ and consequently with $ux, yu \in H$. Thus $ab \in Ha \cap bH$. In the same way we obtain $ba \in Hb \cap aH$.

B) $ba \in S \setminus G$. Then, by hypothesis, $ba = a^h = b^k$ and $ab = (ab)^r$ $(h, k, r > 1)$; hence $ab = a(ba)^{r-1} b = aa^{h(r-1)}b = aba^{h(r-1)} = abb^{k(r-1)} = b^{k(r-1)}ab$.

3. STRUCTURE OF A SUBDUO SEMIGROUP: THE GENERAL CASE

This section deals with subduo semigroups which are not necessarily archimedean. First of all we state the following

Theorem 3.1. *Let S be a non archimedean subduo semigroup. Every Archimedean component of S is either commutative or t-archimedean with an idempotent.*

Proof. Let T be an archimedean component of S. If T is not commutative, it contains two non permutable elements a, b and $H = \langle a, b \rangle \subseteq T \subset S$. Since T is t-archimedean (Proposition ζ) and H is duo, T contains an idempotent in view of Lemma 2.1.

Now we are able to provide the following classification of subduo semigroups.

Theorem 3.2. *Let S be a semigroup of order greater than two. The following conditions are equivalent:*

i) *S is subduo;*

ii) *S is a semilattice of t-archimedean semigroups, every subgroup of S is subduo and, for every $a, b \in S$ with $\langle a, b \rangle \subset S$, we have either $ab = b^p a^q$ $(p, q > 0)$, or $ab = a^h = b^k$ $(h, k > 1)$, or ab lies in a group.*

Proof. i) implies ii). S is a semilattice of t-archimedean semigroups by Theorem 1.11, and obviously every subgroup of S is subduo. Let $a, b \in S$ with $H = \langle a, b \rangle \subset S$. If $ab \neq b^p a^q$ for every pair of positive integers p, q and ab is not in a group, Theorem 1.12 yields that H (being duo) is t-archimedean, and that ba belongs to a subgroup of H; therefore, ba lies also in the maximal subgroup of H, which is an ideal of H. Now the relation $ab \in H^1 a \cap bH^1$, due to the fact that H is duo, leads to $ab \in \langle a \rangle \cap \langle b \rangle$, that is $ab = a^h = b^k$ for some positive integers h, k. Now, if $h = 1$, the relation $ab = a$ and the fact that H is t-archimedean imply that ab is in a group (Proposition β), a contradiction. Thus $h > 1$. Analogously we find that $k > 1$.

ii) implies i). Let $a, b \in S$ with $H = \langle a, b \rangle \subset S$. We may suppose that ab lies in a group, since in the other cases it is obvious that $ab \in H^1 a \cap bH^1$. Consequently, ab belongs to the maximal subgroup G of the t-archimedean component $S_{\alpha\beta}$ of S containing ab. G is an ideal of $S_{\alpha\beta}$ and, denoting by u its identity, we have $au, bu \in G$, and $au = ua$, $bu = ub$. Now, we must observe that ba satisfies one of the following conditions:

$$ba = a^i b^j \quad (i, j > 0)\,, \tag{22}$$

$$ba = a^r = b^s \quad (r, s > 1)\,, \tag{23}$$

$$ba \text{ is in a group}\,. \tag{24}$$

If (22) holds, we have $ba = a^{i-1}abb^{j-1} = (a^{i-1}u)\,ab(ub^{j-1})$, whence $ba \in G$. We arrive at the same conclusion starting from (24). In this case, we consider the subsemigroup $H \cap G$ of G. If $H \cap G$ is commutative, since $aba = abua$ belongs to $H \cap G$, we have $(aba)(ba) = (ba)(aba)$, whence $ab = ba$. Otherwise, $H \cap G$ is a group (Theorem 2.6), and $u \in H \cap G$. Hence $ua, bu \in H \cap G$, whence $ab = uabu = bux = yua$ for some $x, y \in H \cap G$. Thus we may conclude that $ab \in Ha \cap bH$.

Finally, suppose that (23) holds. In this case we have $a, b \in S_{\alpha\beta}$ and there exists a positive integer n such that $a^n, b^n \in G$. Moreover, $buau = bau = a^r u = (au)^r$, and analogously, $buau = (bu)^s$. Hence $bu = (au)^{r-1}$ and $au = (bu)^{s-1}$, whence

$$aubu = buau \tag{25}$$

and $au = (au)^{(r-1)(s-1)}$, $bu = (bu)^{(r-1)(s-1)}$. Putting $t = (r-1)(s-1) - 1$, we have $u = (au)^t = (bu)^t$, whence $u = (au)^{tn} = (bu)^{tn}$ and

$$u = a^{tn} = b^{tn}\,. \tag{26}$$

Now, by (25) and (26), we may conclude that $ab = aubu = buau = b^{tn+1}a^{tn+1}$.

We will conclude this section by a characterization of subduo semigroups without idempotents. In fact, the following holds:

Theorem 3.3. *Let S be a non archimedean semigroup without idempotents. S is subduo if and only if it satisfies one of the following conditions:*

i) *S is quasi hamiltonian;*

ii) *$S = \langle a, b \rangle$ is a semilattice of three t-archimedean semigroups S_α, S_β, $S_{\alpha\beta}$; $S_\alpha \cup S_{\alpha\beta}$ is commutative and the following relations hold:*

$$\begin{aligned} b^2a &= bab^r\,, \quad & ab^2 &= b^s ab\,, \\ bab &= b^{p+1}a\,, \quad & bab &= ab^{q+1} \end{aligned} \tag{27}$$

for some positive integers p, q, r, s.

Proof. Let S be a non archimedean subduo semigroup without idempotents. If S is not quasi hamiltonian, there exist $a, b \in S$ with $ab \neq b^p a^q$ for any pair of positive integers p, q and, since S has no subgroup, we have $\langle a, b \rangle = S$ by Theorem 1.12. Then S has three t-archimedean components S_α, S_β, $S_{\alpha\beta}$ (two of which are surely distinct) that are commutative in view of Theorem 3.1. If $S_\alpha = S_{\alpha\beta}$, $S_\alpha \cup S_{\alpha\beta}$ is trivially commutative. If $S_\alpha \neq S_{\alpha\beta} \neq S_\beta$, we have $\langle c, d \rangle \subset S$ for every $c \in S_{\alpha\beta}$ and $d \in S \setminus S_{\alpha\beta}$; hence $\langle c, d \rangle$ is quasi hamiltonian by Corollary 1.13. Then there exist four positive integers h, k, m, n such that $cd = d^h c^k = c^{mk} d^{nh} = c^{mk-1}(cd)\, d^{nh-1}$.

Since c and cd are in the t-archimedean semigroup $S_{\alpha\beta}$, if $k > 1$, it is easily seen that S has an idempotent, a contradiction. Thus $k = 1$, and therefore

$$cd = d^h c\,. \tag{28}$$

Analogously, we find

$$dc = cd^t \quad (t > 0)\,. \tag{29}$$

Let $c, g \in S_{\alpha\beta}$. Then we have

$$ca = a^r c\,, \quad bg = gb^s \tag{30}$$

for some positive integers r, s. If we suppose $r > 1, s > 1$, we deduce from (30), (29) and (28) and from the fact that $S_{\alpha\beta}$ is commutative that $cgab = cabg = a^r cgb^s = = cga^{\lambda r}b^s = a^{(\lambda r-1)\mu}cgab^s = a^{(\lambda r-1)\mu}b^{(s-1)\nu}cgab$, where λ, μ, ν are positive integers. But this relation again implies the existence of an idempotent (Proposition β); so either $r = 1$ or $s = 1$. It follows that, if there exists $x \in S_{\alpha\beta}$ such that $ax \neq xa$, then $by = yb$ for every $y \in S_{\alpha\beta}$. In other words, either $S_\alpha \cup S_{\alpha\beta}$ or $S_\beta \cup S_{\alpha\beta}$ is commutative. Then we may suppose in any case that $S_\alpha \cup S_{\alpha\beta}$ is commutative. Using (28) and (29), we may find the relations (27) of the statement.

Conversely, if S is quasi hamiltonian, it is obviously subduo. Then let us suppose that S satisfies condition ii). From the first and the fourth identity of (27) it follows that $b^2a = bab^r = ab^{q+r}$, $b^3a = bab^{q+r} = ab^{2q+r}$, and in general, $b^m a = ab^{(m-1)q+r}$ for every integer $m \geqq 2$. Since $m(q-1) \geqq q - r$, we also have $(m-1)\,q + r \geqq m$, whence

$$b^m a = ab^{m+\lambda} \tag{31}$$

for some $\lambda \geqq 0$. In the same way, starting by the second and the third identity of (27), we find

$$ab^m = b^{m+\mu}a \tag{32}$$

for some $\mu \geqq 0$.

Now let us consider an element $b^h ab^k$ with $h, k \geqq 0$ and $h + k > 0$. In view of (31) we have $b^m(b^h ab^k) = b^h(b^m a)\,b^k = b^h ab^{m+\lambda}b^k = (b^h ab^k)\,b^{m+\lambda}$ for every $m \geqq 2$. Moreover, consider $b(b^h ab^k)$. If $k > 0$, then making use of the last identity of (27) we get $b(b^h ab^k) = b^h(bab)\,b^{k-1} = b^h ab^{q+k} = (b^h ab^k)\,b^q$. If $k = 0$, we have $h > 0$, and the first identity of (27) yields that $b(b^h a) = b^{h+1}a = b^{h-1}b^2a = (b^h a)\,b^r$. Analogous results may be found for the symmetric products $(b^h ab^k)\,b^j$. Let us remark that, for every $u, v, w \in S^1$ and every positive integer n, in view of the fact that $S_\alpha \cup S_{\alpha\beta}$ is commutative, we have $b^n(uavaw) = (b^n uav)\,aw = (awb^n)\,uav = (uavaw)\,b^n$. We may conclude that S is subduo.

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