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Demeter Krupka; Věra Mikolášová

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ON THE UNIQUENESS OF SOME DIFFERENTIAL INVARIANTS: d, [,], ∇

DEMETER KRUPKA, VĚRA MIKOLÁŠOVÁ, Brno

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1. Introduction. It is known that Nijenhuis's theory of natural bundles [8], [10], [12] allows us to characterize differential-geometrical (= invariant) operations and mappings as natural transformations of some liftings, or equivalently, as certain differential invariants in the sense of [6]. Moreover, this theory, combined with the prolongation theory of liftings [4], [5], is an important tool for computation of differential-geometrical operations — either by "infinitesimal", or "direct" methods (see e.g. [7], [9]).

The purpose of this paper is to apply Nijenhuis's theory to the problem of uniqueness of the exterior derivative of forms, the Lie bracket of vector fields, and the Levi-Civita connection of metric fields.

As a motivation, consider the exterior derivative $\omega \to d\omega$ of forms on a differential manifold X. It is known that the exterior derivative may be defined by various properties. For example (see [1]), one may require (1) if $\omega = f$ is a function, then df is the differential of f, (2) the domain of definition of $d\omega$ coincides with the domain of definition of ω, (3) the mapping $\omega \to d\omega$ is R-linear, (4) for each p-form ω and each q-form η, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$, and (5) $d(d\omega) = 0$ for each ω. There is also another definition of d (see [11]), based on its behaviour under mappings: d is the unique R-linear mapping such that for each local diffeomorphism of X,

$$d\alpha^*\omega = \alpha^* d\omega . \tag{1.1}$$

Let $\Lambda^p TX$ be the bundle of p-foıms over X, $j^r \Lambda^p TX$ its r-jet prolongation, $\mathscr{D}_n$ the category of smooth n-dimensional manifolds and their embeddings, $\mathscr{D}_X$ its subcategory of morphisms $\alpha: U \to V$, where $U, V \subset X$, and $\mathscr{FB}(\Lambda^p TX)$ the category of local bundle homomorphisms of $\Lambda^p TX$ whose projections are morphisms of $\mathscr{D}_X$. If $\tau_p: \mathscr{D}_X \to \mathscr{FB}(\Lambda^p TX)$ denotes the natural lifting and $j^1\tau_p: \mathscr{D}_X \to \mathscr{FB}(j^1 \Lambda^p TX)$ its 1-jet prolongation, then d may be regarded as a natural transformation of $j^1\tau_p$ to τ_{p+1} or, which is equivalent, a differential invariant $d_0: T^1_n \Lambda^p R^n \to \Lambda^{p+1} R^n$, where

$T_n^1 \Lambda^p R^n$ ($\Lambda^{p+1} R^n$) is the type fiber of $j^1 \Lambda^p TX$ ($\Lambda^{p+1} TX$, respectively); the differential invariant d_0 obtained in this way is a linear mapping. Thus the problem of uniqueness consists in the uniqueness of the differential invariant d_0 or, if we allow higher order liftings, in the uniquenss of d_0: $T_n^r \Lambda^p R^n \to \Lambda^{p+1} R^n$, where $r \geqq 1$ is arbitrary.

The problem of uniqueness of the Lie bracket and the Levi-Civita connection may analogously be formulated in terms of appropriate liftings, their natural transformations, and the corresponding differential invariants. The methods we use are, however, more general, and should be compared with [11] and, in the case of the Levi-Civita connection, with [3], [13].

2. Basic geometrical structures. Throughout this paper, R denotes the field of real numbers, R^n the real, n-dimensional Euclidean space. If R^n is considered with its natural vector-space structure, its dual vector space is denoted by R^{n*}. X denotes a smooth n-dimensional differential manifold. Our notations given below follow [5], [7].

Recall that the *rth differential group* L_n^r of R^n is the group of invertible r-jets with source and target at the origin $0 \in R^n$. Let K_n^r denote the kernel of the canonical projection $L_n^r \to L_n^1$, and let ι_r: $L_n^1 \to L_n^r$ be the canonical homomorphism of the Lie groups. It is known that K_n^r is a nilpotent normal subgroup of L_n^r, diffeomorphic with some Euclidean space, and that $L_n^r = L_n^1 \times_s K_n^r$ (= semi-direct product).

The canonical (global) coordinates $b^i_{j_1}, b^i_{j_2}, \ldots, b^i_{j_1 \ldots j_r}$, $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq \ldots \leqq j_r \leqq n$, on L_n^r are defined as follows. Let $j_0^r \alpha \in L_n^r$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a local diffeomorphism of R^n, let $\alpha^{-1} = (\alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_n^{-1})$ be the inverse diffeomorphism. We set

$$(2.1) \qquad b^i_{j_1}(j_0^r \alpha) = D_{j_1} \alpha_i^{-1}(0), \ldots, b^i_{j_1 \ldots j_r}(j_0^r \alpha) = D_{j_1} \ldots D_{j_r} \alpha_i^{-1}(0),$$

where D_j is the partial derivative operator with respect to the j-th variable in R^n. We define $a^i_j(j_0^r \alpha) = b^i_j(j_0^r \alpha^{-1})$; then $a^i_p b^p_j = \delta^i_j$ (= the Kronecker symbol).

We denote by TX the tangent bundle of X, by $\Lambda^p TX$ ($j^r \Lambda^p TX$) the bundle of p-forms (the r-jet prolongation of the bundle of p-forms, respectively) over X. The type fiber of $\Lambda^p TX$ ($j^r \Lambda^p TX$) is denoted by $\Lambda^p R^n$ ($T_n^r \Lambda^p R^n$, respectively), and the canonical coordinates on $\Lambda^p R^n$ ($T_n^r \Lambda^p R^n$) are denoted by $\omega_{i_1 i_2 \ldots i_p}$, $1 \leqq i_1 < i_2 < \ldots < i_p \leqq n$ ($\omega_{i_1 i_2 \ldots i_p}, \omega_{i_1 i_2 \ldots i_p, k_1}, \ldots, \omega_{i_1 i_2 \ldots i_p, k_1 k_2 \ldots k_r}$, $1 \leqq i_1 < i_2 < \ldots < i_p \leqq n$, $1 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_r \leqq n$, respectively). $\Lambda^p R^n$ ($T_n^r \Lambda^p R^n$) has a natural structure of an L_n^1-module (L_n^{r+1}-module, respectively).

Put

$$(2.2) \qquad Q = R^n \otimes (R^{n*} \odot R^{n*}),$$

where $\odot$ denotes the symmetric tensor product, and denote by Γ^i_{jk}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k \leqq n$, the canonical coordinates on Q. Q may be endowed with the structure of an L_n^2-module as follows. For each $(j_0^2 \alpha, q) \in L_n^2 \times Q$ we set

$$(2.3) \qquad \Gamma^i_{jk}(j_0^2 \alpha \,.\, q) = a^i_p(j_0^2 \alpha) \left(b^q_j(j_0^2 \alpha)\, b^r_k(j_0^2 \alpha)\, \Gamma^p_{qr}(q) + b^p_{jk}(j_0^2 \alpha)\right)$$

or, with the obvious meaning of the symbols,

$$\Gamma^i_{jk} = a^i_p(b^q_j b^r_k \Gamma^p_{qr} + b^p_{jk}) . \tag{2.4}$$

The mapping $(j^2_0\alpha, q) \to j^2_0\alpha . q$ introduced by either of these formulas, defines a left action of L^2_n on Q.

Consider the r-jet prolongation $T^r_n Q$ of Q, i.e. the set of r-jets with source $0 \in R^n$ and target in Q, with its natural manifold structure. Let $\Gamma^i_{jk}, \Gamma^i_{jk,m_1}, \ldots, \Gamma^i_{jk,m_1\ldots m_r}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k \leqq n$, $1 \leqq m_1 \leqq m_2 \leqq \ldots \leqq m_r \leqq n$, be the canonical coordinates on $T^r_n Q$. Recall that for each $p = 1, 2, \ldots, r$,

$$\Gamma^i_{jk,m_1\ldots m_p}(j^r_0 q) = D_{m_1} \ldots D_{m_p}(\Gamma^i_{jk} q)(0) . \tag{2.5}$$

The L^{r+1}_n-module structure of $T^r_n Q$ may be easily described by formal differentiation of (2.4). We obtain

$$\begin{aligned} \Gamma^i_{jk,m} = {} & a^i_s a^t_p b^s_{tm}(b^q_j b^r_k \Gamma^p_{qr} + b^p_{jk}) + \\ & + a^i_p((b^q_{jm} b^r_k + b^q_j b^r_{km})\Gamma^p_{qr} + b^q_j b^r_k b^t_m \Gamma^p_{qr,t} + b^p_{jkm}) , \end{aligned} \tag{2.6}$$

etc.

3. The uniqueness of exterior derivative. The fact that the exterior derivative as defined above (see (1.1)) is unique was proved by Palais [11]. Our proof is based on a generalization of the following lemma proved in [7]: The left L^{r+2}_n-manifold $T^r_n Q$ has the structure of a left principal K^{r+2}_n-bundle. This left principal K^{r+2}_n-bundle is trivial, and its base is diffeomorphic with some Euclidean space. The canonical projection ϱ_r: $T^r_n Q \to T^r_n Q/K^{r+2}_n$ is equivariant in the sense that for each $j^{r+2}_0\alpha \in L^{r+2}_n$, $j^r_0 q \in T^r_n Q$,

$$\varrho_r(j^{r+2}_0\alpha . j^r_0 q) = j^1_0\alpha . \varrho_r(j^r_0 q) . \tag{3.1}$$

Let us reformulate this lemma for spaces arising by prolongations of tensor bundles. Let E be the space of tensors of type (p, q) on R^n, let $t^{i_1\ldots i_p}_{j_1\ldots j_q}$ be the canonical coordinates on E. Consider the product $T^{r-1}_n Q \times T^r_n E$ ($T^r_n E$ = the space of r-jets with source $0 \in R^n$ and target in E), endowed with the natural structure of an L^{r+1}_n-manifold.

Lemma. *The left L^{r+1}_n-manifold $T^{r-1}_n Q \times T^r_n E$ has the structure of a left principal K^{r+1}_n-bundle. This left principal K^{r+2}_n-bundle is trivial, and its base is diffeomorphic with some Euclidean space. The canonical projection π_r: $T^{r-1}_n Q \times T^r_n E \to (T^{r-1}_n Q \times T^r_n E)/K^{r+1}_n$ is equivariant in the sense that for each $j^{r+1}_0\alpha \in L^{r+1}_n$, $(j^{r-1}_0 q, j^r_0\xi) \in T^{r-1}_n Q \times T^r_n E$,*

$$\pi_r(j^{r+1}_0\alpha . (j^{r-1}_0 q, j^r_0\xi)) = j^1_0\alpha . \pi_r(j^{r-1}_0 q, j^r_0\xi) . \tag{3.2}$$

Proof. Following [7] denote by R^i_{jkm} the components of the formal curvature tensor on $T^r_n Q$, by $R^i_{jkm;s_1}$ ($R^i_{jkm;s_1;s_2}$) the comonents of its first (second, respectively) formal covariant derivative, etc. Put

$$(3.3)\qquad \Gamma^i_{jkm_1\dots m_s} = \Gamma^i_{(jk,m_1\dots m_s)},$$

where $(jk, m_1 \dots m_s)$ means the symmetrization of (2.5), and denote by $t^{i_1\dots i_p}_{j_1\dots j_q}$, $t^{i_1\dots i_p}_{j_1\dots j_q;m_1}, \dots, t^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_r}$ the formal covariant derivatives of $t^{i_1\dots i_p}_{j_1\dots j_q}$. For example,

$$(3.4)\qquad t^{i_1\dots i_p}_{j_1\dots j_q;m} = t^{i_1\dots i_r}_{j_1\dots j_q,m} + \Gamma^{i_1}_{sm} t^{si_2\dots i_p}_{j_1\dots j_q} + \dots + \Gamma^{i_p}_{sm} t^{i_1\dots i_{p-1}s}_{j_1\dots j_q} - \Gamma^s_{mj_1} t^{i_1\dots i_p}_{sj_2\dots j_q} - \\ - \Gamma^s_{mj_q} t^{i_1\dots i_p}_{j_1\dots j_{q-1}s}.$$

Then the system of functions $\Gamma^i_{j_1j_2}, \dots, \Gamma^i_{j_1\dots j_{r+1}}$, $1 \leqq i \leqq n$, $1 \leqq j_1 \leqq \dots \leqq j_{r+1} \leqq \leqq n$, $R^i_{jkl}, R^i_{jkl;m_1}, \dots, R^i_{jkl;m_1;\dots;m_{r-2}}$, $t^{i_1\dots i_p}_{j_1\dots j_q}, t^{i_1\dots i_p}_{j_1\dots j_q;m_1}, \dots, t^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_r}$ contains a subsystem defining a global chart on $T^{r-1}_n Q \times T^r_n E$. The coordinate functions belonging to this chart will be referred to as the adapted coordinates. In terms of this chart, the action of L^{r+1}_n on $T^{r-1}_n Q \times T^r_n E$ is expressed by the formulas

$$(3.5)\qquad \bar{\Gamma}^i_{m_1\dots m_{s+2}} = \Gamma^i_{m_1\dots m_{s+2}} + s^i_{m_1\dots m_{s+2}} + b^i_{m_1\dots m_{s+2}},$$
$$\bar{R}^i_{jkl;m_1;\dots;m_s} = R^i_{jkl;m_1;\dots;m_s}, \quad \bar{t}^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_s} = t^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_s},$$

where $s^i_{m_1\dots m_{s+2}}$ is a polynomial in the canonical coordinates on K^{s+1}_n and in the adapted coordinates on $T^{s-1}_n Q$. Consequently, the K^{r+1}_n-orbits in $T^{r-1}_n Q \times T^r_n E$ are defined by the equations

$$(3.6)\qquad R^i_{jkl} = a^i_{jkl}, \dots, R^i_{jkl;m_1;\dots;m_{r-2}} = a^i_{jklm_1\dots m_{r-2}},$$
$$t^{i_1\dots i_p}_{j_1\dots f_q} = b^{i_1\dots i_p}_{j_1\dots j_q}, \dots, t^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_r} = b^{i_1\dots i_p}_{j_1\dots j_q m_1\dots m_r},$$

with arbitrary constants on the right-hand sides.

To prove the lemma it is sufficient to show that the equivalence relation "there exists $j^{r+1}_0\alpha \in L^{r+1}_n$ such that $(j^{r-1}_0 q_1, j^r_0 \xi_1) = j^{r+1}_0\alpha \,.\, (j^{r-1}_0 q_2, j^{r-1}_0 \xi_2)$" is a closed submanifold of the product $(T^{r-1}_n Q \times T^r_n E) \times (T^{r-1}_n Q \times T^r_n E)$, and that the action of K^{r+1}_n on $T^{r-1}_n Q \times T^r_n E$ is free. In the adapted coordinates on the first and the second factor, this eequivalence has the equations $R^i_{jkl} = \bar{R}^i_{jkl}, \dots, R^i_{jkl;m_1;\dots;m_{r-2}} = = \bar{R}^i_{jkl;m_1;\dots;m_{r-2}}$, $t^{i_1\dots i_p}_{j_1\dots j_q} = \bar{t}^{i_1\dots i_p}_{j_1\dots j_q}, \dots, t^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_r} = \bar{t}^{i_1\dots i_p}_{j_1\dots j_q;m_1;\dots;m_r}$ and is therefore closed. Equations (3.5) imply that K^{r+1}_n acts freely on $R^{r-1}_n Q \times T^r_n E$. The remaining assertions are verified in the same manner as in [7].

We are now in position to prove the uniqueness of the exterior derivative. To this purpose we put $E = \Lambda^p R^n$ in the lemma, and denote by $\omega_{j_1\dots j_r}$ the canonical coordinates on $\Lambda^p R^n$.

Theorem 1. *There exists a unique, up to a multiplicative constant factor, linear differential invariant from $T^r_n \Lambda^p R^n$ to $\Lambda^{p+1} R^n$. In canonical coordinates, this differential invariant is expressed by the equations*

$$(3.7)\qquad \omega_{j_1\dots j_p j_{p+1}} = \omega_{[j_1\dots j_p, j_{p+1}]},$$

where $[j_1 \dots j_p, j_{p+1}]$ means antisymmetrization.

Proof. 1) Existence. Using the structure of an L_n^{r+1}-manifold on $T_n^r \Lambda^p R^n$ one directly verifies that (3.7) defines an L_n^{r+1}-equivariant mapping from $T_n^r \Lambda^p R^n$ to $\Lambda^{p+1} R^n$.

2) Uniqueness. Let $d_r \colon T_n^r \Lambda^p R^n \to \Lambda^{p+1} R^n$ be a linear differential invariant. Consider the product $T_n^{r-1} Q \times T_n^r \Lambda^p R^n$ together with its structure of a principal K_n^{r+1}-bundle. We have the diagram

$$\begin{array}{ccccc} T_n^{r-1} Q \times T_n^r \Lambda^p R^n & \xrightarrow{\text{proj}} & T_n^r \Lambda^p R^n & \xrightarrow{d_r} & \Lambda^{p+1} R^n \\ \Big\downarrow \pi_r & & & & \Big\uparrow d_{r,0} \\ (T_n^{r-1} Q \times T_n^r \Lambda^p R^n)/K_n^{r+1} & & \dashrightarrow & & \end{array} \tag{3.8}$$

where proj denotes the projection onto the second factor. Since the mapping $d_r \circ \text{proj}$ is K_n^{r+1}-equivariant, it can be factored through the projection π_r. Denote by $[j_0^{r-1} q, j_0^r \omega]$ the equivalence class of a pair $(j_0^{r-1} q, j_0^r \omega) \in T_n^{r-1} Q \times T_n^r \Lambda^p R^n$. Then by definition

$$d_{r,0}([j_0^{r-1} q, j_0^r \omega]) = d_r(j_0^r \omega)\,. \tag{3.9}$$

Since d_r is linear, we have for each $j_0^{r-1} q, j_0^r \omega_1, j_0^r \omega_2$

$$d_{r,0}([j_0^{r-1} q, j_0^r \omega_1 + j_0^r \omega_2]) = d_{r,0}([j_0^{r-1} q, j_0^r \omega_1]) + d_{r,0}([j_0^{r-1} q, j_0^r \omega_2])\,. \tag{3.10}$$

In coordinates, $d_{r,0}$ is a mapping $(R^i_{jkl}, \ldots, R^i_{jkl;\ldots;m_{r-2}}, \omega_{i_1 \ldots i_p}, \ldots, \omega_{i_1 \ldots i_p;m_1;\ldots;m_r}) \to$ $\to (\omega_{j_1 \ldots j_{p+1}}(R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}, \omega_{i_1 \ldots i_p}, \ldots, \omega_{i_1 \ldots i_p;m_1;\ldots;m_r}))$, and (3.10) means that for each fixed $R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}$, $\omega_{j_1 \ldots j_{p+1}}$ depends linearly on $\omega_{i_1 \ldots i_p}, \ldots$ $\ldots, \omega_{i_1 \ldots i_p;m_1;\ldots;m_r}$. Hence

$$\bar{\omega}_{j_1 \ldots j_{p+1}} = A^{i_1 \ldots i_p}_{j_1 \ldots j_{p+1}} \omega_{i_1 \ldots i_p} + \ldots + A^{i_1 \ldots i_p;m_1;\ldots;m_r}_{j_1 \ldots j_{p+1}} \omega_{i_1 \ldots i_p;m_1;\ldots;m_r} \tag{3.11}$$

where the coefficients depend on $R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}$. Since $\omega_{j_1 \ldots j_{p+1}}$ is an L_n^1-equivariant mapping and $\omega_{i_1 \ldots i_r}, \ldots, \omega_{i_1 \ldots i_p;m_1;\ldots;m_r}$ are tensors, the coefficients in (3.11) must also be tensors: take for example $\omega_{i_1 \ldots i_p;m_1} = 0, \ldots, \omega_{i_1 \ldots i_p;m_1;\ldots;m_r} =$ $= 0$; then after a transformation $j_0^1 \alpha \in L_n^1$,

$$\begin{aligned} \bar{\omega}_{j_1 \ldots j_{p+1}} &= b^{k_1}_{j_1} \ldots b^{k_{p+1}}_{j_{p+1}} \omega_{k_1 \ldots k_{p+1}} = b^{k_1}_{j_1} \ldots b^{k_{p+1}}_{j_{p+1}} A^{i_1 \ldots i_p}_{k_1 \ldots k_{p+1}} \omega_{i_1 \ldots i_p} = \\ &= \bar{A}^{l_1 \ldots l_p}_{j_1 \ldots j_{p+1}} \bar{\omega}_{l_1 \ldots l_p} = \bar{A}^{l_1 \ldots l_p}_{j_1 \ldots j_{p+1}} b^{i_1}_{l_1} \ldots b^{i_p}_{l_p} \omega_{i_1 \ldots i_p} \end{aligned} \tag{3.12}$$

which is possible if and only if

$$\bar{A}^{s_1 \ldots s_p}_{j_1 \ldots j_{p+1}} = a^{s_1}_{i_1} \ldots a^{s_p}_{i_p} b^{k_1}_{j_1} \ldots b^{k_{p+1}}_{j_{p+1}} A^{i_1 \ldots i_p}_{\kappa_1 \ldots k_{p+1}}\,. \tag{3.13}$$

We now apply the condition that the composed mapping $d_r \circ \text{proj} = d_{r,0} \circ \pi_r$ should be independent of $j_0^{r-1} q$. In coordinates this means that the expression (3.11) rewritten in the canonical coordinates on $T^{r-1}_u Q \times T^r_n \Lambda^r R^n$, does not depend on $\Gamma^i_{jk}, \dots, \Gamma^i_{jk,m_1\dots m_{r-1}}$. Applying the formulas for the formal covariant derivatives analogous to (3.4) one easily obtains that $A^{i_1\dots i_p;m_1;\dots;m_r}_{j_1\dots j_{p+1}}$ (= the coefficient at $\omega_{i_1\dots i_p,m_1\dots m_r}$) is independent of $\Gamma^i_{jk}, \dots, \Gamma^i_{jk,m_1\dots m_{r-1}}$ (unless $r = 1$); the classical invariant theory then says, however, that $A^{i_1\dots i_p;m_1;\dots;m_r}_{j_1\dots j_{p+1}} = 0$ (see e.g. [2]). Repeating this argument we obtain

$$A^{i_1\dots i_p}_{j_1\dots j_{p+1}} = 0, \dots, A^{i_1\dots i_p;m_1;m_2}_{j_1\dots j_{p+1}} = 0\,. \tag{3.14}$$

Hence

$$\omega_{j_1\dots j_{p+1}} = A^{i_1\dots i_p;m}_{j_1\dots j_{p+1}}\omega_{i_1\dots i_p;m} = \tag{3.15}$$
$$= A^{i_1\dots i_p;m}_{j_1\dots j_{p+1}}(\omega_{i_1\dots i_p,m} - \Gamma^s_{i_1 m}\omega_{s i_1\dots i_p} - \Gamma^s_{i_p m}\omega_{i_1\dots i_{p-1}s})\,.$$

Hence $A^{i_1\dots i_p;m}_{j_1\dots j_{p+1}}$ is a constant tensor. Since it is antisymmetric in the subscripts, it must be a multiple of the peımutation tensor $\varepsilon^{i_1\dots i_p m}_{j_1\dots j_{p+1}}$, and we have

$$\omega_{j_1\dots j_{p+1}} = c\,.\,\omega_{[j_1\dots j_p;j_{p+1}]}\,.$$

This completes the proof.

4. The uniqueness of the Lie bracket. Using the notation of the fundamental lemma of Section 3, put $E = R^n \times R^n$, and examine the bilinear differential invariants from $T^r_n(R^n \times R^n)$ to R^n. The canonical coordinates on $T^r_n(R^n \times R^n)$ (on R^n) are denoted by $\xi^i, \zeta^i, \xi^i_{j_1}, \zeta^i_{j_1}, \dots, \xi^i_{j_1\dots j_r}, \zeta^i_{j_1\dots j_r}$ (λ^i, respectively). As before, we shall also consider the left principal K^{r+1}_n-bundle $\pi_r\colon T^{r-1}_n Q \times T^r_n(R^n \times R^n) \to (T^{r-1}_n Q \times T^r_n(R^n \times R^n))/K^{r+1}_n$ and denote by $\xi^i, \zeta^i, \xi^i_{;j_1}, \zeta^i_{;j_1}, \dots, \xi^i_{;j_1;\dots;j_r}, \zeta^i_{;j_1;\dots;j_r}$ the adapted coordinates on the total space of this bundle.

Theorem 2. *There exists a unique, up to a multiplicative constant, bilinear differential invariant from $T^r_n(R^n \times R^n)$ to R^n. In canonical coordinates, this differential invariant is expressed by the equations*

$$\lambda^i = \xi^k\zeta^i_k - \zeta^k\xi^i_k\,. \tag{4.1}$$

Proof. 1) Existence. The mapping $(j^r_0\xi, j^r_0\zeta) \to \lambda(j^r_0\xi, j^r_0\zeta)$ defined by (4.1) has all the properties required.

2) Uniqueness. Let $\lambda_r\colon T^r_n(R^n \times R^n) \to R^n$ be a bilinear differential invariant. In the diagram

$$\begin{array}{ccccc} T^{r-1}_n Q \times T^r_n(R^n \times R^n) & \xrightarrow{\text{proj}} & T^r_n(R^n \times R^n) & \xrightarrow{\lambda_r} & R^n \\ \Big\downarrow \pi_r & & & & \Big\uparrow \lambda_{r,0} \\ (T^{r-1}_n Q \times T^r_n(R^n \times R^n))/K^{r+1}_n & & \text{-----} & & \text{-----} \end{array} \tag{4.2}$$

the mapping $\lambda_r \circ \text{proj}$ can be factored through π_r. Hence λ_r is, in the sense of the classical invariant theory, a vector (= element of R^n endowed with the standard action of L_n^1) depending on the tensors $R^i_{jkl}, \ldots, R^i_{jkl;m_1;\ldots;m_{r-2}}, \xi^i, \zeta^i, \ldots, \xi^i_{;j_1;\ldots;j_r}, \zeta^i_{;j_1;\ldots;j_r}$; by the bilinearity assumption, the components λ^k of λ_r are linear combinations of tensors $\xi^i, \zeta^j, \xi^i\zeta^j_{;l_1}, \ldots, \xi^i\zeta^j_{;l_1;\ldots;l_r}, \ldots, \xi^i_{;s_1;\ldots;s_r}\zeta^j_{;l_1;\ldots;l_r}$, whose coefficients are constant tensors. Therefore the only non-vanishing coefficients are at the tensors $\xi^i\zeta^j_{;l_1}, \xi^j_{;l_1}\zeta^i$, and λ^k must have the form

$$\lambda^k = A^{kl}_{ij}\xi^i\zeta^j_{;l} + B^{kl}_{ij}\zeta^i\xi^j_{;l} = A^{kl}_{ij}\xi^i(\zeta^j_l + \Gamma^j_{lm}\zeta^m) + B^{kl}_{ij}\zeta^i(\xi^j_l + \Gamma^j_{lm}\xi^m). \tag{4.3}$$

Since λ^k does not depend on $\Gamma^i_{jk}, \ldots, \Gamma^i_{jk,l_1\ldots l_{r-1}}$, we have

$$A^{kl}_{ij}\delta^i_p\delta^m_q + B^{kl}_{ij}\delta^i_q\delta^m_p + A^{km}_{ij}\delta^i_p\delta^l_q + B^{km}_{ij}\delta^i_q\delta^l_p = 0. \tag{4.4}$$

But

$$A^{kl}_{ij} = a\delta^k_i\delta^l_j + b\delta^k_j\delta^l_i, \quad B^{kl}_{ij} = a'\delta^k_i\delta^l_j + b'\delta^k_j\delta^l_i \tag{4.5}$$

for some $a, b, a', b' \in R$ so that

$$\begin{aligned}(a\delta^k_i\delta^l_j + b\delta^k_j\delta^l_i)\,\delta^i_p\delta^m_q + (a'\delta^k_i\delta^l_j + b'\delta^k_j\delta^l_i)\,\delta^i_q\delta^m_p + \\ + (a\delta^k_i\delta^m_j + b\delta^k_j\delta^m_i)\,\delta^i_p\delta^l_q + (a'\delta^k_i\delta^m_j + b'\delta^k_j\delta^m_i)\,\delta^i_q\delta^l_p = \\ = a(\delta^l_j\delta^k_p\delta^m_q + \delta^k_p\delta^m_j\delta^l_q) + b(\delta^k_j\delta^l_p\delta^m_q + \delta^k_j\delta^m_p\delta^l_q) + \\ + a'(\delta^k_q\delta^l_j\delta^m_p + \delta^k_q\delta^m_j\delta^l_p) + b'(\delta^k_j\delta^l_q\delta^m_p + \delta^k_j\delta^m_q\delta^l_p) = \\ = (b + b')(\delta^k_j\delta^l_p\delta^m_q + \delta^k_j\delta^m_p\delta^l_q) + a\delta^k_p(\delta^l_j\delta^m_q + \delta^m_j\delta^l_q) + a'\delta^k_q(\delta^l_j\delta^m_p + \delta^m_j\delta^l_p) = 0.\end{aligned} \tag{4.6}$$

If $n = 1$, then (4.3) is rewritten in the form $\lambda = A\xi(\zeta' + \Gamma\zeta) + B\zeta(\xi' + \Gamma\xi) = A\xi\zeta' + B\zeta\xi' + (A + B)\Gamma\xi\zeta$, and the independence of Γ means that $A + B = 0$, i.e., λ_r is of the form (4.1). Let $n > 1$. Take for example $j = k = 1$, $p = q = m = l = 2$. Then (4.6) gives

$$b + b' = 0. \tag{4.7}$$

Similarly take $k = p = 1$, $l = j = m = q = 2$. In this way we get

$$a = 0, \quad a' = 0. \tag{4.8}$$

Returing back to (4.3) we get

$$\lambda^k = A^{kl}_{ij}\xi^i\zeta^j_l + B^{kl}_{ij}\zeta^i\xi^j_l = b\delta^k_j\delta^l_i\xi^i\zeta^j_l + b'\delta^k_j\delta^l_i\zeta^i\xi^j_l = b(\xi^l\zeta^k_l - \zeta^l\xi^k_l). \tag{4.9}$$

This completes the proof.

5. The uniqueness of the Levi-Civita connection. Let us consider the space $E = R^{n*} \odot R^{n*}$ of tensors of type (0, 2) on R^n and denote by g_{ij}, $1 \leqq i \leqq j \leqq n$, the canonical coordinates on E. Let $E_0 = E \setminus \{g \in E \mid \det(g_{ij}(g)) = 0\}$ be the subset of *regular* tensors; E_0 is endowed with the natural structure of an L^1_n-manifold. It is of great interest to know all differential invariants from L^{k+1}_n-invariant open subsets of $T^r_n E_0$ to Q which correspond in a well-known manner to linear connections which are geometric objects (= concominants) of the metric tensor and its derivatives of order $\leqq r$.

An example of a connection of this type is the Levi-Civita connection ∇, for which $r = 1$. For $r > 1$, however, some other examples may be given which differ from the Levi-Civita connection. For this reason we shall consider the problem of uniqueness only for $r = 1$. We note that the problem of uniqueness of the Levi-Civita connection has been stated and solved in a different way in [3] and [13].

Let $g_{ij}, g_{ij,k}$ be the canonical coordinates on $T^r_n E_0$, and let (g^{ij}) denote the inverse matrix of (g_{ij}). As before, let Γ^i_{jk} be the canonical coordinates on Q.

Theorem 3. *There exists a unique differential invariant from $T^1_n E_0$ to Q. In canonical coordinates, this differential invariant is expressed by the equations*

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}) . \tag{5.1}$$

Proof. The action of the group L^2_n on $T^1_n E_0$ and Q is expressed by

$$\begin{aligned} \bar{g}_{ij} &= b^p_i b^q_j g_{pq} , \\ \bar{g}_{ij,m} &= (b^p_{im} b^q_j + b^p_i b^q_{jm})\, g_{pq} + b^p_i b^q_j b^l_m g_{pq,l} \end{aligned} \tag{5.2}$$

and by (2.4). The fundamental vector fields on $T^1_n E_0$ relative to this action, are

$$\begin{aligned} \xi^q_p &= \left(\frac{\partial \bar{g}_{ij}}{\partial b^p_q}\right)_e \frac{\partial}{\partial g_{ij}} + \left(\frac{\partial \bar{g}_{ij,m}}{\partial b^p_q}\right)_e \frac{\partial}{\partial g_{ij,m}} = \\ &= (\delta^q_i g_{pj} + \delta^q_j g_{ip}) \frac{\partial}{\partial g_{ij}} + (\delta^q_i g_{pj,m} + \delta^q_j g_{ip,m} + \delta^q_m g_{ij,p}) \frac{\partial}{\partial g_{ij,m}} , \\ \xi^{qr}_p &= \left(\frac{\partial \bar{g}_{ij,m}}{\partial b^p_{qr}}\right)_e \frac{\partial}{\partial g_{ij,m}} = g_{ip}\left(\frac{\partial}{\partial g_{iq,r}} + \frac{\partial}{\partial g_{ir,q}}\right), \end{aligned} \tag{5.3}$$

and the fundamental vector fields on Q relative to (2.4), are

$$\begin{aligned} \Xi^q_p &= \left(\frac{\partial \bar{\Gamma}^i_{jk}}{\partial b^p_q}\right)_e \frac{\partial}{\partial \Gamma^i_{jk}} = (-\delta^i_p \Gamma^q_{jk} + \delta^q_j \Gamma^i_{pk} + \delta^q_k \Gamma^i_{jp}) \frac{\partial}{\partial \Gamma^i_{jk}} , \\ \Xi^{qr}_p &= \left(\frac{\partial \bar{\Gamma}^i_{jk}}{\partial b^p_{qr}}\right)_e \frac{\partial}{\partial \Gamma^i_{jk}} = \frac{\partial}{\partial \Gamma^p_{qr}} , \end{aligned} \tag{5.4}$$

where e denotes $j^2_0\,\mathrm{id}$ (= the identity of L^2_n). Hence each L^2_n-equivariant mapping

$(g_{ij}, g_{ij,k}) \rightarrow (\Gamma^p_{qr}(g_{ij}, g_{ij,k}))$ of $T^1_n E_0$ to Q must obey the system of equations

$$(5.5) \qquad (\delta^q_a g_{pb} + \delta^q_b g_{ap}) \frac{\partial \Gamma^i_{jk}}{\partial g_{ab}} + (\delta^q_a g_{pb,m} + \delta^q_b g_{ap,m} + \delta^q_m g_{ab,p}) \frac{\partial \Gamma^i_{jk}}{\partial g_{ab,m}} = = -\delta^i_p \Gamma^q_{jk} + \delta^q_j \Gamma^i_{pk} + \delta^q_k \Gamma^i_{jp},$$

$$(5.6) \qquad g_{ap} \frac{\partial \Gamma^i_{jk}}{\partial g_{aq,r}} + \frac{\partial \Gamma^i_{jk}}{\partial g_{ar,q}} = \tfrac{1}{2}\delta^i_p(\delta^q_j \delta^r_k + \delta^q_k \delta^r_j).$$

Consider the system (5.6). Multiplying both sides by g^{ps} and rewriting the system for cyclic permutations of the subscripts of the independent variable one gets

$$(5.7) \qquad \frac{\partial \Gamma^i_{jk}}{\partial g_{sq,r}} + \frac{\partial \Gamma^i_{jk}}{\partial g_{sr,q}} = \tfrac{1}{2} g^{is}(\delta^q_j \delta^r_k + \delta^q_k \delta^r_j),$$

$$\frac{\partial \Gamma^i_{jk}}{\partial g_{rs,q}} + \frac{\partial \Gamma^i_{jk}}{\partial g_{rq,s}} = \tfrac{1}{2} g^{ir}(\delta^s_j \delta^q_k + \delta^s_k \delta^q_j),$$

$$-\frac{\partial \Gamma^i_{jk}}{\partial g_{qr,s}} - \frac{\partial \Gamma^i_{jk}}{\partial g_{qs,r}} = \tfrac{1}{2} g^{iq}(\delta^r_j \delta^s_k + \delta^r_k \delta^s_j),$$

which implies

$$(5.8) \qquad \frac{\partial \Gamma^i_{jk}}{\partial g_{sr,q}} = \tfrac{1}{4}(g^{is}(\delta^q_j \delta^r_k + \delta^q_k \delta^r_j) + g^{ir}(\delta^s_j \delta^q_k + \delta^s_k \delta^q_j) - g^{iq}(\delta^r_j \delta^s_k + \delta^r_k \delta^s_j).$$

Hence

$$(5.9) \qquad \Gamma^i_{jk} = \tfrac{1}{4}(g^{is}(g_{sk,j} + g_{sj,k}) + g^{ir}(g_{jr,k} + g_{kr,j}) - - g^{iq}(g_{kj,q} + g_{jk,q})) + \gamma^i_{jk} = \tfrac{1}{2} g^{is}(g_{sj,k} + g_{sk,j} - g_{jk,s}) + \gamma^i_{jk},$$

where γ^i_{jk} does not depend on $g_{pq,r}$. Substituting this expression into (5.5) one obtains

$$(5.10) \qquad \frac{\partial \gamma^i_{jk}}{\partial g_{aq}} = 0,$$

which means that $\gamma^i_{jk} \in R$. This shows that each differential invariant from $T^1_n E_0$ to Q must belong to the family of mappings (5.9), where γ^i_{jk} are real numbers. Consider the transformation properties of these mappings. It is directly seen that among these mappings there is one and only one which is L^2_n-equivariant; this is the mapping for which $\gamma^i_{jk} = 0$. This completes the proof.

References

[1] *R. L. Bishop, R. J. Crittenden:* Geometry of Manifolds, Academic Press, New York and London, 1964.

[2] *J. A. Dieudonné, J. B. Carrel:* Invariant Theory, Old and New, Academic Press, New York and London, 1971.

[3] *D. B. Epstein:* Natural tensors on Riemannian manifolds, J. Differential Geom. *10* (1975), 631—645.

[4] *I. Kolář:* On the prolongations of geometric object fields, An. Sti. Univ. "Al. I. Cuza" Iasi, *17* (1971), 437—446.

[5] *D. Krupka:* Differential Invariants, Lecture Notes, Faculty of Science, Purkyně University, Brno, 1979 (preprint).

[6] *D. Krupka:* Elementary theory of differential invariants, Arch. Math. *XIV* (1978), 207—214.

[7] *D. Krupka:* Local invariants of a linear connection, Colloquia Mathematica Societatis János Bolyai, 31. Differential Geometry, Budapest (Hungary), 1979, North-Holland, 1982, 349—369.

[8] *D. Krupka:* Reducibility theorems for differentiable liftings in fiber bundles. Arch. Math. *XV* (1979), 93—106.

[9] *V. Mikolášová:* On the functional independence of scalar invariants of curvature for dimensions $n = 2, 3, 4$, Math. Slovaca *32*, 1982, 349—354.

[10] *A. Nijenhuis:* Natural bundles and their general properties, Differential Geometry, in honour of K. Yano, Kinokunyia, Tokyo, 1972, 317—334.

[11] *R. S. Palais:* Natural operations on differential forms, Trans. Amer. Math. Soc. *92* (1959), 125—141.

[12] *R. S. Palais, C. L. Terng:* Natural bundles have finite order, Topology *16* (1977), 271—277.

[13] *P. Stredder:* Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups, J. Differential Geom. *10* (1975), 647—660.

[14] *C. L. Terng:* Natural vector bundles and natural differential operators, Amer. J. Math. *100* (1978), 775—828.

Authors' address: 611 37 Brno, Kotlářská 2, ČSSR (Univerzita J. E. Purkyně).