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Tadeusz Bromek; Maria Moszyńska; Krzysztof Prażmowski

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# CONCERNING BASIC NOTIONS OF THE MEASUREMENT THEORY

T. BROMEK, M. MOSZYŃSKA, K. PRAŻMOWSKI, Warszawa

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**Introduction.** This paper was inspired by our cooperation with statisticians who have been interested in the measurement theory as a tool for describing phenomena appearing in statistical practice and research. Applications to statistics will be the subject of another paper. The aim of this paper is to present the basic notions of the measurement theory in a precise and slightly generalized form, to clarify the role of relational structures, and to show to what extent these structures can be modified without changing the type of a scale.

Section 0 contains model-theoretical preliminaries. For a comprehensive treatment the reader is referred to Chang, Keisler [2].

In the classical approach, the measurement theory deals with classes of homomorphisms of relational structures.

In Section 1 we generalize the basic notions of the measurement theory (cf. e.g. Roberts [3]) replacing the set of homomorphisms $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ by an arbitrary class $\mathscr{F}$ of functions from the universe $\Omega_0$ of $\boldsymbol{B}_0$ to the universe $\Omega$ of $\boldsymbol{B}$.

Section 2 concerns the family $\mathscr{F} = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ for arbitrary structures $\boldsymbol{B}_0, \boldsymbol{B}$. We examine the regularity and the type of a scale and their dependence on the underlying structures.

In Section 3 we consider simple measurement scales: nominal, ordinal, interval, and ratio. We first show that under the assumption of regularity these four kinds of scales are types in our sense. Next we try to answer the question what possible structure $\boldsymbol{B}$ may appear when $f: \boldsymbol{B}_0 \to \boldsymbol{B}$ is a scale of one of these four types. In 3.12 we give some comments on Bartoszyński's paper [1].

Section 4 concerns composed measurement scales, i.e. scales of the form $f = (f_1, \ldots, f_n)$ with all $f_i$ being simple scales.

**0. Preliminaries.** Let us start with some basic notions of universal algebra.

Let $\boldsymbol{B} = (\Omega, \mathscr{R}, \Sigma)$ be a relational structure with the universe $\Omega$, the indexed family of relations $\mathscr{R} = (R_\lambda)_{\lambda \in \Lambda}$, and the indexed family of operations $\Sigma = (\sigma_\mu)_{\mu \in M}$.

The *signature of $\boldsymbol{B}$* is understood to be the pair of sequences of natural numbers,

$$s(\boldsymbol{B}) := ((m_\lambda)_{\lambda\in\Lambda}, (n_\mu)_{\mu\in M}) \quad (m_\lambda, n_\mu \geqq 1)$$

where $m_\lambda$ is the arity of $R_\lambda$ and $n_\mu$ is the arity of $\sigma_\mu$.

Let $\boldsymbol{B}_0 = (\Omega_0, \mathscr{R}_0, \Sigma_0)$ and $s(\boldsymbol{B}_0) = s(\boldsymbol{B})$. A *homomorphism* of $\boldsymbol{B}_0$ into $\boldsymbol{B}$ is a function $f\colon \Omega_0 \to \Omega$ satisfying the following conditions:

$$(*) \qquad \forall\lambda\in\Lambda \quad R_{0\lambda}(x_1, \ldots, x_m) \Leftrightarrow R_\lambda(f(x_1), \ldots, f(x_m))$$

and

$$(**) \qquad \forall\mu\in M \quad x = \sigma_{0\mu}(x_1, \ldots, x_n) \Rightarrow f(x) = \sigma_\mu(f(x_1), \ldots, f(x_n))\,.$$

Clearly, every $n$-ary operation $\sigma$ may be treated as an $(n+1)$-ary relation:

$$R(x_1, \ldots, x_{n+1}) :\Leftrightarrow x_{n+1} = \sigma(x_1, \ldots, x_n)\,,$$

which will be referred to as *induced by $\sigma$*. Notice that

**0.1.** *If $R_0, R$ are induced by $\sigma_0, \sigma$, respectively, then for every injective $f$,*

$$f \in \mathrm{Hom}((\Omega_0, \sigma_0), (\Omega, \sigma)) \Rightarrow f \in \mathrm{Hom}((\Omega_0, R_0), (\Omega, R))\,.$$

However, if $f$ is not injective, then the above implication fails.

Indeed, according to the terminology adopted e.g. in Chang, Keisler [2], the condition $(*)$ defines a strong homomorphism.

Setting

$$f(R) := \{(f(x_1), \ldots, f(x_m));\ R(x_1, \ldots, x_m)\}$$

for an arbitrary $m$-ary relation $R$ on $\Omega$, we can replace $(*)$ by the equivalent condition

$$(*') \qquad \forall\lambda\in\Lambda \quad f(R_{0\lambda}) = R_\lambda \,|\, f(\Omega_0)\,.$$

We use the symbol $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ to denote the set of homomorphisms from $\boldsymbol{B}_0$ to $\boldsymbol{B}$. We often write "$f\colon \boldsymbol{B}_0 \to \boldsymbol{B}$" to denote a homomorphism in $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$.

As usual, $f\colon \boldsymbol{B}_0 \to \boldsymbol{B}$ is an *isomorphism* iff there is a $g\colon \boldsymbol{B} \to \boldsymbol{B}_0$ inverse to $f$. We denote by $\mathrm{Iso}(\boldsymbol{B}_0, \boldsymbol{B})$ the set of isomorphisms from $\boldsymbol{B}_0$ to $\boldsymbol{B}$ and by $\mathrm{Aut}\,\boldsymbol{B}$ the set of automorphisms of $\boldsymbol{B}$, i.e.

$$\mathrm{Aut}\,\boldsymbol{B} = \mathrm{Iso}(\boldsymbol{B}, \boldsymbol{B})\,.$$

Since any function $f \in \Omega^{\Omega_0}$ is treated here as a subset $\{(x, f(x));\ x \in \Omega_0\}$ of the Cartesian product $\Omega_0 \times \Omega$ (not as a triple $(f, \Omega_0, \Omega)$), we assume that

$$\Omega' \subset \Omega \Rightarrow (\Omega')^{\Omega_0} \subset \Omega^{\Omega_0}\,.$$

When dealing with definability (cf. 2.6 and 2.7), we use formulae of the first order predicate language $\mathscr{L}_{\boldsymbol{B}}$ without identity. Its specific symbols are those for operations and relations in $\boldsymbol{B}$. Hence, whenever the symbol "=" appears in a formula of $\mathscr{L}_{\boldsymbol{B}}$, it is treated as the predicate for the identity relation $=$ which consequently has to be an element of the family $\mathscr{R}$ in $\boldsymbol{B}$.

For any formula $\alpha$ of $\mathscr{L}_{\boldsymbol{B}}$, we denote by "$\alpha^{\boldsymbol{B}}$" the relation in $\boldsymbol{B}$ defined by $\alpha$. Similarly, for any term $\tau$ of $\mathscr{L}_{\boldsymbol{B}}$, we denote by "$\tau^{\boldsymbol{B}}$" the operation in $\boldsymbol{B}$ defined by $\tau$. That is,

$$\alpha^{\boldsymbol{B}}(x_1, \ldots, x_k) \quad \text{iff} \quad \boldsymbol{B} \models \alpha(x_1, \ldots, x_k)$$
$$(\text{i.e. } \alpha \text{ is satisfied in } \boldsymbol{B} \text{ by } (x_1, \ldots, x_k) \in \Omega^k)$$

and

$$x = \tau^{\boldsymbol{B}}(x_1, \ldots, x_l) \quad \text{iff} \quad \boldsymbol{B} \models (x = \tau(x_1, \ldots, x_l)).$$

**1. Regular and homogeneous families of functions.** Let $\Omega_0$ and $\Omega$ be two non-empty sets and let $\mathscr{F} \subset \Omega^{\Omega_0}$.

**1.1. Definition.** A function $f \in \mathscr{F}$ is *regular with respect to* $\mathscr{F}$ iff

$$\forall g \in \mathscr{F} \quad \exists \varphi : f(\Omega_0) \to \Omega \quad (g = \varphi f).$$

The class $\mathscr{F}$ is *regular* iff all functions in $\mathscr{F}$ are regular with respect to $\mathscr{F}$.

It is easy to show (cf. Roberts [3, p. 60])

**1.2.** *A function $f \in \mathscr{F}$ is regular with respect to $\mathscr{F}$ iff*

$$\forall g \in \mathscr{F} \quad \forall x, y \in \Omega_0 \quad [f(x) = f(y) \Rightarrow g(x) = g(y)].$$

As an immediate consequence of 1.2 we obtain

**1.3.** *Every injection $f \in \Omega^{\Omega_0}$ is regular with respect to $\mathscr{F}$.*

**1.4. Definition.** Let $f \in \mathscr{F}$. A function $\varphi : f(\Omega_0) \to \Omega$ is *admissible for $f$ with respect to* $\mathscr{F}$ iff $\varphi f \in \mathscr{F}$. The set of all functions admissible for $f$ with respect to $\mathscr{F}$ will be denoted by $\Phi_{\mathscr{F}}(f)$ (or simply $\Phi(f)$ if it does not lead to a confusion).

**1.5. Definition.** The class $\mathscr{F}$ is *homogeneous* iff

$$\forall f, g \in \mathscr{F} \quad \Phi_{\mathscr{F}}(f) = \Phi_{\mathscr{F}}(g).$$

The next two propositions concern the connections between the *homogeneity* and the properties of admissible functions, under the assumption of regularity. Notice that the latter (1.7) is almost converse to the former (1.6).

**1.6. Proposition.***) *If there is $f \in \mathscr{F}$ regular with respect to $\mathscr{F}$ with $\Phi_{\mathscr{F}}(f)$ being*

*) For particular cases of 1.6 see Roberts [3, p. 67, Th. 2.2.].

*a transformation group of* $f(\Omega_0)$ *then*

(i) $\mathscr{F}$ *is regular;*

(ii) $\Phi_{\mathscr{F}}(g)$ *is a group for every* $g \in \mathscr{F}$*;*

(iii) $\mathscr{F}$ *is homogeneous.*

Proof. Take $g \in \mathscr{F}$. Since $\Phi(f)$ is a group, obviously $g(\Omega_0) = f(\Omega_0)$. Since $f$ is regular, there exists $\varphi \in \Phi(f)$ such that

$$g = \varphi f . \tag{1}$$

(i): Let $h \in \mathscr{F}$. There is $\psi \in \Phi(f)$ such that

$$h = \psi f . \tag{2}$$

By (1) and (2), since $\Phi(f)$ is a group,

$$h = (\psi\varphi^{-1})\, g .$$

Thus $g$ is regular, which proves (i).

(ii): Notice that

$$\Phi(g) = \Phi(f)\, \varphi^{-1} . \tag{3}$$

Indeed,

$$\psi \in \Phi(g) \Leftrightarrow \psi\varphi \in \Phi(f) \Leftrightarrow \psi = (\psi\varphi)\, \varphi^{-1} \in \Phi(f)\, \varphi^{-1} .$$

This proves (ii).

(iii): We have

$$\Phi(f)\, \varphi^{-1} = \Phi(f) , \tag{4}$$

which combined with (3) proves (iii). ∎

**1.7. Proposition.** *If* $\mathscr{F}$ *is regular and homogeneous, then for every* $f \in \mathscr{F}$*, the set* $\Phi_{\mathscr{F}}(f)$ *is a group of transformations of* $f(\Omega_0)$.

Proof. By homogeneity of $\mathscr{F}$

$$f(\Omega_0) = g(\Omega_0) \quad \text{for every} \quad f, g \in \mathscr{F} .$$

Take $f \in \mathscr{F}$. Clearly $\mathrm{id}_{f(\Omega_0)} \in \Phi(f)$.

Take $\varphi, \psi \in \Phi(f)$. Then $\psi \in \Phi(\varphi f)$, because $\varphi f \in \mathscr{F}$ and $\mathscr{F}$ is homogeneous. Thus $\psi\varphi \in \Phi(f)$. Finally, take $\varphi \in \Phi(f)$ and let

$$g = \varphi f . \tag{1}$$

Since $g$ is regular, there is $\psi \in \Phi(g)$ such that

$$f = \psi g . \tag{2}$$

By (1) and (2), $g = \varphi\psi g$ and $f = \psi\varphi f$, whence $\varphi\psi = \mathrm{id}_{f(\Omega_0)} = \psi\varphi$. Thus $\psi = \varphi^{-1}$. Since $\Phi(g) = \Phi(f)$, it follows that $\psi \in \Phi(f)$. ∎

Let us introduce the following relations on the class $\mathscr{F}$:

**1.8. Definition.** Let $f, g \in \mathscr{F}$.

(i) $f \prec g :\Leftrightarrow \exists \varphi : f(\Omega_0) \to \Omega \ (g = \varphi f)$;

(ii) $f \approx g :\Leftrightarrow f \prec g \wedge g \prec f$.

Evidently

**1.9.** *$\prec$ is a quasi-order and thus $\approx$ is an equivalence relation.*

**1.10. Definition.** The *type* $[f]$ of $f$ is an equivalence class of $f$ with respect to $\approx$

**1.11.** (i) *If $f$ is regular with respect to $\mathscr{F}$, then $f \prec g$ for every $g \in \mathscr{F}$.*

(ii) *If $\mathscr{F}$ is regular, then $\mathscr{F} = [f]$ for every $f \in \mathscr{F}$.*

Let us notice that

**1.12.** *For arbitrary families $\mathscr{F}_1$, $\mathscr{F}_2$ of functions from $\Omega_0$ to $\Omega$,*

(i) $\mathscr{F}_1 \subset \mathscr{F}_2 \Rightarrow \forall f \in \mathscr{F}_1 \ \Phi_{\mathscr{F}_1}(f) \subset \Phi_{\mathscr{F}_2}(f)$;

(ii) *if $f \in \bigcap_i \mathscr{F}_i$ then*

$$\Phi_{\bigcup_i \mathscr{F}_i}(f) = \bigcup_i \Phi_{\mathscr{F}_i}(f) \quad \text{and} \quad \Phi_{\bigcap_i \mathscr{F}_i}(f) = \bigcap_i \Phi_{\mathscr{F}_i}(f) .$$

**1.13. Definition.** $\Phi(\mathscr{F}) := \bigcup_{f \in \mathscr{F}} \Phi_{\mathscr{F}}(f)$.

Then obviously

**1.14.** *If $\mathscr{F}$ is homogeneous, then $\Phi(\mathscr{F}) = \Phi_{\mathscr{F}}(f)$ for every $f \in \mathscr{F}$.*

As an example let us consider the family defined as follows.

**1.15. Definition.** For any $f_0 \in \Omega^{\Omega_0}$ and any group of transformations $G$, of $f_0(\Omega_0)$, let

$$F(G, f_0) := \{\varphi f_0;\ \varphi \in G\} .$$

**1.16.** *$F(G, f_0)$ is regular.*

Proof. Take $f, g \in F(G, f_0)$. There exist $\varphi, \psi \in G$ such that $f = \varphi f_0$ and $g = \psi f_0$. Then $f_0 = \varphi^{-1} f$ and thus

$$g = (\psi\varphi^{-1}) f , \quad \text{where} \quad \psi\varphi^{-1} \in G .$$

Hence $f$ is regular. ■

**1.17.** $\mathsf{F}(G, f_0)$ *is homogeneous. Moreover,*

$$\text{if} \quad \mathscr{F} = \mathsf{F}(G, f_0), \quad \text{then} \quad \Phi_{\mathscr{F}}(f) = G \quad \text{for every} \quad f \in \mathscr{F}.$$

Proof. By 1.4 and 1.15, for every $f \in \mathscr{F}$ we have

$$\psi \in \Phi_{\mathscr{F}}(f) \Leftrightarrow \psi f \in \mathscr{F} \Leftrightarrow \exists \varphi \in G \quad (\psi f = \varphi f_0),$$

and $f = \varphi' f_0$ for some $\varphi' \in G$. Thus

$$\psi \varphi' f_0 = \varphi f_0,$$

whence $\psi \varphi' = \varphi$. Therefore $\psi = \varphi(\varphi')^{-1} \in G$; hence $\Phi_{\mathscr{F}}(f) \subset G$.

On the other hand, let $\psi \in G$ and $f \in \mathscr{F}$. Then $f = \varphi' f_0$ for some $\varphi' \in G$, whence $\psi f = (\psi \varphi') f_0$ with $\psi \varphi' \in G$, i.e. $\psi f \in \mathscr{F}$. Thus $\psi \in \Phi_{\mathscr{F}}(f)$; hence $G \subset \Phi_{\mathscr{F}}(f)$. ∎

As a direct corollary of 1.7 we obtain

**1.18.** *If $\mathscr{F}$ is regular and homogeneous, $f_0 \in \mathscr{F}$ and $G = \Phi_{\mathscr{F}}(f_0)$, then $G$ is a group and*

$$F = \mathsf{F}(G, f_0).$$

Let us now consider the product of families $\mathscr{F}_1, \ldots, \mathscr{F}_n$:

**1.19. Proposition.** *Let $\mathscr{F}_i \subset (\Omega_i)^{\Omega_0}$, $i = 1, \ldots, n$, and let*

$$\mathscr{F} := \mathscr{F}_1 \times \ldots \times \mathscr{F}_n \subset (\Omega_1 \times \ldots \times \Omega_n)^{\Omega_0}.$$

*If $f = (f_1, \ldots, f_n) \in \mathscr{F}$, $f_i$ are regular with respect to $\mathscr{F}_i$ and $f_i(\Omega_0) = \Omega_i$, then*

(i) $\Phi_{\mathscr{F}}(f) = \Phi_{\mathscr{F}_1}(f_1) \times \ldots \times \Phi_{\mathscr{F}n}(f_n)$

*and*

(ii) *$f$ is regular with respect to $\mathscr{F}$.*

Proof. (i): The inclusion $\supset$ is evident. Let us prove $\subset$. Take $\varphi \in \Phi_{\mathscr{F}}(f)$. Then $\varphi f \in \mathscr{F}_1 \times \ldots \times \mathscr{F}_n$, i.e. $\varphi f = (g_1, \ldots, g_n)$ for some $g_i \in \mathscr{F}_i$, $i = 1, \ldots, n$. Since $f_i$ is regular,

$$g_i = \psi_i f_i \quad \text{for some} \quad \psi_i \in \Phi_{\mathscr{F}_i}(f_i), \quad i = 1, \ldots, n.$$

Thus

$$\varphi(f_1, \ldots, f_n) = (\psi_1 f_1, \ldots, \psi_n f_n) \tag{1}$$

and, clearly,

$$\varphi = (\varphi_1, \ldots, \varphi_n) \quad \text{for some} \quad \varphi_i : f_1(\Omega_0) \times \ldots \times f_n(\Omega_0) \to \Omega_1 \times \ldots \times \Omega_n. \tag{2}$$

By (1) and (2), $\varphi = (\psi_1, \ldots, \psi_n)$ and thus $\varphi_i$ depends only on the $i^{\text{th}}$ coordinate. Therefore $\varphi \in \Phi_{\mathscr{F}_1}(f_1) \times \ldots \times \Phi_{\mathscr{F}n}(f_n)$.

Proof of (ii) is routine. ∎

Let us turn to the particular case described by 1.15.

**1.20.** *For $f_i \in (\Omega_i)^{\Omega_0}$, let $G_i$ be a group of transformations of $f_i(\Omega_0)$, $i = 1, \ldots, n$, and let $G$ be the direct product of $G_1, \ldots, G_n$.*

$$G = G_1 \times \ldots \times G_n .$$

*If $f = (f_1, \ldots, f_n): \Omega_0 \to \Omega_1 \times \ldots \times \Omega_n$, then*

$$\mathsf{F}(G, f) = \mathsf{F}(G_1, f_1) \times \ldots \times \mathsf{F}(G_n, f_n) .$$

Proof. By 1.15,

$$\mathsf{F}(G_i, f_i) = \{\varphi_i f_i;\ \varphi_i \in G_i\} \quad \text{for} \quad i = 1, \ldots, n ,$$

and

$$\mathsf{F}(G, f) = \{\varphi f;\ \varphi \in G\} .$$

Since every $\varphi \in G$ is of the form

$$\varphi = (\varphi_1, \ldots, \varphi_n) \quad \text{with} \quad \varphi_i \in G_i ,$$

we have

$$\mathsf{F}(G, f) = \{(\varphi_1 f_1, \ldots, \varphi_n f_n);\ \varphi_i \in G_i\} = \mathsf{F}(G_1, f_1) \times \ldots \times \mathsf{F}(G_n, f_n) . \quad \blacksquare$$

Of course, 1.19 and 1.20 are valid for products of arbitrary collections $(\mathcal{F}_i)_{i \in I}$ and $(G_i)_{i \in I}$, not necessarily finite.

**2. Families of homomorphisms.** We are going to investigate the case which is classical for the measurement theory, when $\mathcal{F} = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ for some relational structures $\boldsymbol{B}_0 = (\Omega_0, \mathcal{R}_0, \Sigma_0)$ and $\boldsymbol{B} = (\Omega, \mathcal{R}, \Sigma)$ of the same signature.

In particular, a homomorphism $f: \boldsymbol{B}_0 \to \boldsymbol{B}$ is called a *(measurement) scale* whenever $\Omega \subset (\mathrm{Re})^n$ for some $n \geqq 1$. *)

Let us start with two simple examples.

**2.1. Example.** Let $\boldsymbol{B}_0 = \boldsymbol{B} = (\mathrm{Re}, \max)$, i.e. $\mathcal{R}_0 = \emptyset = \mathcal{R}$ and $\Sigma_0\ (= \Sigma)$ consists of one binary operation max. Define $f$: Re → Re by the formula

$$f(x) := \begin{cases} 0 & \text{if } |x| \leqq 1 , \\ x - 1 & \text{if } x \geqq 1 , \\ x + 1 & \text{if } x \leqq -1 . \end{cases}$$

Then $f \in \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$, because

$$f(\max(x_1, x_2)) = \max(f(x_1), f(x_2)) .$$

By 1.2, the function $f$ is irregular with respect to $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$, because for $g =$

*) Following Roberts [3] we use the symbol Re for reals.

$= \mathrm{id} : \mathrm{Re} \to \mathrm{Re}$ we have

$$g \in \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B}) \quad \text{and} \quad g(-1) \neq g(1),$$

while $f(-1) = f(1)$.

By 1.3, $g$ is regular with respect to $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$. ∎

**2.2. Example.** Let $\boldsymbol{B}_0 = ((\mathrm{Re})^2, \equiv)$ with $\equiv$ defined by the formula

$$(x_1, x_2) \equiv (y_1, y_2) :\Leftrightarrow x_1 = y_1,$$

and let $\boldsymbol{B} = (\mathrm{Re}, =)$.

Let $f\colon (\mathrm{Re})^2 \to \mathrm{Re}$ be the projection

$$f(x_1, x_2) := x_1 .$$

Then $f$ is regular with respect to $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$, though $f$ is not injective (counter-images of points are equivalence classes of $\equiv$). ∎

In what follows we refer to a scale $f$ as *regular* if it is regular with respect to $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$.

**2.3. Proposition.** *Let $f \in \Omega^{\Omega_0}$ and let $R$ be a relation on $\Omega$. If*

$$R_0 := f^{-1}(R),$$

*then*

(i) *$R_0$ is the unique relation on $\Omega_0$ such that $f \in \mathrm{Hom}((\Omega_0, R_0), (\Omega, R))$;*
(ii) *if $R \subset \Omega^2$ is an equivalence (a weak order) then $R_0$ is also an equivalence (a weak order);**) **)
(iii) *if $R$ is reflexive and antisymmetric (in particular if $R$ is the identity), then $f$ is regular.*

Proof. (i) follows directly from the definition of homomorphism.
(ii): If $R$ is reflexive, symmetric, transitive, connected, then so is, respectively, $R_0$*). This proves (ii).
(iii): Take $g\colon (\Omega_0, R_0) \to (\Omega, R)$.

If $R$ is reflexive and antisymmetric, then

$$f(x) = f(y) \Rightarrow R(f(x), f(y)) \wedge R(f(y), f(x)) \Rightarrow$$
$$\Rightarrow R_0(x, y) \wedge R_0(y, x) \Rightarrow R(g(x), g(y)) \wedge R(g(y), g(x)) \Rightarrow g(x) = g(y).$$

Thus, by 1.2, $f$ is regular. ∎

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*) *Weak order* is understood as a transitive and connected binary relation.
**) In general, if a universal sentence $\beta$ of the 1st order language without $=$) is true in $(\Omega, R)$, then $\beta$ is true in $(\Omega_0, R_0)$ as well.

For $\mathscr{F} = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$, the set $\Phi_{\mathscr{F}}(f)$ and the homogeneity of $\mathscr{F}$ are characterized by the following

**2.4. Proposition.** *Let* $\mathscr{F} = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$. *Then*

(i) $\Phi_{\mathscr{F}}(f) = \mathrm{Hom}(\boldsymbol{B} \mid f(\Omega), \boldsymbol{B})$ *for every* $f \in \mathscr{F}$;

(ii) $\mathscr{F}$ *is homogeneous if and only if*

$$f(\Omega_0) = g(\Omega_0) \quad \textit{for every} \quad f, g \in \mathscr{F}\,.$$

Proof. (i): By 1.4, $(*')$ and $(**)$ from Section 0,

$$\varphi \in \Phi_{\mathscr{F}}(f) \Leftrightarrow \varphi f \in \mathrm{Hom}(B_0, B) \Leftrightarrow$$

$$\Leftrightarrow [\forall \lambda \in \Lambda \ \ \varphi f(R_{0\lambda}) = R_\lambda \mid \varphi f(\Omega_0) \wedge \forall \mu \in M \ \ \forall x \in (\Omega_0)^{n_\mu} \ \varphi f\, \sigma_{0\mu}(x) = \sigma_\mu \varphi\, f(x)] \Leftrightarrow$$

$$\Leftrightarrow \forall \lambda, \mu \ \ [\varphi(R_\lambda) = R_\lambda \mid \varphi f(\Omega_0) \wedge \varphi \sigma_\mu f = \sigma_\mu \varphi f] \Leftrightarrow \varphi \in \mathrm{Hom}(\boldsymbol{B} \mid f(\Omega_0), \boldsymbol{B})\,.$$

(ii) follows immediately from (i). ∎

**2.5. Proposition.** *If* $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ *is regular, then the following are equivalent:*

(i) $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ *consists of surjections;*

(ii) $\Phi(f) = \mathrm{Aut}\, \boldsymbol{B}$ *for some* $f\colon \boldsymbol{B}_0 \to \boldsymbol{B}$;

(iii) $\Phi(f) = \mathrm{Aut}\, \boldsymbol{B}$ *for every* $f\colon \boldsymbol{B}_0 \to \boldsymbol{B}$.

Proof. (i) $\Rightarrow$ (iii): If $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ consists of surjections, then, by 2.4, $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ is homogeneous, whence by 1.7, $\Phi(g)$ is a group for every $g\colon \boldsymbol{B}_0 \to \boldsymbol{B}$, and thus, by 2.4, $\Phi(g) = \mathrm{Aut}\, \boldsymbol{B}$.

The implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious. ∎

The problem arises, to what extent the set of functions admissible for a given function $f\colon \Omega_0 \to \Omega$ depends on the choice of the relational structures for which $f$ has to be a homomorphism. More precisely, let

$$\boldsymbol{B}_0 = (\Omega_0, \mathscr{R}_0, \Sigma_0)\,, \quad \boldsymbol{B} = (\Omega, \mathscr{R}, \Sigma)$$

and

$$\boldsymbol{B}_0' = (\Omega_0, \mathscr{R}_0', \Sigma_0')\,, \quad \boldsymbol{B}' = (\Omega, \mathscr{R}', \Sigma')\,,$$

and let

$$\mathscr{F} = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B}) \quad \text{and} \quad \mathscr{F}' = \mathrm{Hom}(\boldsymbol{B}_0', \boldsymbol{B}')\,.$$

Under what assumption on the pairs $(\boldsymbol{B}_0, \boldsymbol{B})$ and $(\boldsymbol{B}_0', \boldsymbol{B}')$ the equality

$$\Phi(\mathscr{F}) = \Phi(\mathscr{F}')$$

holds?

To answer this question, let us introduce the following relations $\geqq^\alpha$ and $\equiv^{\alpha,\alpha'}$ for relational structures with the same universe:

**2.6. Definition.** Let $\boldsymbol{B} = (\Omega, \mathscr{R}, \Sigma)$ and $\boldsymbol{B}' = (\Omega, \mathscr{R}', \Sigma')$.

(i) $\boldsymbol{B} \geqq^{\alpha} \boldsymbol{B}'$ iff $[(\mathscr{R}', \Sigma')$ is definable in terms of $(\mathscr{R}, \Sigma)$ by means of the family $\alpha$ of formulae and terms of $\mathscr{L}_{\boldsymbol{B}}]$;
(ii) $\boldsymbol{B} \equiv^{\alpha,\alpha'} \boldsymbol{B}'$ iff $\boldsymbol{B} \geqq^{\alpha} B' \wedge B' \geqq^{\alpha'} B$.

Let us prove

**2.7. Lemma.** *If $\boldsymbol{B}_0 \geqq^{\alpha} \boldsymbol{B}_0'$ and $\boldsymbol{B} \geqq^{\alpha} \boldsymbol{B}'$ (for the same $\alpha$!), and either all formulae in $\alpha$ are quantifier-free or $f$: $\boldsymbol{B}_0 \to \boldsymbol{B}$ is surjective, then*

$$f \in \operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B}) \Rightarrow f \in \operatorname{Hom}(\boldsymbol{B}_0', \boldsymbol{B}') .$$

Proof. Let $f \in \operatorname{Hom}(\boldsymbol{B}_0\ \boldsymbol{B})$. First take operations $\sigma_{0\mu}'$ in $\Sigma_0'$ and $\sigma_\mu'$ in $\Sigma'$. By the assumption both $\sigma_{0\mu}'$ and $\sigma_\mu'$ are definable by a symbol $\tau_\mu$ or by a formula $\alpha_\mu$ of $\mathscr{L}_{\boldsymbol{B}}$, i.e. either

$$(1) \qquad \sigma_{0\mu}' = \tau_\mu^{\boldsymbol{B}_0} \quad \text{and} \quad \sigma_\mu' = \tau_\mu^{\boldsymbol{B}}$$

or for every $x_1, \ldots, x_m, x \in \Omega_0$ ($m$ being the arity of $\sigma_\mu'$),

$$(2) \qquad \begin{cases} \sigma_{0\mu}'(x_1, \ldots, x_m) = x \Leftrightarrow \alpha_\mu^{\boldsymbol{B}_0}(x_1, \ldots, x_m, x) \\ \text{and} \\ \sigma_\mu'(f(x_1), \ldots, f(x_m)) = f(x) \Leftrightarrow \alpha_\mu^{\boldsymbol{B}}(f(x_1), \ldots, f(x_m), f(x)) \end{cases}$$

(cf. Section 0).

Assume (1).

To prove that $f(\sigma_{0\mu}'(x_1, \ldots, x_k)) = \sigma_\mu'(f(x_1), \ldots, f(x_k))$ it suffices to show that

$$(3) \qquad f(\tau_\mu^{\boldsymbol{B}_0}(x_1, \ldots, x_m)) = \tau_\mu^{\boldsymbol{B}}(f(x_1), \ldots, f(x_m)) .$$

If $\tau_\mu$ is one of the atomic terms*) in $\mathscr{L}_{\boldsymbol{B}}$, i.e. $\tau_\mu^{\boldsymbol{B}_0} = \sigma_{0\nu}$ and $\tau_\mu^{\boldsymbol{B}} = \sigma_\nu$ for some $\nu \in M$, then (3) holds because $f \in \operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$. It also holds if $\tau_\mu$ is a composition of atomic terms. Thus (3) holds for arbitrary $\tau_\mu$.

Now assume (2). To prove that $f\sigma_{0\mu}'(x_1, \ldots, x_m) = \sigma_\mu'(f(x_1), \ldots, f(x_m))$ it suffices to show that $f(R_0') = R' \mid f(\Omega_0)$ for the relations $R_0'$ and $R'$ induced by $\sigma_{0\mu}'$ and $\sigma_\mu'$. Clearly, in this case we may include $R_0'$ and $R'$ in $\mathscr{R}_0'$ and $\mathscr{R}'$, respectively.

Now take relations $R_{0\lambda}'$ in $\mathscr{R}_0'$ and $R_\lambda'$ in $\mathscr{R}'$. By the assumption, there is a formula $\alpha_\lambda$ of $\mathscr{L}_{\boldsymbol{B}}$ such that for every $x_1, \ldots, x_n \in \Omega_0$ ($n$ being the arity of $R_\lambda'$)

$$(4) \qquad R_{0\lambda}'(x_1, \ldots, x_n) \Leftrightarrow \alpha_\lambda^{\boldsymbol{B}_0}(x_1, \ldots, x_n)$$

and

$$(5) \qquad R_\lambda'(f(x_1), \ldots, f(x_n)) \Leftrightarrow \alpha_\lambda^{\boldsymbol{B}}(f(x_1), \ldots, f(x_n)) .$$

To prove that $f(R_{0\lambda}') = R_\lambda'$ (i.e. the left hand sides of (4) and (5) are equivalent) it suffices to show that

$$(6) \qquad \alpha_\lambda^{\boldsymbol{B}_0}(x_1, \ldots, x_n) \Leftrightarrow \alpha_\lambda^{\boldsymbol{B}}(f(x_1), \ldots, f(x_n)) .$$

*) More precisely, $\tau_\mu(x_1, \ldots, x_m)$ is an atomic term.

We prove (6) by induction on the complexity of $\alpha_\lambda$. If $\alpha_\lambda$ is an atomic formula, i.e. $\alpha_\lambda^{B_0} = R_{0\kappa}$ and $\alpha_\lambda^{B} = R_\kappa$ for some $\kappa \in \Lambda$, then (6) holds because $f(R_{0\kappa}) = R_\kappa \mid f(\Omega_0)$. Thus, by (3), if $\alpha_\lambda^{B_0}(x_1, \ldots, x_n) \Leftrightarrow R_{0\kappa}(\tau_{\mu_1}^{B_0}(x_1, \ldots, x_n), \ldots, \tau_{\mu_{m\kappa}}^{B_0}(x_1, \ldots, x_n))$ for some $\tau_i$, $i = 1, \ldots, m_\kappa$, then (6) holds as well.

If $\alpha_\lambda(x_1, \ldots, x_n)$ is of the form $x_n = \tau(x_1, \ldots, x_{n-1})$ for some $\tau$ in $\mathscr{L}_B$, then, by the assumption on $\mathscr{L}_B$ (cf. Section 0), the identity relation is in $\mathscr{R}$ and in $\mathscr{R}_0$; however, since $=$ is in $\mathscr{R}_0$, it follows that $f$ preserves $=$, i.e. $f$ is injective. Therefore (6) also holds (cf. 0.1). Further, if (6) holds for some formula $\alpha_\lambda$, it also holds for its negation; if (6) holds for two formulae, it also holds for their conjunction.

If (6) holds for some $\alpha_\lambda$, it also holds for $\beta_\lambda$ defined by

$$\beta_\lambda(y) :\Leftrightarrow \forall x \ \alpha_\lambda(x, y),$$

because then $f$ is a surjection (since $\alpha$ contains a formula with quantifiers).

In conclusion, (6) holds for arbitrary $\alpha_\lambda$. This completes the proof. ∎

As a direct consequence of 2.7 we obtain

**2.8. Corollary.** *Let* $\mathscr{F} = \{f \in \mathrm{Hom}(B_0, B);\ f(\Omega_0) = \Omega\}$ and $\mathscr{F}' = \{f \in \mathrm{Hom}(B_0', B');\ f(\Omega_0) = \Omega\}$. *If* $B_0 \equiv^{\alpha,\alpha'} B_0'$ *and* $B \equiv^{\alpha,\alpha'} B'$ *for some* $\alpha$, $\alpha'$, *then for every* $f \in \mathscr{F} \cap \mathscr{F}'$

(i) $\Phi_{\mathscr{F}}(f) = \Phi_{\mathscr{F}'}(f)$;
(ii) *$f$ is regular with respect to $\mathscr{F}$ iff $f$ is regular with respect to $\mathscr{F}'$;*
(iii) *the type of $f$ with respect to $\mathscr{F}$ coincides with the type of $f$ with respect to $\mathscr{F}'$.*

Proof. By 2.7, $\mathscr{F} = \mathscr{F}'$; this implies (i), (ii) and (iii). ∎

**2.9. Corollary.** *Let* $\mathscr{F} = \mathrm{Hom}(B_0, B)$ *and* $\mathscr{F}' = \mathrm{Hom}(B_0', B')$. *If* $B_0 \equiv^{\alpha,\alpha'} B_0'$ *and* $B \equiv^{\alpha,\alpha'} B'$ *for some* $\alpha$, $\alpha'$, *and all formulae in* $\alpha$ *and* $\alpha'$ *are quantifier-free, then for every* $f \in \mathscr{F} \cap \mathscr{F}'$

(i) $\Phi_{\mathscr{F}}(f) = \Phi_{\mathscr{F}'}(f)$;
(ii) *$f$ is regular with respect to $\mathscr{F}$ iff $f$ is regular with respect to $\mathscr{F}'$;*
(iii) *the type of $f$ with respect to $\mathscr{F}$ coincides with the type of $f$ with respect to $\mathscr{F}'$.*

Proof. By 2.7, $\mathscr{F} = \mathscr{F}'$; this implies (i), (ii) and (iii). ∎

**3. Simple measurement scales: nominal, ordinal, interval, and ratio.** Let $B_0 = (\Omega_0, \mathscr{R}_0, \Sigma_0)$ and $B = (\Omega, \mathscr{R}, \Sigma)$. A homomorphism $f$: $B_0 \to B$ is said to be a *simple (measurement) scale* whenever $\Omega \subset \mathrm{Re}$.

**3.1. Definition.** Let $f$: $B_0 \to B$ be a simple scale and let $\mathscr{F} = \mathrm{Hom}(B_0, B)$.

(i) $f$ is a *nominal scale* iff $\Phi_{\mathscr{F}}(f)$ consists of all injections from $f(\Omega_0)$ to $\Omega$;
(ii) $f$ is an *ordinal scale* iff $\Phi_{\mathscr{F}}(f)$ consists of all strictly increasing functions from $f(\Omega_0)$ to $\Omega$;

(iii) $f$ is an *interval scale* iff $\Phi_{\mathscr{F}}(f)$ consists of all increasing similarities from $f(\Omega_0)$ to $\Omega$ ($\varphi(x) = ax + b$ for some $a, b \in \mathrm{Re}$, $a > 0$);

(iv) $f$ is a *ratio scale* iff $\Phi_{\mathscr{F}}(f)$ consists of all increasing homotheties with centre 0*) ($\varphi(x) = ax$ for some $a > 0$).

The above four classes of scales are usually referred to as types of scales (cf. Roberts [3]). We are going to show that under the assumption of regularity of $\mathrm{Hom}(B_0, B)$ this terminology agrees with that introduced in 1.10 (cf. 3.3). Let us start with

**3.2. Theorem.** *Let* $f, g \in \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ *with* $\Omega \in \mathrm{Re}$ *and let* $g \succ f$. *Then*

(i) *$f$ is nominal $\Rightarrow$ $g$ is nominal;*

(ii) *$f$ is ordinal $\Rightarrow$ $g$ is ordinal;*

(iii) *$f$ is interval $\Rightarrow$ $g$ is interval;*

(iv) *$f$ is a ratio scale $\Rightarrow$ $g$ is a ratio scale.*

Proof. Since $g \succ f$, there exists $\varphi_0 \in \Phi(f)$ such that $g = \varphi_0 f$. Thus

$$\psi \in \Phi(g) \Leftrightarrow \psi\varphi_0 \in \Phi(f) . \tag{1}$$

(i): Let $f$ be a nominal scale, i.e.

$$\Phi(f) = \{\varphi : f(\Omega_0) \to \Omega;\ \varphi \text{ is injective}\} . \tag{2}$$

By (1) and (2), $\varphi_0$ is injective and

$$\psi \in \Phi(g) \Leftrightarrow \psi\varphi_0 \text{ is injective} \Leftrightarrow \psi \text{ is injective} .$$

Thus $g$ is a nominal scale.

Proofs of (i)-(iv) are analogous. ∎

As a direct consequence of 3.2 and 1.11 (ii) we obtain

**3.3. Corollary.** *If* $\mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ *is regular, then the classes of nominal, ordinal, interval, and ratio scales from* $\boldsymbol{B}_0$ *to* $\boldsymbol{B}$ *are types of scales.*

The assumption of regularity in 3.3 is esential; suitable examples may be found in Roberts [3, p. 68].

The natural question arises, what are the possible structures of nominal, ordinal, interval or ratio scales as homomorphisms. A partial answer is given by 3.5, 3.6 and 3.8–3.11.

In what follows $\Omega \subset \mathrm{Re}$, $\leqq$ is the natural order in Re, and the quaternary relation $\varrho$ on Re is defined by the formula

$$\varrho(x_1, x_2, y_1, y_2) :\Leftrightarrow x_1 - x_2 = y_1 - y_2 .$$

The restrictions of $\leqq$ and $\varrho$ to a subset of Re will be denoted by the same symbols.

*) Called increasing homotheties in the sequel.

**3.4. Lemma.** *For any connected* $\Omega' \subset \mathrm{Re}$ *with* $\operatorname{card} \Omega' > 1$, *and for any* $\varphi\colon \Omega' \to \Omega \subset \mathrm{Re}$,

(i) *if* $\varphi$ *preserves* $\leqq$ *and* $\varrho$, *then* $\varphi$ *is an increasing similarity,*
(ii) *if* $\varphi$ *preserves* $0$, $\leqq$, *and* $\varrho$, *then* $\varphi$ *is an increasing homothety.*

Proof. (i): Let $\varphi$ preserve $\leqq$ and $\varrho$. It is easy to see that if $x_2 = \frac{1}{2}(x_1 + x_3)$, then $(x_i, \varphi(x_i))$, $i = 1, 2, 3$, are collinear. Further, for any set $Z$ of integers and every sequence $(x_i)_{i \in Z}$, if

$$(1) \qquad x_i = \tfrac{1}{2}(x_{i-1} + x_{i+1}) \quad \text{for every} \quad i \in Z\,,$$

then all $(x_i, \varphi(x_i))$ are collinear.

For every natural $n$, there are a set of integers $Z_n$ and a sequence $(x_i^{(n)})_{i \in Z_n}$ such that

$$(1') \qquad x_{i+1}^{(n)} - x_i^{(n)} = \frac{1}{2^n}$$

and

$$(2) \qquad \forall x \in \Omega' \; \exists i \in Z_n \; \left|x - x_i^{(n)}\right| \leqq \frac{1}{2^n}\,;$$

moreover, we may require that

$$(3) \qquad \{x_i^{(n-1)};\ i \in Z_{n-1}\} \subset \{x_i^{(n)};\ i \in Z_n\} \quad \text{for every} \quad n\,.$$

Thus, the set $\Omega'' = \bigcup_n \{x_i^{(n)};\ i \in Z_n\}$ is dense in $\Omega'$. Since $\Omega''$ is a subset of a line, the function $\varphi \mid \Omega''$ is an (increasing) similarity and thus its unique monotone extension over $\Omega'$ is also an (increasing) similarity.

(ii) follows from (i). ∎

**3.5. Theorem.** *Let* $\Omega \subset \mathrm{Re}$ *and let* $f \in \operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$, *where* $\Omega$ *is the universe of* $\boldsymbol{B}$.

(i) *If* $\boldsymbol{B} \equiv^{\alpha,\alpha'} (\Omega, =)$ *and* $\boldsymbol{B} \mid f(\Omega_0) \equiv^{\alpha,\alpha'} (f(\Omega_0), =)$ *for some quantifier-free* $\alpha, \alpha'$, *then* $f$ *is a nominal scale;*
(ii) *if* $\boldsymbol{B} \equiv^{\alpha,\alpha'} (\Omega, \leqq)$ *and* $\boldsymbol{B} \mid f(\Omega_0) \equiv^{\alpha,\alpha'} (f(\Omega_0), \leqq)$ *for some quantifier-free* $\alpha, \alpha'$, *then* $f$ *is an ordinal scale.*

*If, moreover,* $f(\Omega_0)$ *is connected and* $\operatorname{card} f(\Omega_0) > 1$, *then*

(iii) *if* $\boldsymbol{B} \equiv^{\alpha,\alpha'} (\Omega, \leqq, \varrho)$ *and* $\boldsymbol{B} \mid f(\Omega_0) \equiv^{\alpha,\alpha'} (f(\Omega_0), \leqq, \varrho)$ *for some quantifier-free* $\alpha, \alpha'$, *then* $f$ *is an interval scale;*
(iv) *if* $\boldsymbol{B} \equiv^{\alpha,\alpha'} (\Omega, 0, \leqq, \varrho)$ *and* $\boldsymbol{B} \mid f(\Omega_0) \equiv^{\alpha,\alpha'} (f(\Omega_0), 0, \leqq, \varrho)$ *for some quantifier-free* $\alpha, \alpha'$, *then* $f$ *is a ratio scale.*

Proof. By 2.4,

$$(1) \qquad \Phi(f) = \operatorname{Hom}(\boldsymbol{B} \mid f(\Omega_0), \boldsymbol{B})\,.$$

By 2.8,

$$(2) \qquad \operatorname{Hom}(\boldsymbol{B} \mid f(\Omega_0), \boldsymbol{B}) = \begin{cases} \operatorname{Hom}((f(\Omega_0), =), (\Omega, =)) & \text{for} \quad (\mathrm{i})\,, \\ \operatorname{Hom}(f(\Omega_0), \leqq), (\Omega, \leqq)) & \text{for} \quad (\mathrm{ii})\,, \\ \operatorname{Hom}(f(\Omega_0), \leqq, \varrho), (\Omega, \leqq, \varrho)) & \text{for} \quad (\mathrm{iii})\,, \\ \operatorname{Hom}(f(\Omega_0), 0, \leqq, \varrho), (\Omega, 0, \leqq, \varrho)) & \text{for} \quad (\mathrm{iv})\,. \end{cases}$$

Notice that for an arbitrary $\varphi: f(\Omega_0) \to \Omega$

(3) $\varphi$ preserves $=$ iff $\varphi$ is an injection,

and

(4) $\varphi$ preserves $\leqq$ iff $\varphi$ is strictly increasing.

Clearly, if $\varphi$ is an increasing similarity (homothety) then it preserves $\leqq$ and $\varrho$ (0, $\leqq$, and $\varrho$). Moreover, since $f(\Omega_0)$ is connected and $\operatorname{card} f(\Omega_0) > 1$, 3.4 implies that

(5) $\varphi$ preserves $\leqq$ and $\varrho$ iff $\varphi$ is an increasing similarity

and

(6) $\varphi$ preserves 0, $\leqq$, and $\varrho$ iff $\varphi$ is an increasing homothety.

Now, (1) and (2) combined with (3) prove (i), combined with (4) prove (ii), combined with (5) prove (iii), and combined with (6) prove (iv). ∎

We are now interested in the following question: are the conditions given in 3.5 also necessary for $f$ to be, respectively, a nominal, ordinal, interval, or ratio scale?

As we shall prove, the answer is "almost affirmative" for nominal and ordinal surjective scales (with $\leqq^{\alpha}$ instead of $\equiv^{\alpha,\alpha'}$), while it is negative for interval and ratio scales (cf. 3.8–3.11).

Let us start with the following

**3.6. Definition.** Let $G$ be a group of transformations of $\Omega$ onto itself and let $(x_1, \ldots, x_n) \in \Omega^n$. The set

$$G(x_1, \ldots, x_n) := \{(\varphi(x_1), \ldots, \varphi(x_n)); \varphi \in G\}$$

is called the *orbit of* $(x_1, \ldots, x_n)$ *with respect to* $G$.

**3.7. Lemma.** *For an arbitrary* $G$,

(i) *every orbit with respect to* $G$ *is an invariant of* $G$, *i.e. for every* $(x_1, \ldots, x_n) \in \in \Omega^n$, $n \in N$, *and* $\psi \in G$,

$$\psi(G(x_1, \ldots, x_n)) = G(x_1, \ldots, x_n)\,;$$

(ii) *every invariant of* $G$ *is a union of orbits*; *moreover, for every* $R \subset \Omega^n$, $n \in N$,

$$[\forall \varphi \in G\ \varphi(R) \subset R] \Rightarrow R = \bigcup\{G(x_1, \ldots, x_n); (x_1, \ldots, x_n) \in R\}\,.$$

Proof. (i): Take $\psi \in G$. On the one hand

$$x' \in \psi(G(x_1 \dots x_n)) \Leftrightarrow \exists \varphi \in G \; x' = (\psi\,\varphi(x_1), \dots, \psi\,\varphi(x_n)) \Rightarrow x' \in G(x_1, \dots, x_n)$$

because $\psi\varphi \in G$; thus $\psi\, G(x) \subset G(\dot{x})$ for every $x = (x_1, \dots, x_n) \in \Omega^n$.

On the other hand,

$$x' \in G(x_1, \dots, x_n) \Leftrightarrow \exists \varphi \in G \; x' = (\varphi(x_1), \dots, \varphi(x_n)) \Leftrightarrow$$
$$\Leftrightarrow \exists \varphi \in G \; x' = (\psi\psi^{-1}\,\varphi(x_1), \dots, \psi\psi^{-1}\,\varphi(x_n)) \in \psi(G(x_1, \dots, x_n)),$$

because $\psi^{-1}\varphi \in G$; thus $G(x) \subset \psi(G(x))$ for every $x \in \Omega^n$.

(ii): Take $R \subset \Omega^n$ and let $\varphi(R) \subset R$ for every $\varphi \in G$. Evidently $(x_1, \dots, x_n) \in G(x_1, \dots, x_n)$ because $\mathrm{id}_\Omega \in G$; thus $R \subset \bigcup\{G(x_1, \dots, x_n); (x_1, \dots, x_n) \in R\}$. Conversely, let $x' \in \bigcup\{G(x_1, \dots, x_n); (x_1, \dots, x_n) \in R\}$; then there is $(x_1, \dots, x_n) \in R$ such that $x' = (\varphi(x_1), \dots, \varphi(x_n))$ for some $\varphi \in G$; but $\varphi(R) \subset R$, whence $x' \in R$; thus $\bigcup\{G(x_1, \dots, x_n); (x_1, \dots, x_n) \in R\} \subset R$. ■

We are now ready to prove

**3.8. Theorem.** (i) *If there is a surjective nominal scale* $f: \boldsymbol{B}_0 \to \boldsymbol{B}$, *then* $(\Omega, =) \geqq^\alpha \boldsymbol{B}$ *for some quantifier-free* $\alpha$.

(ii) *If there is a surjective ordinal scale* $f: \boldsymbol{B}_0 \to \boldsymbol{B}$, *and* $\Omega$ *is a connected open subset of* Re *then* $(\Omega, \leqq) \geqq^\alpha \boldsymbol{B}$ *for some quantifier-free* $\alpha$.

Proof. Let $f: \boldsymbol{B}_0 \to \boldsymbol{B}$ be a surjection; then, by 2.4, $\Phi(f) = \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{B})$ and thus $\Phi(f) \supset \mathrm{Aut}\,\boldsymbol{B}$. Let

$$G := \mathrm{Aut}\,\boldsymbol{B}\,.$$

Then, evidently, every $R_\lambda$ in $\mathscr{R}$ and every $\sigma_\mu$ in $\Sigma$ are invariants of $G$. Hence, by 3.7 (ii), they are unions of orbits with respect to $G$.

(i): If $f$ is nominal, then $G$ is the group of bijections of $\Omega$. Let us look for orbits with respect to $G$. Take $x = (x_1, \dots, x_n) \in \Omega^n$ and let $\gamma$ be the conjunction of all the formulae $x_i = x_j$ and $x_k \neq x_l$ which are satisfied by $x$. It is evident that every element of $G(x)$ satisfies $\gamma$. On the other hand, if $\gamma(y_1, \dots, y_n)$ holds in $(\Omega, =)$, then there is a bijection $\varphi_0: \{x_1, \dots, x_n\} \to \{y_1, \dots, y_n\}$. Since $\mathrm{card}\,(\Omega - \{x_1, \dots, x_n\}) = \mathrm{card}\,(\Omega - \{y_1, \dots, y_n\})$, $\varphi_0$ can be extended to a bijection $\varphi: \Omega \to \Omega$. Thus $(y_1, \dots, y_n) \in G(x)$. In conclusion

$$(\Omega, =) \models \gamma(y_1, \dots, y_n) \quad \text{iff} \quad (y_1, \dots, y_n) \in G(x)\,,$$

i.e. $G(x)$ is definable in terms of $=$ by means of $\gamma$. The number of orbits is finite, because the number of $\gamma$'s is finite for a given $n$. Thus every relation and every operation in $\boldsymbol{B}$, as finite unions of orbits, are definable by means of $(\Omega, =)$.

(ii): If $f$ is ordinal, then $G$ is the group of increasing bijections of $\Omega$. Let us again look for orbits with respect to $G$. Take $x = (x_1, \dots, x_n) \in \Omega^n$ and let now $\gamma$ be the conjunction of all the formulae $x_i = x_j$ and $x_k < x_l$ which are satisfied by $x$. Clearly, every element of $G(x)$ also satisfies $\gamma$. On the other hand, if $\gamma(y_1, \ldots, y_n)$ holds in $(\Omega, =, <)$, then there is an increasing bijection $\varphi_0: \{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_n\}$. By the assumption on $\Omega$, $\varphi_0 \subset \varphi \in G$, whence $(y_1, \ldots, y_n) \in G(x)$. In conclusion, $G(x)$ is definable in terms of $=$ and $<$. Again, the number of obrbits in finite, whence $\boldsymbol{B}$ is definable by $=$ and $<$. Obviously, $(=, <)$ and $\leqq$ are mutually definable; thus $\boldsymbol{B}$ is definable by $\leqq$. ■

Let us show that in 3.8 the assumption of the existence of a surjective scale is essential.

**3.9. Example.** Let $\Omega_0 = \{1, \ldots, n\}$ and $\Omega = \mathrm{Re}$; then obviously there is no surjection $f: \Omega_0 \to \Omega$. Further, let $\Sigma_0 = \emptyset = \Sigma$ and let $\mathscr{R} = \{R\}$, $\mathscr{R}_0 = \{R \mid \Omega_0\}$ with the relation $R$ defined by the formula

$$R(x_1, \ldots, x_{n+1}) :\Leftrightarrow [(x_1 = x_2 \wedge \neq (x_2, \ldots, x_{n+1})) \vee x_1 < x_2 < \ldots < x_{n+1}] .$$

The function $f: \Omega_0 \to \Omega$ defined by

$$f(x) = x \quad \text{for every} \quad x \in \Omega_0$$

is obviously a homomorphism of $\boldsymbol{B}_0 = (\Omega_0, \mathscr{R}_0, \Sigma_0)$ in $\boldsymbol{B} = (\Omega, \mathscr{R}, \Sigma)$. Notice that $\Phi(f)$ consists of all injections of $\Omega_0$ into $\Omega$, because $\Phi(f) = \mathrm{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ and

$$R_0(x_1, \ldots, x_{n+1}) \Leftrightarrow R(x_1, \ldots, x_n) \Leftrightarrow (x_1 = x_2 \wedge \neq (x_2, \ldots, x_{n+1})) .$$

Thus $f$ is a nominal scale. But it is easy to show that $R$ is not definable by means of $=$. ■

**3.10. Example.** Let $\Omega_0$, $\Omega$ and $f$ be as in 3.9. Replacing $R$ and $R_0 = R \mid \Omega_0$ in 3.9 by

$$R' \quad \text{and} \quad R'_0 = R' \mid \Omega_0 ,$$

where

$$R'(x_1, \ldots, x_{n+1}) :\Leftrightarrow [x_1 = x_2 < x_3 < \ldots < x_{n+1} \vee$$
$$\vee (\neq (x_1, \ldots, x_{n+1}) \wedge (\exists \lambda > 0 \ x_i = \lambda^{i-1} x_1 \quad \text{for} \quad i = 1, \ldots, n+1))] ,$$

we obtain a homomorphism $f: \boldsymbol{B}'_0 = (\Omega_0, \{R'_0\}) \to (\Omega, \{R'\}) = \boldsymbol{B}'$, which is an ordinal scale, though $R'$ is not definable by $\leqq$. ■

Let us observe that there is no analogue of 3.8 for interval and ratio scales.

**3.11. Example.** Let $\boldsymbol{B}_0 = \boldsymbol{B} = (\mathrm{Re}, \mathscr{R}, \emptyset)$ with $\mathscr{R}$ being the family of all relations invariant under the group of similarities (homotheties). Then among the relations in $\mathscr{R}$ there are those defined by the formula

$$D_\lambda(x_1, x_2, x_3) :\Leftrightarrow x_2 - x_1 = \lambda(x_2 - x_3) \quad \text{for} \quad \lambda \in \mathrm{Re} .$$

Since the family $\{D_\lambda\}_{\lambda \in \mathrm{Re}}$ has cardinality $\mathfrak{c}$, we infer that $R$ has cardinality at least $\mathfrak{c}$.

On the other hand, there is only countably many relations on $\Omega$ which are definable in terms of $\leqq$ and $\varrho$ (of 0, $\leqq$ and $\varrho$). ∎

The following problem remains open:

As was shown by K. Rudnik in [4], if there is a surjective nominal scale $f: \boldsymbol{B}_0 \to \boldsymbol{B}$, the $(\Omega, =)$ is definable in terms of $\boldsymbol{B}$. Using this fact he proved that every surjective nominal scale is regular (cf. [4]).

**3.12. Remark.** An example of a ratio scale is given in Bartoszynski [1], however, the author did not formulate his results in terms of scales. Those results can be presented as follows. The structure $(\Omega_0, x_0, p_0)$ under consideration consists of a non-empty set $\Omega_0$, an $x_0 \in \Omega_0$ and a function $p: (\Omega_0)^2 \to [0, 1]$, satisfying Axioms 1–8. Let us quote only the first three of them:

Axiom 1. $\forall x_1, x_2 \ p(x_1, x_2) + p(x_2, x_1) = 1$.
Axiom 2. $\forall x_1, x_2, x_3 \ (p(x_1, x_2) \geqq \frac{1}{2} \wedge p(x_2, x_3) \geqq \frac{1}{2} \Rightarrow p(x_1, x_3) \geqq$
$\geqq \max(p(x_1, x_2), p(x_2, x_3))$.
Axiom 3. $\forall x \ p(x_0, x) \geqq \frac{1}{2}$.

In fact, the author is interested in another structure

$$\boldsymbol{B}_0 = (\Omega_0, x_0, \preccurlyeq, \varrho_0)\,,$$

with the relations $\preccurlyeq$ and $\varrho_0$ defined by the formulae

$$(*) \qquad x_1 \preccurlyeq x_2 :\Leftrightarrow p(x_1, x_2) \geqq \tfrac{1}{2}$$

and

$$(**) \qquad \varrho_0(x_1, x_2; y_1, y_2) :\Leftrightarrow 0 < p(x_1, x_2) = p(y_1, y_2) < 1\,.$$

By Axioms 1 and 2, the relation $\preccurlyeq$ is a linear order, whence the relation $\sim$ defined by

$$x_1 \sim x_2 :\Leftrightarrow (x_1 \preccurlyeq x_2 \wedge x_2 \preccurlyeq x_1)$$

is an equivalence relation (cf. Prop. 1).

By Axiom 3, $x_0$ is evidently a minimal element in $(\Omega_0, \preccurlyeq)$. Let

$$\boldsymbol{B} = (\mathrm{Re}^+, 0, \leqq, \varrho)\,,$$

where $\varrho$ is a quaternary relation defined by the formula

$$\varrho(t_1, t_2; s_1, s_2) :\Leftrightarrow t_1 - t_2 = s_1 - s_2\,.$$

The author proves that (by Axioms 1–8) there exists a homomorphism*) $f: \boldsymbol{B}_0 \to \boldsymbol{B}$

*) In fact, the author proves that $f(\preccurlyeq) = \leqq$ and $f(\varrho_0) \subset \varrho$; however, by Axioms 2 and 5 we infer that $f(\varrho_0) = \varrho$.

such that

$$\operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B}) = \{\alpha \cdot f;\ \alpha > 0\}.$$

Thus $f$ is a regular ratio scale. Consequently, by 3.2 (iv), $\operatorname{Hom}(B_0, B)$ consists of ratio scales. ∎

**4. Composed measurement scales.** Let $\Omega_i \subset \mathrm{Re}$ for $i = 1, \ldots, n$ and let $\Omega = \Omega_1 \times \ldots \times \Omega_n$. For $i = 1, \ldots, n$, let

$$\mathscr{F}_i = \operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B}_i), \quad \text{where} \quad \boldsymbol{B}_0 = (\Omega, \mathscr{R}_0, \Sigma_0) \quad \text{and} \quad \boldsymbol{B}_i = (\Omega_i, \mathscr{R}_i, \Sigma_i), \quad i = 1, \ldots, n.$$

Consider the family

$$\mathscr{F} := \mathscr{F}_1 \times \ldots \times \mathscr{F}_n$$

as in 1.19.

Every $f \in \mathscr{F}$ is referred to as a *composed measurement scale.*

In particular, let $n = 2$, i.e. $\Omega = \Omega_1 \times \Omega_2 \subset (\mathrm{Re})^2$. The following terminology is commonly used:

$f : \boldsymbol{B}_0 \to \boldsymbol{B} = (\Omega, \mathscr{R}, \Sigma)$ is *nominal-nominal* iff $f = (f_1, f_2)$ with $f_i$-nominal for $i = 1, 2$; $f$ is *nominal-ordinal* iff $f = (f_1, f_2)$ with $f_1$ nominal and $f_2$ ordinal, etc.

By 1.19 (i), a nominal-nominal scale $f$ with $f_1, f_2$ surjective regular is characterized by $\Phi(f)$ which consists of $(\varphi_1, \varphi_2)$ with both $\varphi_1, \varphi_2$ being injections; a nominal-ordinal scale $f$ (with $f_1, f_2$ surjective regular) – by $\Phi(f)$ consisting of $(\varphi_1, \varphi_2)$ with $\varphi_1$ an injection and $\varphi_2$ an increasing injection, and so on.

By 1.19 (ii), if all the components of $f$ are regular, then $f$ itself is regular.

Finally, if $\operatorname{Hom}(\boldsymbol{B}_0, \boldsymbol{B})$ consists of scales with regular components, then by 3.3 the above mentioned classes of composed scales (nominal-nominal, nominal-ordinal, nominal-interval, ordinal-nominal etc.) are types of scales.

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*Authors' addresses:* T. Bromek, Institute of Computer Science, PAS, 00-901 Warszawa, PKiN, Poland; M. Moszyńska, K. Prażmowski, Institute of Mathematics, Warsaw University, 00–901 Warszawa, PKiN, Poland.