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SPACES OF OBSERVABLES

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Suppose that Q is a quantum system, $L(Q)$ its logic (see [7], [17]), and q a physical quantity. Then any statement which an observer can make on Q may be viewed as an assertion specifying that the value of q lies in a certain set of an "evaluating" structure M. Here M may be the space of reals or a more general space. If we denote by $x(A)$ the statement that the value of q lies in the set A, $A \subset M$, we obtain a mapping $x: A \to x(A)$ of subsets of M to $L(Q)$. Such a mapping is called an *observable*. In the axiomatic approach the sets A, $A \subset M$ are assumed to be Borel sets and x is moreover supposed to respect the set-theoretic operations in a certain manner.

In this paper we assume that M is a Banach space and $L(= L(Q))$ is an orthomodular poset. We then consider the question of when (and how) one can introduce a linear and a topological structure into certain sets of observables. This question – obviously important for both physical and mathematical reasons – has been considered in [6] and [17] for $M = R^n$ and L a lattice logic. While the generalization to Banach spaces roots mostly in the mathematical curiosity, the generalization to non-lattice logics is also dictated by physical considerations. The reason is that the assumption on L to be a lattice is the only one which does not seem to be motivated physically (see [1], [17], etc.).

The paper consists of three parts. In the first part we assume that M is separable and show that certain sets of observables can be endowed so that we obtain a Banach space. Although we had to refine some technical procedures and adequately alter a few notions, the central line follows the pattern of the paper [6]. As a more explicit novelty one may offer a new proof of the basic technical tool (Lemma 1.2) and a version of a theorem "on simultaneous testability" (Theorem 1.1). In the second part we introduce compact observables as a natural generalization of real bounded observables. In the third part we consider the observables for M possibly non-separable. We show that a potential generalization, if there is any, requires that the topological character of M be less than the continuum.

1. COMPATIBLE SETS OF BOUNDED OBSERVABLES

Let us first recall the basic notions (see e.g. [6]).

Definition 1.1. A logic (of a quantum mechanical system) is a partially ordered set L with first and last elements 0, 1, respectively, and an operation $a \to a'$ which satisfies the conditions

(i) $(a')' = a$ for all $a \in L$,
(ii) $a \leqq b$ implies $b' \leqq a'$,
(iii) $a \vee a'$ always exists in L and equals 1,
(iv) $\bigvee a_i$ exists in L for any disjoint sequence a_i, $i \in N$ (disjointness: $a_i \leqq a_j'$ for $i \neq j$).

A mapping p: $L_1 \to L_2$ between two logics is called a *σ-homomorphism* if

(i) $p(0) = 0$,
(ii) $p(a') = p(a)'$ for all $a \in L_1$,
(iii) $p(\bigvee a_i) = \bigvee p(a_i)$ for any disjoint sequence $a_i \in L$, $i \in N$.

As an example of a logic we may introduce a σ-algebra or a lattice of projectors of a Hilbert space (see also [5], [13]). In what follows we shall deal with a fixed logic L.

Definition 1.2. Let M be a separable Banach space and let $\mathscr{B}(M)$ denote the σ-algebra of Borel subsets of M. By an observable we mean a σ-homomorphism x: $\mathscr{B}(M) \to L$.

Let us agree that the letters x, y, possibly with indices, $x_1, x_2, x_\lambda, \ldots$, will always mean an observable x: $\mathscr{B}(M) \to L$, where M will be a given separable Banach space.

Proposition 1.1. *If x is an observable then $x(\mathscr{B}(M)) = \{k \in L \mid k = x(A)$ for an $A \in \mathscr{B}(M)\}$ is a Boolean σ-subalgebra of L.*

Proof. Obvious (see also [6]).

Proposition 1.2. *Let x be an observable and let f: $M \to M$ be a Borel measurable mapping. Then the mapping y: $\mathscr{B}(M) \to L$ defined by the formula $y(A) = x(f^{-1}(A))$, $A \in \mathscr{B}(M)$ is an observable. (We shall write $y = xf^{-1}$).*

Proof. Obvious (see also [6]).

Proposition 1.3. *If two observables x_1, x_2: $\mathscr{B}(M) \to L$ coincide on a base for the open sets of M then they are identical.*

Proof. Since M is separable the observables x_1, x_2 must coincide on the collection $\mathcal{O}(M)$ of all open sets of M. One can then continue by induction since $\mathscr{B}(M)$ is known to be the least collection containing $\mathcal{O}(M)$ that is closed under the formation of complements and disjoint countable unions (see [10]).

In the sequel we determine those sets of observables which map $\mathscr{B}(M)$ into a σ-subalgebra of L. It turns out that a finite character condition will guarantee this property.

Definition 1.3. Let $\mathscr{S} = \{x_\lambda \mid \lambda \in I\}$ be a set of observables. Then $\mathscr{S}$ is called *compatible* if any set $\{x_{\lambda_1}(A_1), x_{\lambda_2}(A_2), \ldots, x_{\lambda_n}(A_n)\} \subset L$, $\lambda_i \in I$, $A_i \in \mathscr{B}(M)$ generates a Boolean subalgebra of L.

One may remark that there are also intrinsic definitions of compatibility which are perhaps better for physical verification (see [4], [6]). If a collection of observables is found compatible then the corresponding part of the experiment may be reduced to a classical one. For a detail explanation, see [6], [17].

Proposition 1.4. *If $\{x_\lambda \mid \lambda \in I\}$ is a compatible set of observables, then there is such a σ-subalgebra B of L that $x(A_\lambda) \in B$ for any $\lambda \in I$, $A \in \mathscr{B}(M)$.*

Proof. See [2], [4]. It may be worth while to note that "pairwise compatibility" used in [17] does not ensure the validity of Proposition 1.4 for logics. For this and other algebraic problems concerning compatibility, consult [4] and [9].

The following theorem will enable us to introduce the linear and topological structures into certain sets of compatible observables. The fact that the Borel mappings guaranteed by the theorem can be chosen "uniformly" is perhaps of some interest by itself.

Theorem 1.1. *Let M be a separable Banach space. Then there exist Borel measurable mappings $f_n\colon M \to M$, $n \in N$ such that the following statement holds true: If $\{x_n \mid n \in N\}$ is a sequence of compatible observables then $x_n = zf_n^{-1}$ for an observable $z\colon \mathscr{B}(M) \to L$.*

Proof. We need two lemmas.

Lemma 1.1. (Loomis-Sikorski theorem). *Let B be a Boolean σ-algebra. Let us denote the Stone space of B by S. In other words. let us take such a compact totally disconnected space S that B is Boolean isomorphic to the Boolean algebra $C(S)$ of all clopen subspaces of S. Let (S, Σ) stand for the σ-algebra generated by $C(S)$. Then there exists a σ-homomorphism $h\colon (S, \Sigma) \to B$ whose kernel is the σ-ideal of meagre subsets of S.*

Proof. See e.g. [17], § 2.

Lemma 1.2. *Let $x\colon \mathscr{B}(M) \to L$ be an observable. Put $B = x(\mathscr{B}(M))$. Let $h\colon (S, \Sigma) \to \to B$ have the same meaning as in the previous lemma. Then there exists a (measurable) mapping $g\colon S \to M$ such that $x = hg^{-1}$.*

Proof. If $M = R$ then the theorem has been proved by R. Sikorski (see [14], § 29) and re-proved by V. Varadarajan (see [17], Chap. 1). The proofs use the ordering of R and require rather complicated reasoning. By a classical theorem of the separable descriptive theory, we have $\mathscr{B}(R) = \mathscr{B}(M)$ for any separable Banach space (see [8], § 33, Th. 2). This would give us a proof of Lemma 1.2. The fact is that there is a fairly transparent direct proof whose sketch we shall present here. The method is also applicable to more general situations (see [12]). The details of the following proof may be found in [11].

Let us consider an observable $x: \mathscr{B}(M) \to L$ and put $B = x(\mathscr{B}(M))$. Let the mapping $h: (S, \Sigma) \to B$ be the Loomis-Sikorski σ-homomorphism. Before starting the construction of the mapping $g: S \to M$, let us make a few observations.

(i) With a harmless abuse of notation we can regard B as a collection of subsets of S. For a set $\{B_n | n \in N\} \subset B$, the symbol $\bigvee_{n=1}^{\infty} B_n$ will then mean the least upper bound in B and $\bigcup_{n=1}^{\infty} B_n$ the set-theoretical union. Thus, $\overline{\bigcup_{n=1}^{\infty} B_n} \in \Sigma$.

(ii) If $\{A_n | n \in N\} \subset \mathscr{B}(M)$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $x(A) = \overline{\bigcup_{n=1}^{\infty} x(A_n)}$, where the latter operation means the topological closure in the Stone space S. Therefore the set $x(A) - \bigcup_{n=1}^{\infty} x(A_n)$ is always meagre in S.

(iii) Let us denote by $\mathscr{K}_n^{\wedge}$ the covering of M consisting of all open $1/n$-balls. Take a countable uniform refinement of $\mathscr{K}_n^{\wedge}$ and denote it by $\mathscr{K}_n$. Thus $\mathscr{K}_n = \{K_k^n | k \in N\}$. Since $\bigcup_{k=1}^{\infty} x(K_k^n) = x(M) = 1$, we obtain that the sets $S_n = S - \bigcup_{k=1}^{\infty} x(K_k^n)$ are meagre in S. Put $V = \bigcup_{n=1}^{\infty} S_n$. Then V is also meagre and moreover, if $\mathscr{U} = \{U_\alpha | \alpha \in J\}$ is a uniform covering of M then $\bigcup_{\alpha \in J} x(U_\alpha) \supset S - V$.

(iv) We may suppose that there is a point $r \in M$ such that $x(r) = \emptyset$. Otherwise x would be a σ-isomorphism and we should have nothing to prove.

Let us now return to the construction of $g: S \to M$. If $s \in V$ then we set $g(s) = r$. Suppose now that $s \in S - V$. Put $\mathscr{K} = \bigcup_{n=1}^{\infty} \mathscr{K}_n$ and consider the collection $\mathscr{L}$ of all finite intersections of elements of $\mathscr{K}$. Put $\mathscr{F}_s = \{D \in \mathscr{L} | s \in x(D)\}$. Then $\mathscr{F}_s$ is a Cauchy filter base. Denote by m_s the closure point of $\mathscr{F}_s$ in M. Define $g(s) = m_s$ for any $s \in S - V$. Thus we have completely defined the mapping $g: S \to M$ and one can show that $x = hg^{-1}$. (It suffices to show this identity for all open sets.)

Let us return to the proof of Theorem 1.1. Denote by $\mathscr{B}(M)^{\omega_0}$ the countable product of the σ-algebras $\mathscr{B}(M)$. Let $i: M^{\omega_0} \to M$ be such a mapping between the countable product of M and M that $i^{-1}: \mathscr{B}(M) \to \mathscr{B}(M)^{\omega_0}$ is a Boolean σ-isomorphism (see [8]). Then the required mappings f_n can be defined so that $f_n = \pi_n i^{-1}$, where π_n stand for the respective projections of M^{ω_0} onto the n-th copy of M. To show that, let $\{x_n | n \in N\}$ be a set of compatible observables and let B be a Boolean σ-algebra containing $\bigcup_{n=1}^{\infty} x_n(\mathscr{B}(M))$. Then there is such a countable set of measurable mappings

$g_n: S \to M$ that $x_n = hg_n^{-1}$ (Lemma 1.2). Let us set $z = hg^{-1}i^{-1}$, where $g: S \to M^{\omega_0}$ is the mapping defined by the formula $g(s) = (g_1(s), g_2(s), \ldots)$. Then we obtain that $zf_n^{-1} = hg^{-1}i^{-1}i\pi_n^{-1} = hg^{-1}\pi_n^{-1} = hg_n^{-1}$, and this was to prove.

The foregoing theorem suggests the way of introducing the linear structure into certain sets of observables. If x_1, x_2 are compatible observables then we can write $x_1 = zf_1^{-1}$, $x_2 = zf_2^{-1}$ for an observable z and Borel mappings f_1, f_2. This leads us to setting $x_1 \dotplus x_2 = z(f_1 + f_2)^{-1}$. If $r \in R$, then the scalar multiplication $r \circ x$ is defined as follows: $r \circ x = xj_r^{-1}$, where $j_r(a) = ra$, $a \in M$. Naturally, the observable $o = 0 \circ x$ will serve as the zero.

Now we have to determine suitable sets of observables which will be closed under the formation of the operations $\dotplus, \circ$. This is done in the following definition.

Definition 1.4. Let $\mathcal{S} = \{x_\lambda | \lambda \in I\}$ be a set of compatible observables and let B be the least Boolean σ-subalgebra of L which contains the set $\{x_\lambda(A) | \lambda \in I, A \in \mathcal{B}(M)\}$. We say that $\mathcal{S}$ is *exhaustive* if the following requirement is fulfilled: If x is an observable such that $x(\mathcal{B}(M)) \subset B$ then $x \in \mathcal{S}$.

Theorem 1.2. *Any exhaustive set of observables forms a linear space with the operations* $\dotplus, \circ$.

Proof. We shall show that the operations $\dotplus, \circ$ are correctly defined. The verification of the axioms for a linear space then becomes routine and we leave it to the reader.

Let x_1, x_2 be compatible observables and let f_1, f_2, g_1, g_2 be measurable mappings with $x_1 = zf_1^{-1} = ug_1^{-1}$, $x_2 = zf_2^{-1} = ug_2^{-1}$ for observables z, u. We are to show that $z(f_1 + f_2)^{-1} = u(g_1 + g_2)^{-1}$ holds true for all open subsets of M. Let K be an open subset of M and let $\{G_n | n \in N\}$ be a base for open sets in M. Put $\mathcal{P} = \{(G_{n_1}, G_{n_2}) | G_{n_1} + G_{n_2} \subset K\}$. It is clear that $(f_1 + f_2)^{-1}(K) = \bigcup (f_1^{-1}(G_{n_1}) \cap \cap f_2^{-1}(G_{n_2})$, where $(G_{n_1}, G_{n_2}) \in \mathcal{P}$, and similarly for $g_1 + g_2$. We must show that $z(f_1^{-1}(G_{n_1}) \cap f_2^{-1}(G_{n_2})) = u(g_1^{-1}(G_{n_1}) \cap g_2^{-1}(G_{n_2})$ for any $(G_{n_1}, G_{n_2}) \in \mathcal{P}$. This is easily verified since $z(f_1^{-1}(G_{n_1}) \cap f_2^{-1}(G_{n_2})) = z\, f_1^{-1}(G_{n_1}) \cap z\, f_2^{-1}(G_{n_2}) = u\, g_1^{-1}(G_{n_1}) \cap \cap u\, g_2(G_{n_2}) = u(g_1^{-1}(G_{n_1}) \cap g_2^{-1}(G_{n_2})$.

It follows from the last theorem that if we are given M and L, we have certain subsets of the set of all observables which admit the linear structure. The subsets may be relatively small (e.g. the ones generated by exactly one observable) or relatively big (e.g. the maximal compatible sets). The endowing of the entire set of observables does not seem possible without violating the natural intrinsic properties (see also [7], [12]).

A topology is introduced into observables via the notion of the spectrum (see [6]).

Definition 1.5. The *spectrum* of an observable x is the least closed subset F of M such that $x(F) = 1$. We denote the spectrum of x by $\omega(x)$.

Proposition 1.5. *Any observable has its spectrum.*

Proof. Let $\mathscr{F} = \{F_\alpha | \alpha \in I\}$ be the collection of all closed subsets of M such that $x(F) = 1$. Put $F = \bigcap_{\alpha \in I} F_\alpha$. Then the collection $\{M - F_\alpha | \alpha \in I\}$ is an open covering of (a separable metric) space $M - F$ and therefore there exists a countable collection $\{F_n | n \in N\} \subset \mathscr{F}$ such that $\bigcup_{n=1}^{\infty} (M - F_n) = M - F$. It follows that $F = \bigcap_{n=1}^{\infty} F_n$ and therefore $x(F) = 1$.

Definition 1.6. An observable $x\colon \mathscr{B}(M) \to L$ is called *bounded* if $\omega(x)$ is a bounded subset of M. If x is bounded then $|x| = \sup \{\|m\| \mid m \in \omega(x)\}$ is called *the norm* of x.

Proposition 1.6. *Suppose that x is bounded. Then*

(i) $|x| = 0$ *if and only if* $x = 0$,

(ii) $|r \circ x| = |r|\,|x|$, $r \in R$,

(iii) $|x \mathbin{\#} y| \leqq |x| + |y|$ *for any two compatible bounded observables* x, y.

Proof. Let us indicate the proof of (iii). Since x, y are compatible we can write $x = zf^{-1}$, $y = zg^{-1}$ for an observable $z\colon \mathscr{B}(M) \to L$ and measurable mappings $f, g\colon M \to M$. First we observe that there exist (measurable) mappings $f_1, g_1\colon M \to M$ such that $x = zf_1^{-1}$, $y = zg_1^{-1}$ and moreover, $\omega(x) = \overline{f_1(M)}$, $\omega(y) = \overline{g_1(M)}$. Showing this for f_1, one puts $f_1 = f$ on the set $f^{-1}(\omega(x))$ and defines $g_1(M - \omega(x)) = p$ for an arbitrarily chosen point $p \in \omega(x)$. We can now write $(x \mathbin{\#} y) = z(f_1 + g_1)^{-1}$. Since we have $\omega(x) = \overline{f_1(M)}$, $\omega(y) = \overline{g_1(M)}$ and, for obvious reasons, $\omega(x \mathbin{\#} y) \subset \subset \overline{(f_1 + g_1)(M)}$, we only need to show that $\sup\{\|k\| \mid k \in \overline{(f_1 + g_1)(M)} \leqq \leqq \sup\{\|k\| \mid k \in \overline{f_1(M)}\} + \sup\{\|k\| \mid k \in \overline{g_1(M)}\}$. But this is easy.

Theorem 1.3. *Let $\mathscr{S}$ be an exhaustive set of compatible bounded observables. If we endow the linear space $\mathscr{S}$ with the norm $|\ |$ then $\mathscr{S}$ becomes a Banach space.*

Prof. By Proposition 1.6, the linear space endowed with $|\ |$ is a normed space. We need to show that $\mathscr{S}$ is complete. This immediately follows from the following lemma.

Lemma 1.3. *Let $\{x_n | n \in N\}$ be a sequence of bounded compatible observables. Take a sequence $\{f_n | n \in N\}$ such that $x_n = zf_n^{-1}$ for all $n \in N$ and for an observable z. Then x_n is a Cauchy sequence in the norm $|\ |$ if and only if the sequence $\{f_n | n \in N\}$ is Cauchy on $M - Z$ for a set $Z \in \mathscr{B}(M)$ with $z(Z) = 0$.*

Proof. Suppose that $x_n = zf_n^{-1}$ is Cauchy in the norm $|\ |$. This means that for any $k \in N$ there exists $n_k \in N$ such that, for all $n, p \geqq n_k$, we have $1/k \geqq |x_n - x_p| = = |z(f_n - f_p)^{-1}|$. Put $Z^k_{n,p} = \{m \in M | \|(f_n - f_p)(m)\| > 1/k\}$. Since $|z(f_n - f_p)^{-1}| \leqq \leqq 1/k$ we see that $z(Z^k_{n,p}) = 0$. Put $Z = \bigcup_{k \in N} (\bigcup_{n,p \geqq n_k} Z^k_{n,p})$. As z preserves the countable unions, we obta n that $z(Z) = 0$ and the construction of Z yields that f_n is Cauchy on $M - Z$.

Conversely, let $\{f_n | n \in N\}$ be Cauchy up to a set Z, $z(Z) = 0$. Then for any $k \in N$ there exists $n_k \in N$ such that $\{m \in M | \|f_n(m) - f_p(m)\| > 1/k\} \subset Z$ for all $n, p \geqq n_k$. This implies that $x_n = zf_n^{-1}$ is Cauchy in the norm $| \ |$. Lemma 1.3 is proved.

According to the previous lemma, if $x_n = zf_n^{-1}$ is Cauchy in the norm $| \ |$ then f_n is Cauchy "almost everywhere" with respect to a set Z with $z(Z) = 0$. Then f_n converges to a (Borel) mapping f uniformly on $M - Z$. If we define f on Z e.g. as a constant, we see that x_n converges in the normed space $\mathscr{S}$ to $x = zf^{-1}$. The proof of Theorem 1.3 is complete.

One can easily check that if M is a Banach lattice, algebra, etc., then we can similarly "structure" certain sets of observables, thus obtaning a Banach lattice, algebra, etc. The structural properties of M are generally transferred to the sets of observables. The intrinsic cardinal or topological properties need not be preserved. For instance, if we take $M = R$, $L = \exp N$ the the space of all bounded observables equals $l_\infty(N)$. Although R is separable and reflexive, the space $l_\infty(N)$ is not.

In conclusion of the first part of the paper let us remark that there has been another approach to introducing the linear and convergent structure into sets of observables (see [2], [15]). This technique used a generalization of Caratheodory's notion of scale.

2. COMPACT OBSERVABLES

Let us now introduce a more special class of observables which may be viewed as generalizations of bounded real observables.

Definition 2.1. An observable x is called *compact* if $\omega(x)$ is a compact subspace of M.

Proposition 2.1. (i) *Let x, y be compatible observables and let y be compact. Then $\omega(x \dotplus y) \subset \omega(x) + \omega(y)$.*

(ii) *If x_n are compatible and compact and if x_n converges to x in the norm $| \ |$ then x is compact.*

Proof. (i) By the same argument as in the proof of Proposition 1.6, (iii), we have mappings $g_1, g_2 : M \to M$ such that $x = zg_1^{-1}$, $y = zg_2^{-1}$ and $\overline{g_1(M)} = \omega(x)$, $\overline{g_2(M)} = \omega(y)$. We can write $(x \dotplus y) = z(g_1 + g_2)^{-1}$. Since $\omega(x \dotplus y) \subset \subset \overline{(g_1 + g_2)(M)}$, it suffices to show that $\overline{(g_1 + g_2)(M)} \subset \overline{g_1(M)} + \overline{g_2(M)}$. Suppose that $c \in \overline{(g_1 + g_2)(M)}$. Then there exists a sequence c_n, $c_n \in M$, such that $(g_1 + g_2)(c_n) \to c$. Since $\overline{g_2(M)}$ is compact, there exists such a subsequence c_{n_k} of c_n that $g_2(c_{n_k}) \to d$. Therefore $g_1(c_{n_k}) \to c - d$ and this completes the proof.

(ii) It follows from Lemma 1.3 and the fact that a set arbitrarily close to a precompact set must be precompact.

The above proposition allows us to convert sets of compact observables to Banach spaces.

Theorem 2.1. *Let $\mathcal{S}$ be an exhaustive set of compatible compact observables. If we endow $\mathcal{S}$ with the linear operations $\mp, \circ$ and the norm $|\,|$ then $\mathcal{S}$ becomes a Banach space.*

Proof. Use Proposition 2.1 and Theorem 1.2.

Compact observables seem to be the most natural generalization of the real bounded observables. Since the bounded real observables to the Hilbert space logic $L(H)$ can be identified with the self-adjoint operators on H, one may try to look for an analogous representation of compact observables $x: \mathscr{B}(M) \to L(H)$, M being e.g. a Banach algebra. We do not know of any kind of such a representation yet.

Let us conclude this part by giving a simple example showing that Proposition 2.1 (i) is not valid for bounded observables. Put $L = \exp N$ and $M = l_\infty(N)$. Denote by δ_k the element of $l_\infty(N)$ whose all but the k-th coordinate are 0, the k-th one being 1. Let us define $f, g: N \to l_\infty(N)$ by setting $f(n) = \delta_n$, $g(n) = \delta_n(-1 + 1/n)$. Put $x = f^{-1}$, $y = g^{-1}$. Then $0 \in \omega(x \mp y)$ but $0 \notin \omega(x) + \omega(y)$.

3. PROSPECTS OF A NON-SEPARABLE THEORY

The previous investigations established a strategy for dealing with the "separable" observables. Let us now allow the space M to be non-separable and try to "structure" sets of observables. The following two statements show that we cannot hope for any generalizations unless there is a base for open sets of M which contains no more than 2^{ω_0} elements. (The class of M which remains uncovered by this article will be analysed elsewhere.)

The first statement points out one type of obstacles. Let us express it in the language of Lemma 1.2. Prior to that, recall that a set I is called measurable if there is a non-trivial two-valued σ-additive measure $\mu: \exp I \to \{0, 1\}$ which vanishes at all points (see [16]).

Theorem 3.1. *Suppose that M is a metric space which possesses a discrete subspace of measurable cardinality. Then there is a logic L and an observable $x: \mathscr{B}(M) \to L$ such that $x = hg^{-1}$ for no measurable mapping $g: S \to M$.*

Proof. Let $L = \{0, 1\}$ be viewed as a two-point Boolean algebra. Let $\mu: \exp I \to \{0, 1\}$ be a measure making a discrete set I, $I \subset M$, measurable. Take such an observable $x: \mathscr{B}(M) \to L$ that $x(A) = \mu(A \cap I)$ for any $A \in \mathscr{B}(M)$. Since $h = \mathrm{id}$, the equality $x = hg^{-1}$ would mean that μ is concentrated at a point of I — a contradiction.

It is known that if measurable cardinals exist in an "ordinary" theory of sets, they must be "extremely big" (see [16]). It follows that we could harmlessly ignore them within the framework of quantum theories. Then an analogue of Lemma 1.2 may hold, and often really does, but unfortunately the nonmeasurability of M still appears too weak to make an analogue of Theoıem 1.1 possible. Before showing that, let us agree that a cardinal $\alpha(M)$ for a metric space M will denote the topological character of M ($\alpha(M)$ = the infimum of the cardinalities of the bases for open sets of M).

Theorem 3.2. *Let M be a Banach space of nonmeasurable cardinality and let $\alpha(M) > 2^{\omega_0}$. Then there exist a logic L and two compatible observables $x, y: \mathscr{B}(M) \to \to L$ such that the equalities $x = zf_1^{-1}$, $y = zf_2^{-1}$ do not simultaneously hold for any two Borel mappings $f_1, f_2: M \to M$ and any observable $z: \mathscr{B}(M) \to L$.*

Proof. Let $L = \mathscr{B}(M) \times \mathscr{B}(M)$, that is, let L be the σ-algebra of subsets of $M \times M$ generated by all rectangles $A \times B$, $A \in \mathscr{B}(M)$, $B \in \mathscr{B}(M)$. Let us now define the observables x, y. Since $\alpha(M) > 2^{\omega_0}$, the space M must contain a discrete subspace D such that card $D > 2^{\omega_0}$. Take two points $a, b \in M$ such that the set $J = \{m \in M | \, m = = \lambda a + (1 - \lambda) b, \lambda \in \langle 0, 1 \rangle\}$ is disjoint with D. Put $K = D \cup J$ and pick up an arbitrary point $c \in M - K$. We set $x = p^{-1}$, $y = q^{-1}$, where the mappings $p, q: M \times \times M \to M$ are defined as follows. If $d \in K \times K$ then $p(d) = \pi_1(d)$, $q(d) = \pi_2(d)$, where π_1, π_2 denote the respective projections. If $d \notin K$ then $p(d) = q(d) = c$. One easily sees that p, q are measurable and hence they define observables. Seeking a contradiction, let us suppose that there exist two Borel measurable mappings f_1, f_2 and an observable z such that $x = zf_1^{-1}$, $y = zg_2^{-1}$. Since any metric space of non-measurable cardinality is realcompact (see [12]), we obtain that $z = k^{-1}$ for a mapping $k: M \times M \to M$ (see [12]). We shall show first that k restricted to $K \times K$ is an injective (measurable) mapping onto K. Indeed, if $(t_1, t_2) \in K \times K$, $(u_1, u_2) \in \in K \times K$ and if e.g. $t_1 \neq u_1$ then $k^{-1}(f_1^{-1}(t_1)) \cap k^{-1}(f_1^{-1}(u_1)) = p^{-1}(t_1) \cap \cap p^{-1}(u_1) = \pi_1^{-1}(t_1) \cap \pi_1^{-1}(u_1) = \emptyset$ and therefore $k(t_1, t_2) \neq k(u_1, u_2)$.

We now complete the proof by showing that $k/K \times K$ cannot be measurable. Since card $D > 2^{\omega_0}$, there exists a point $d \in D$ such that $k(\{d\} \times J) \subset D$. We obtain that every subset of $\{d\} \times J$ is Borel which is obviously absurd. The proof of Theorem 3.2 is complete.

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